
MATHEMATICS
Class XI
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MATHEMATICS

CLASS XI **VOLUME-1**

As per the latest revised syllabus prescribed by CBSE for Class XI under 10+2 Pattern of Senior School Certificate Examination

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CHAPTER 1

SETS

1.1 SETS

It is a well known fact that any attempt to define a set has always led mathematicians to unsurmountable difficulties. For example, suppose one defines the term set as *"a well defined collection of objects"*. One may then ask what is meant by a collection. If one answers that a collection is an aggregate of objects or things. What is then an aggregate? Perhaps then one may define that an aggregate is a class of things. What is then a class? Now, one may define a class as a collection. In this manner question after question, since our language is finite, we find that after some time we will have to use some words which have already been questioned. The definition thus becomes circular and worthless. Thus, mathematicians realized that there must be some undefined (or primitive) terms. In this chapter, we start with two undefined (or primitive) terms — "element" and "set". We assume that the word "set" is synonymous with the words "collection", "aggregate", "class" and is comprised of elements. The words "element", "object", "member" are synonymous.

If a is an element of a set A, then we write $a \in A$ and say a belongs to A or a is in A or a is a member of A. If a does not belong to A, then we write $a \notin A$. It is assumed here that if A is any set and a is any element, then either $a \in A$ or $a \notin A$ and the two possibilities are mutually exclusive. Thus, one cannot say "consider the set A of some positive integers", because it is not sure whether $3 \in A$ or $3 \notin A$.

Throughout this chapter we shall denote sets by capital alphabets e.g. A, B, C, X, Y, Z etc. and the elements by the small alphabets e.g. a, b, c, x, y, z etc.

The following are some illustrations of sets:

ILLUSTRATION 1 *The collection of vowels in English alphabets. This set contains five elements, namely, a, e, i, o, u.*

ILLUSTRATION 2 *The collection of first five prime natural numbers is a set containing the elements 2, 3, 5, 7, 11.*

ILLUSTRATION 3 *The collection of all States in the Indian Union is a set.*

ILLUSTRATION 4 *The collection of past presidents of the Indian union is a set.*

ILLUSTRATION 5 *The collection of cricketers in the world who were out for 99 runs in a test match is a set.*

ILLUSTRATION 6 *The collection of good cricket players of India is not a set, since the term "good player is vague and it is not well defined".*

Similarly, collection of good teachers in a school is not a set. However, the collection of all teachers in a school is a set.

In this chapter we will have frequent interaction with some sets, so we reserve some letters for these sets as listed below:

- N : for the set of natural numbers.
- Z : for the set of integers.
- Z^+ : for the set of all positive integers.
- Q : for the set of all rational numbers.

Q^+ : for the set of all positive rational numbers.
R : for the set of all real numbers.
R^+ : for the set of all positive real numbers.
C : for the set of all complex numbers.

EXERCISE 1.1

1. What is the difference between a collection and a set? Give reasons to support your answer?
2. Which of the following collections are sets? Justify your answer:
   (i) A collection of all natural numbers less than 50.
   (ii) The collection of good hockey players in India.
   (iii) The collection of all girls in your class.
   (iv) The collection of most talented writers of India. [NCERT]
   (v) The collection of difficult topics in Mathematics.
   (vi) The collection of novels written by Munshi Prem Chand. [NCERT]
   (vii) The collection of all months of a year beginning with the letter J. [NCERT]
   (viii) The collection of all questions in this chapter. [NCERT]
   (ix) A collection of most dangerous animals of the world. [NCERT]
   (x) The collection of prime integers.
3. If $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then insert the appropriate symbol $\in$ or $\notin$ in each of the following blank spaces:
   (i) $4 \ldots\ldots A$ (ii) $-4 \ldots\ldots A$ (iii) $12 \ldots\ldots A$
   (iv) $9 \ldots\ldots A$ (v) $0 \ldots\ldots A$ (vi) $-2 \ldots\ldots A$

ANSWERS

1. Every set is a collection but a collection is not necessarily a set. Only well defined collections are sets. For example, group of good cricket players is a collection but it is not a set.
2. (i), (iii), (vi), (vii), (viii), (x)
3. (i) $\in$ (ii) $\notin$ (iii) $\notin$ (iv) $\in$ (v) $\in$ (vi) $\notin$

HINTS TO SELECTED PROBLEMS

2. (iv) The collection of most talented writers of India is not a set as there is no specific criterion to determine whether a writer is talented or not.
   (vi) The collection of all months of a year beginning with the letter J is a set given by {January, June July}.
   (vii) The collection of novels written by Munshi Prem Chand is a set because one can determine whether a novel is written by him or not.
   (viii) The collection of all questions in this chapter is a set because if a question is given one can easily decide whether it is a question of this chapter or not.
   (ix) The collection of most dangerous animals of the world is not a set because there is no criterion to determine whether an animal is most dangerous or not.

1.2 DESCRIPTION OF A SET

A set is often described in the following two forms. One can make use of any one of these two ways according to his (her) convenience.

(i) Roster form or Tabular form (ii) Set-builder form

Let us now discuss these forms.

1.2.1 ROSTER FORM

In this form a set is described by listing elements, separated by commas, within braces { }.

ILLUSTRATION 1 *The set of vowels of English Alphabet may be described as {a, e, i, o, u}.*

ILLUSTRATION 2 *The set of even natural numbers can be described as {2, 4, 6, ...}. Here the dots stand for 'and so on'.*

ILLUSTRATION 3 *If A is the set of all prime numbers less than 11, then A = {2, 3, 5, 7}.*

NOTE *The order in which the elements are written in a set makes no difference. Thus, {a, e, i, o, u} and {e, a, i, o, u} denote the same set. Also, the repetition of an element has no effect. For example, {1, 2, 3, 2} is the same set as {1, 2, 3}.*

1.2.2 SET-BUILDER FORM

In this form, a set is described by a characterizing property $P(x)$ of its elements x. In such a case the set is described by $\{x : P(x)$ holds $\}$ or, $\{x \mid P(x)$ holds$\}$, which is read as 'the set of all x such that $P(x)$ holds'. The symbol '|' or ':' is read as 'such that'.

In other words, in order to describe a set, a variable x (say) (to denote each element of the set) is written inside the braces and then after putting a colon the common property $P(x)$ possessed by each element of the set is written within the braces.

ILLUSTRATION 4 *The set E of all even natural numbers can be written as*

$$E = \{x : x \text{ is a natural number and } x = 2n \text{ for } n \in N\}$$

or, $E = \{x : x \in N, x = 2n, n \in N\}$ or, $E = \{x \in N : x = 2n, n \in N\}$

ILLUSTRATION 5 *The set A = {1, 2, 3, 4, 5, 6, 7, 8} can be written as* $A = \{x \in N : x \leq 8\}$.

ILLUSTRATION 6 *The set of all real numbers greater than –1 and less than 1 can be described as* $\{x \in R : -1 < x < 1\}$.

ILLUSTRATION 7 *The set A = {0, 1, 4, 9, 16, ...} can be written as* $A = \{x^2 : \in Z\}$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I ON DESCRIBING OR REPRESENTING SETS IN TABULAR FORM OR ROSTER FORM

EXAMPLE 1 *Describe the following sets in Roster form:*

(i) *The set of all letters in the word 'MATHEMATICS'*

(ii) *The set of all letters in the word 'ALGEBRA'*

(iii) *The set of all vowels in the word 'EQUATION'*

(iv) *The set of all natural numbers less than 7.*

(v) *The set of squares of integers.*

SOLUTION (i) We observe that distinct letters in the word 'MATHEMATICS' are:

M, A, T, H, E, I, C, S

Since the order in which the elements of a set are written is immaterial and the repetition of elements has no effect. So, required set can be described as $\{M, A, T, H, E, I, C, S\}$.

(ii) We find that the word 'ALGEBRA' has following distinct letters: A, L, G, E, B, R

Hence, required set can be described in Roster form as follows: $\{A, L, G, E, B, R\}$.

(iii) Clearly, word 'EQUATION' has vowels: A, E, I, O, U.

So, required set can be described as $\{A, E, I, O, U\}$.

(iv) Natural numbers less than 7 are: 1, 2, 3, 4, 5, 6. Hence, required set can be described as follows: $\{1, 2, 3, 4, 5, 6\}$.

(v) Since square of a negative integer is same as the square of its absolute value. Therefore, squares of integers are 0, 1, 4, 9, 16, 25, ... Hence, required set is $\{0, 1, 4, 9, 16, ...\}$.

Type II ON DESCRIBING OR REPRESENTING SETS IN SET-BUILDER FORM

EXAMPLE 2 *Describe the following sets in set-builder form:*

(i) *The set of all letters in the word 'PROBABILITY'.* (ii) *The set of reciprocals of natural numbers.*
(iii) *The set of all odd natural numbers.* (iv) *The set of all even natural numbers.*

SOLUTION (i) Given set in set-builder form can be described as follows:

$\{x : x$ is a letter in the word 'PROBABILITY'$\}$

(ii) Given set can be described in set-builder form as follows:

$\{x : x$ is reciprocal of a natural number$\}$ or, $\left\{x : x = \frac{1}{n}, n \in N\right\}$ or, $\left\{\frac{1}{n} : n \in N\right\}$

(iii) An odd natural number can be written in the form $(2n - 1)$. So, given set can be described as:

$\{x : x = 2n - 1, n \in N\}$ or, $\{2n - 1 : n \in N\}$.

(iv) An even natural number can be written as $2n$, where $n \in N$. Therefore, set of all even natural numbers can be written in the form $\{x : x = 2n, n \in N\}$ or, $\{2n : n \in N\}$

EXAMPLE 3 *Write the set of all integers whose cube is an even integer.*

SOLUTION We know that the cube of an even integer is also an even integer. Hence, the required set is the set of all even integers which can also be written in the set-builder form as $\{2n : n \in Z\}$.

EXAMPLE 4 *Write the set of all real numbers which cannot be written as the quotient of two integers in the set-builder form.*

SOLUTION We know that all rational numbers are expressible as the quotient of two integers. Therefore, the required set is the set of all irrational numbers which can be written as

$\{x : x$ is real and irrational$\}$ or, $\{x : x \in R$ but $x \notin Q\}$.

Type III ON DESCRIBING A SET IN ROSTER FORM WHEN IT IS GIVEN IN SET-BUILDER FORM

EXAMPLE 5 *Describe each of the following sets in Roster form*

(i) $\{x : x$ *is a positive integer and a divisor of* $9\}$ (ii) $\{x : x \in Z$ *and* $|x| \leq 2\}$

(iii) $\{x : x$ *is a letter of the word 'PROPORTION'*$\}$ (iv) $\left\{x : x = \frac{n}{n^2+1} \text{ and } 1 \leq n \leq 3, \text{ where } n \in N\right\}$

(v) $\{x : x$ *is a positive integer less than 10 and* $2^x - 1$ *is an odd number*$\}$ [NCERT EXEMPLAR]

(vi) $\{x : x$ *is a positive factor of the number* $2^{p-1}(2^p - 1)$, *where* $2^p - 1$ *is a prime number*$\}$.

[NCERT EXEMPLAR]

SOLUTION (i) Since x is a positive integer and a divisor of 9. So, x can take values 1, 3, 9.

$\therefore$ $\{x : x$ is a positive integer and a divisor of 9$\} = \{1, 3, 9\}$

(ii) We find that x is an integer satisfying $|x| \leq 2$ and, $|x| = 0, 1, 2 \Rightarrow x = 0, \pm 1, \pm 2$.

So, x can take values $-2, -1, 0, 1, 2$. Thus, $\{x : x \in Z$ and $|x| \leq 2\} = \{-2, -1, 0, 1, 2\}$.

(iii) We find that distinct letters in the word 'PROPORTION' are P, R, O, T, N, I. So, x can be P, R, O, T, I, N. Hence, $\{x : x$ is a letter in the word '*PROPORTION*'$\} = \{P, R, O, T, I, N\}$

(iv) We have,

$$x = \frac{n}{n^2+1} \text{ where } n \in N \text{ and } 1 \leq n \leq 3.$$

$$\therefore \quad x = \frac{n}{n^2+1}, \text{ where } n = 1, 2, 3 \Rightarrow x = \frac{1}{1^2+1}, \frac{2}{2^2+1}, \frac{3}{3^2+1} \Rightarrow x = \frac{1}{2}, \frac{2}{5}, \frac{3}{10}$$

Hence, $\left\{x : x = \dfrac{n}{n^2+1} \text{ and } 1 \le n \le 3, \text{ where } n \in N\right\} = \left\{\dfrac{1}{2}, \dfrac{2}{5}, \dfrac{3}{10}\right\}$

(v) We find that $2^x - 1$ is an odd number for all positive integral values of x. In particular, $2^x - 1$ is an odd number for $x = 1, 2, 3, \ldots, 9$. Hence, required set is $\{1, 2, 3, \ldots, 9\}$.

(vi) We find that the positive factor of $2^{p-1}(2^p - 1)$ are $1, 2, 2^2, 2^3, \ldots 2^{p-1}, 2^p - 1, 2(2^p - 1), 2^2(2^p - 1), \ldots, 2^{p-1}(2^p - 1)$. Therefore, the given set in roster form is $\left\{1, 2, 2^2, \ldots, 2^{p-1}, (2^p - 1), 2(2^{p-1}) \ldots, 2^{p-1}(2^p - 1)\right\}$.

EXAMPLE 6 *Write the set of all vowels in English alphabet which precede s.*

SOLUTION The vowels in English alphabet which precede s are a, e, i, o. So, the set $A = \{a, e, i, o\}$ is the set of all vowels in English alphabet which precede s.

EXAMPLE 7 *Write the set $A = \{x : x \in Z, x^2 < 20\}$ in the roster form.*

SOLUTION We observe that the integers whose squares are less than 20 are: $0, \pm 1, \pm 2, \pm 3, \pm 4$. Therefore, the set A in roster form is $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

EXAMPLE 8 *Match each of the set on the left described in the roster form with the same set on the right described in the set-builder form.* [NCERT]

(i) $\{P, R, I, N, C, A, L\}$	(a) $\{x : x$ *is a positive integer and is a divisor of* $18\}$
(ii) $\{0\}$	(b) $\{x : x$ *is an integer and* $x^2 - 9 = 0\}$
(iii) $\{1, 2, 3, 6, 9, 18\}$	(c) $\{x : x$ *is an integer and* $x + 1 = 1\}$
(iv) $\{-3, 3\}$	(d) $\{x : x$ *is a letter of the word 'PRINCIPAL'*$\}$

SOLUTION (i) Clearly, $\{P, R, I, N, C, A, L\}$

$= \{P, R, I, N, C, I, P, A, L\} = \{x : x$ is a letter of the word 'PRINCIPAL'$\}$

Hence, (i) matches with (d).

(ii) $\{0\} = \{x : x$ is an integer equal to zero$\} = \{x : x$ is an integer and $x + 1 = 1\}$

Hence, (ii) matches with (c).

(iii) $\{1, 2, 3, 6, 9, 18\}$

= Set of all positive divisors of 18 $= \{x : x$ is a positive integer and is a divisor of $18\}$

Hence, (iii) matches with (a).

(iv) Clearly, $\{-3, 3\} = \{x : x$ is an integer and $x^2 - 9 = 0\}$. Hence, (iv) matches with (b).

Type IV ON DESCRIBING A SET IN SET-BUILDER FORM WHEN IT IS GIVEN IN ROSTER FORM

EXAMPLE 9 *Write the set* $\left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}, \dfrac{6}{7}, \dfrac{7}{8}, \dfrac{8}{9}, \dfrac{9}{10}\right\}$ *in the set-builder form.* [NCERT]

SOLUTION We observe that each element in the given set has the denominator one more than the numerator. Also, the numerator begins from 1 and do not exceed 9. Hence, in the set-builder form the given set can be written as $\left\{x : x = \dfrac{n}{n+1}, n \in N, n \le 9\right\}$.

EXAMPLE 10 *Write the set* $X = \left\{1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \dfrac{1}{25}, \ldots\right\}$ *in the set -builder form.*

SOLUTION We observe that the elements of set X are the reciprocals of the squares of all natural numbers. So, the set X in set builder form is $X = \left\{\dfrac{1}{n^2} : n \in N\right\}$.

EXAMPLE 11 *Write the following sets in Roster form:*

(i) $A = \{a_n : n \in N, a_{n+1} = 3a_n \text{ and } a_1 = 1\}$ (ii) $B = \{a_n : n \in N, a_{n+2} = a_{n+1} + a_n, a_1 = a_2 = 1\}$

SOLUTION (i) We have, $a_1 = 1$ and $a_{n+1} = 3a_n$ for all $n \in N$

Putting $n = 1$ in $a_{n+1} = 3a_n$, we get : $a_2 = 3a_1 = 3 \times 1 = 3$. $[\because a_1 = 1]$

Putting $n = 2$ in $a_{n+1} = 3a_n$, we get : $a_3 = 3a_2 = 3 \times 3 = 3^2$. $[\because a_2 = 3]$

Putting $n = 3$ in $a_{n+1} = 3a_n$, we get : $a_4 = 3a_3 = 3 \times 3^2 = 3^3$. $[\because a_3 = 3]$

Similarly, we obtain : $a_5 = 3a_4 = 3 \times 3^3 = 3^4$, $a_6 = 3a_5 = 3 \times 3^4 = 3^5$ and so on.

Hence, $A = \{a_1, a_2, a_3, a_4, a_5, a_6, \ldots\} = \{1, 3, 3^2, 3^3, 3^4, 3^5, \ldots\ldots\}$

(ii) We have, $a_1 = 1, a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$. Putting $n = 1, 2, 3, 4, \ldots\ldots$ in $a_{n+2} = a_{n+1} + a_n$, we get

$$a_3 = a_2 + a_1 = 1 + 1 = 2;\ a_4 = a_3 + a_2 = 2 + 1 = 3;\ a_5 = a_4 + a_3 = 3 + 2 = 5;$$
$$a_6 = a_5 + a_4 = 5 + 3 = 8 \text{ and so on.}$$

Hence, $B = \{a_1, a_2, a_3, a_4, a_5, a_6, \ldots\ldots\} = \{1, 1, 2, 3, 5, 8, \ldots\ldots\} = \{1, 2, 3, 5, 8, \ldots\ldots\}$

EXAMPLE 12 *State which of the following statements are true and which are false. Justify your answer.*

(i) $35 \in \{x : x$ *has exactly four positive factors*$\}$

(ii) $128 \in \{y :$ *the sum of all positive factors of* y *is* $2y\}$

(iii) $3 \notin \{x : x^4 - 5x^3 + 2x^2 - 112x + 6 = 0\}$

(iv) $496 \notin \{y :$ *the sum of all the positive factors of* y *is* $2y\}$ [NCERT EXEMPLAR]

SOLUTION (i) Positive factors of 35 are 1, 5, 7, 35. Thus, 35 has four positive factors. Therefore, $35 \in \{x : x$ has exactly four positive factors$\}$ is a true statement.

(ii) Positive factors of 128 are $1, 2, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7$. The sum of these factors is

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^7 = \left(\frac{2^8 - 1}{2 - 1}\right) = 255, \text{ which is not equal to } 2 \times 128.$$

So, the given statement is not true.

(iii) We find that $3^4 - 5 \times 3^3 + 2 \times 3^2 - 112 \times 3 + 6 \neq 0$. Therefore,

$3 \notin \{x : x^4 - 5x^3 + 2x^2 - 112x + 6 = 0\}$. Hence, the given statement is true.

(iv) Positive factors of $y = 496$ are : 1, 2, 4, 8, 16, 31, 62, 124, 248 and 496. Clearly, their sum is $2 \times 496 = 2y$. Therefore, 496 belongs to the given set. Hence, the given statement is not true.

EXAMPLE 13 *Given that* $E = \{2, 4, 6, 8, 10\}$. *If* n *represents any member of* E, *then, write the following sets containing all numbers represented by* (i) $n + 1$ (iii) n^2. [NCERT EXEMPLAR]

SOLUTION Given $E = \{2, 4, 6, 8, 10\}$.

(i) Let $A = \{x : x = n + 1, n \in E\}$. Then, $A = \{2 + 1, 4 + 1, 6 + 1, 8 + 1, 10 + 1\} = \{3, 5, 7, 9, 11\}$

(ii) Let $B = \{x : x = n^2, n \in E\}$. Then, $B = \left\{2^2, 4^2, 6^2, 8^2, 10^2\right\} = \{4, 16, 36, 64, 100\}$

EXERCISE 1.2

BASIC

1. Describe the following sets in Roster form:
   (i) $\{x : x$ is a letter before e in the English alphabet$\}$.
   (ii) $\{x \in N : x^2 < 25\}$.

(iii) $\{x \in N : x \text{ is a prime number}, 10 < x < 20\}$.
(iv) $\{x \in N : x = 2n, n \in N\}$. (v) $\{x \in R : x > x\}$.
(vi) $\{x : x \text{ is a prime number which is a divisor of } 60\}$.
(vii) $\{x : x \text{ is a two digit number such that the sum of its digits is } 8\}$.
(viii) The set of all letters in the word 'Trigonometry'.
(ix) The set of all letters in the word 'Better'.
(x) $\{x : x \in R,\ 2x + 11 = 15\}$ (xi) $\{x : x^2 = x, x \in R\}$
(xii) $\{x : x \text{ is a positive factor of prime number } p\}$
(xiii) $\{t : t^3 = t, t \in R\}$ (xiv) $\left\{w : \dfrac{w-2}{w+3} = 3, w \in R\right\}$

Describe the following sets in set-builder form:
(i) $A = \{1, 2, 3, 4, 5, 6\}$ (ii) $B = \{1, 1/2, 1/3, 1/4, 1/5, ...\}$ (iii) $C = \{0, 3, 6, 9, 12, ...\}$
(iv) $D = \{10, 11, 12, 13, 14, 15\}$ (v) $E = \{0\}$ (vi) $\{1, 4, 9, 16, \ldots, 100\}$
(vii) $\{2, 4, 6, 8, ...\}$ (viii) $\{5, 25, 125, 625\}$

List all the elements of the following sets:
(i) $A = \{x : x^2 \le 10, x \in Z\}$ (ii) $B = \left\{x : x = \dfrac{1}{2n-1}, 1 \le n \le 5\right\}$
(iii) $C = \left\{x : x \text{ is an integer}, -\dfrac{1}{2} < x < \dfrac{9}{2}\right\}$
(iv) $D = \{x : x \text{ is a vowel in the word "EQUATION"}\}$
(v) $E = \{x : x \text{ is a month of a year not having 31 days}\}$
(vi) $F = \{x : x \text{ is a letter of the word "MISSISSIPPI"}\}$

Match each of the sets on the left in the roster form with the same set on the right described in the set-builder form:

(i) $\{A, P, L, E\}$	(i) $\{x : x + 5 = 5, x \in Z\}$
(ii) $\{5, -5\}$	(ii) $\{x : x \text{ is a prime natural number and a divisor of } 10\}$
(iii) $\{0\}$	(iii) $\{x : x \text{ is a letter of the word "RAJASTHAN"}\}$
(iv) $\{1, 2, 5, 10\}$	(iv) $\{x : x \text{ is a natural number and divisor of } 10\}$
(v) $\{A, H, J, R, S, T, N\}$	(v) $\{x : x^2 - 25 = 0\}$
(vi) $\{2, 5\}$	(vi) $\{x : x \text{ is a letter of the word "APPLE"}\}$

Write the set of all vowels in the English alphabet which precede q.

Write the set of all positive integers whose cube is odd.

Write the set $\left\{\dfrac{1}{2}, \dfrac{2}{5}, \dfrac{3}{10}, \dfrac{4}{17}, \dfrac{5}{26}, \dfrac{6}{37}, \dfrac{7}{50}\right\}$ in the set-builder form.

If $X = \{1, 2, 3\}$ and n represents any member of X, write the following sets containing all numbers represented by (i) $4n$ (ii) $n + 6$ (iii) $\dfrac{n}{2}$ (iv) $n - 1$.

State which of the following statements are true and which are false. Justify your answer.
(i) $37 \notin \{x : x \text{ has exactly two positive factors}\}$
(ii) $128 \in \{y : \text{the sum of all positive factors of } y \text{ is } 2y\}$
(iii) $7747 \in \{t : t \text{ is a multiple of } 37\}$

Let $X = \{1, 2, 3, 4, 5, 6\}$. If n represents any member of X, express the following as sets :

(i) $n \in X$ but $2n \notin X$ (ii) $n + 5 = 8$ (iii) n is greater than 4.

If $Y = \{1, 2, 3, \ldots, 10\}$ and a represents any element of Y, write the following sets, containing all the elements satisfying the given conditions :

(i) $a \in Y$ and $a^2 \notin Y$ (ii) $a + 1 = 6, a \in Y$ (iii) $a < 6$ and $a \in Y$

ANSWERS

1. (i) $\{a, b, c, d\}$ (ii) $\{1, 2, 3, 4\}$ (iii) $\{11, 13, 17, 19\}$ (iv) $\{2, 4, 6, 8, \ldots\}$
(v) ϕ (vi) $\{2, 3, 5\}$ (vii) $\{17, 26, 35, 44, 53, 62, 71, 80\}$
(viii) $\{T, R, I, G, O, N, M, E, Y\}$ (ix) $\{B, E, T, R\}$ (x) $\{2\}$
(xi) $\{0, 1\}$ (xii) $\{1, p\}$ (xiii) $\{-1, 0, 1\}$ (xiv) $\left\{-\frac{11}{2}\right\}$

2. (i) $\{x : x \in N, x < 7\}$ (ii) $\{x : x = 1/n, n \in N\}$ (iii) $\{x : x = 3n, n \in W\}$
(iv) $\{x : x \in N, 9 < x < 16\}$ (v) $\{x : x = 0\}$ (vi) $\{x^2 : x \in N, 1 \le x \le 10\}$
(vii) $\{x : x = 2n, n \in N\}$ (viii) $\{5^n : n \in N, 1 \le n \le 4\}$

3. (i) $A = \{0, \pm 1, \pm 2, \pm 3\}$ (ii) $B = \left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\right\}$
(iii) $C = \{0, 1, 2, 3, 4\}$ (iv) $D = \{A, E, I, O, U\}$
(v) $E = \{$Feb., April, June, Sept., November$\}$ (vi) $F = \{M, I, S, P\}$

4. (i)$\to$ (vi) ; (ii)$\to$ (v); (iii) $\to$ (i) ; (iv) $\to$ (iv); (v) $\to$ (iii); (vi) $\to$ (ii)

5. $\{a, e, i, o\}$ 6. $\{2n + 1 : n \in Z, n \ge 0\}$ 7. $\left\{\frac{n}{n^2 + 1} : n \in N, n \le 7\right\}$

8. (i) $\{4, 8, 12\}$ (ii) $\{7, 8, 9\}$ (iii) $\left\{\frac{1}{2}, 1, \frac{3}{2}\right\}$ (iv) $\{0, 1, 2\}$

9. (i) False (ii) True (iii) False

10. (i) $\{4, 5, 6\}$ (ii) $\{3\}$ (iii) $\{5, 6\}$

11. (i) $\{4, 5, 6, 7, 8, 9, 10\}$ (ii) $\{5\}$ (iii) $\{1, 2, 3, 4, 5\}$

TYPES OF SETS

A set is said to be empty or null or void set if it has no element and it is denoted by ϕ. In Roster method, ϕ is denoted by $\{\ \}$.

It follows from this definition that a set A is an empty set if the statement $x \in A$ is not true for any x.

ILLUSTRATION 1 $\{x \in R : x^2 = -2\} = \phi$.

ILLUSTRATION 2 $\{x \in N : 5 < x < 6\} = \phi$.

The set A given by $A = \{x : x$ is an even prime number greater than $2\}$ is an empty set because 2 is the only even prime number.

A set consisting of at least one element is called a non-empty or non-void set.

If A and B are any two empty sets, then $x \in A$ iff (if and only if) $x \in B$ is satisfied because there is no element x in either A or B to which the condition may be applied. Thus, $A = B$. Hence, there is only one empty set and we denote it by ϕ. Therefore, article 'the' is used before empty set.

A set consisting of a single element is called a singleton set.

ILLUSTRATION 4 *The set {5} is a singleton set.*

ILLUSTRATION 5 *The set* $\{x : x \in N \text{ and } x^2 = 9\}$ *is a singleton set equal to* $\{3\}$.

FINITE SET *A set is called a finite set if it is either void set or its elements can be listed (counted, labelled) by natural numbers 1, 2, 3, ... and the process of listing terminates at a certain natural number n (say).*

CARDINAL NUMBER OF A FINITE SET *The number n in the above definition is called the cardinal number or order of a finite set A and is denoted by* $n(A)$.

INFINITE SET *A set whose elements cannot be listed by the natural numbers 1, 2, 3, ..., n, for any natural number n is called an infinite set.*

ILLUSTRATION 6 *Each one of the following sets is a finite set:*

(i) *Set of even natural numbers less than 100.* (ii) *Set of soldiers in Indian army.*
(iii) *Set of even prime natural numbers.* (iv) *Set of all persons on the earth.*

ILLUSTRATION 7 *Each one of the following sets is an infinite set:*

(i) *Set of all points in a plane.* (ii) *Set of all lines in a plane.* (iii) $\{x \in R : 0 < x < 1\}$.

EQUIVALENT SETS *Two finite sets A and B are equivalent if their cardinal numbers are same. i.e.* $n(A) = n(B)$.

EQUAL SETS *Two sets A and B are said to be equal if every element of A is a member of B, and every element of B is a member of A.*

If sets A and B are equal, we write $A = B$ and $A \neq B$ when A and B are not equal.

If $A = \{1, 2, 5, 6\}$ and $B = \{5, 6, 2, 1\}$. Then $A = B$, because each element of A is an element of B and vice-versa. Note that the elements of a set may be listed in any order.

A set does not change if one or more elements of the set are repeated. For example, the sets $A = \{1, 2, 3\}$ and $B = \{2, 2, 1, 3, 3\}$ are equal because each element of A is in B and vice-versa. That is why we generally do not repeat any element in describing a set.

It follows from the above definition and the definition of equivalent sets that equal sets are equivalent but equivalent sets need not be equal.

For example, $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ are equivalent sets but not equal sets.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **ON IDENTIFYING WHETHER GIVEN SET IS EMPTY OR NOT**

EXAMPLE 1 *Which of the following sets are empty sets?*

(i) $A = \{x : x^2 - 3 = 0 \text{ and } x \text{ is rational}\}$ (ii) $B = \{x : x \text{ is an even prime number}\}$
(iii) $C = \{x : 4 < x < 5, x \in N\}$ (iv) $D = \{x : x^2 = 25, \text{ and } x \text{ is an odd integer}\}$

SOLUTION (i) We know that there is no rational number whose square is 3. So, $x^2 - 3 = 0$ is not satisfied by any rational number. Hence, A is an empty set.

(ii) We know that 2 is the only even prime number. Therefore, $B = \{2\}$. So, B is not an empty set.

(iii) Since there is no natural number between 4 and 5. So, C is an empty set.

(iv) Since $x = 5, -5$ satisfy $x^2 = 25$ and ± 5 are odd integers. Therefore, $D = \{-5, 5\}$. Thus, D is a non-empty set.

Type II ON EQUAL SETS

EXAMPLE 2 *Find the pairs of equal sets, from the following sets, if any, giving reasons:*

$A=\{0\}$, $B=\{x : x>15$ *and* $x<5\}$, $C=\{x : x-5=0\}$, $D=\{x : x^2=25\}$

$E=\{x : x$ *is an integral positive root of the equation* $x^2-2x-15=0\}$.

SOLUTION We have, $A = \{0\}$, $B=\{x : x>15$ and $x<5\}=\phi$, $C=\{x : x-5=0\}=\{5\}$,

$D=\{x : x^2=25\}=\{-5, 5\}$, and, $E = \{5\}$. Clearly, $C = E$.

EXAMPLE 3 *Which of the following pairs of sets are equal ? Justify your answer.*

(i) $A=\{x : x$ *is a letter in the word "LOYAL"*$\}$, $B=\{x : x$ *is a letter of the word "ALLOY"* $\}$

(ii) $A=\{x : x \in Z$ and $x^2 \leq 8\}$, $B=\{x : x \in R$ *and* $x^2-4x+3=0\}$

SOLUTION (i) We have, $A=\{L, O, Y, A, L\}=\{L, O, Y, A\}$ and, $B=\{A, L, L, O, Y\}=\{L, O, Y, A\}$
Clearly, $A=B$.

(ii) $A=\{x : x \in Z$ and $x^2 \leq 8\}=\{-2, -1, 0, 1, 2\}$ and, $B=\{x : x \in R$ and $x^2-4x+3=0\}=\{1, 3\}$.
We observe that $0 \in A$ but $0 \notin B$. So, $A \neq B$.

Type III ON FINITE AND INFINITE SETS

EXAMPLE 4 *State which of the following sets are finite and which are infinite:*

(i) $A = \{x : x \in Z$ and $x^2-5x+6 = 0\}$ (ii) $B = \{x : x \in Z$ and x^2 is even $\}$

(iii) $C = \{x : x \in Z$ and $x^2 = 36\}$ (iv) $D = \{x : x \in Z$ and $x>-10\}$

SOLUTION (i) $A = \{x : x \in Z$ and $x^2-5x+6=0\} = \{2, 3\}$. So, A is a finite set

(ii) $B = \{x : x \in Z$ and x^2 is even$\} = \{...., -6, -4, -2, 0, 2, 4, 6, ...\}$. Clearly, B is an infinite set.

(iii) $C = \{x : x \in Z$ and $x^2 = 36\} = \{6, -6\}$. Clearly, A is a finite set.

(iv) $D = \{x : x \in Z$ and $x>-10\} = \{-9, -8, -7, ...\}$. Clearly, D is an infinite set.

EXERCISE 1.3

BASIC

1. Which of the following are examples of empty set ?
   (i) Set of all even natural numbers divisible by 5. (ii) Set of all even prime numbers.
   (iii) $\{x : x^2-2 = 0$ and x is rational $\}$.
   (iv) $\{x : x$ is a natural number, $x<8$ and simultaneously $x>12\}$.
   (v) $\{x : x$ is a point common to any two parallel lines $\}$.
2. Which of the following sets are finite and which are infinite ?
   (i) Set of concentric circles in a plane. (ii) Set of letters of the English Alphabets.
   (iii) $\{x \in N : x>5\}$ (iv) $\{x \in N : x<200\}$ (v) $\{x \in Z : x<5\}$ (vi) $\{x \in R : 0<x<1\}$.
3. Which of the following sets are equal ?
   (i) $A=\{1, 2, 3\}$ (ii) $B=\{x \in R : x^2-2x+1=0\}$
   (iii) $C=\{1, 2, 2, 3\}$ (iv) $D=\{x \in R : x^3-6x^2+11x-6=0\}$.
4. Are the following sets equal ?
   $A=\{x : x$ is a letter in the word reap$\}$, $B=\{x : x$ is a letter in the word paper$\}$,
   $C=\{x : x$ is a letter in the word rope$\}$.

5. From the sets given below, pair the equivalent sets: $A = \{1, 2, 3\}$, $B = \{t, p, q, r, s\}$, $C = \{\alpha, \beta, \gamma\}$, $D = \{a, e, i, o, u\}$.
6. Are the following pairs of sets equal ? Give reasons.
   (i) $A = \{2, 3\}$, $B = \{x : x \text{ is a solution of } x^2 + 5x + 6 = 0\}$
   (ii) $A = \{x : x \text{ is a letter of the word "WOLF"}\}$, $B = \{x : x \text{ is a letter of the word "FOLLOW"}\}$ [NCERT]
7. From the sets given below, select equal sets and equivalent sets.
   $A = \{0, a\}$, $B = \{1, 2, 3, 4\}$ $C = \{4, 8, 12\}$, $D = \{3, 1, 2, 4\}$, $E = \{1, 0\}$, $F = \{8, 4, 12\}$
   $G = \{1, 5, 7, 11\}$, $H = \{a, b\}$.
8. Which of the following sets are equal?
   $A = \{x : x \in N, x < 3\}$, $B = \{1, 2\}$, $C = \{3, 1\}$, $D = \{x : x \in N,\ x \text{ is odd}, x < 5\}$,
   $E = \{1, 2, 1, 1\}$, $F = \{1, 1, 3\}$.
9. Show that the set of letters needed to spell "CATARACT" and the set of letters needed to spell "TRACT" are equal. [NCERT]
10. Let $T = \left\{x : \dfrac{x+5}{x-7} - 5 = \dfrac{4x-40}{13-x}\right\}$. Is T an empty set? Justify your answer.

ANSWERS

1. (iii), (iv), (v) 2. (i) Infinite (ii) finite (iii) Infinite (iv) Finite (v) Infinite (vi) Infinite.
3. $A = C = D$ 4. No 5. A, C ; B, D 6. (i) No (ii) Yes
7. Equal sets : $B = D, C = F$ Equivalent sets : A, E, H ; B, D, G ; C, F
8. $A = B = E$, $C = D = F$ 10. Yes

HINTS TO SELECTED PROBLEMS

6. (ii) We have, $A = \{x : x \text{ is a letter of the word "WOLF"}\} = \{W, O, L, F\}$
   $B = \{x : x \text{ is a letter of the word "FOLLOW"}\} = \{W, O, L, F\}$. Clearly, $A = B$.
9. $A =$ Set of letters of the word "CATARACT" $= \{A, C, R, T\}$,
   $B =$ Set of letters of the word "TRACT" $= \{A, C, R, T\}$. Clearly, $A = B$.

1.4 SUBSETS

SUBSETS *Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B.*

If A is a subset of B, we write $A \subseteq B$, which is read as "A is a subset of B" or "A is contained in B".

Thus, $A \subseteq B$ iff $a \in A \Rightarrow a \in B$.

The symbol "$\Rightarrow$" stands for "implies".

If A is a subset of B, we say that B contains A or, B is a super set of A and we also write $B \supseteq A$.

If A is not a subset of B, we write $A \not\subset B$.

Obviously, every set is a subset of itself and the empty set is subset of every set. A subset A of a set B is called a *proper subset* of B, if $A \neq B$ and we write $A \subset B$. In such a case, we also say that B is a super set of A. An improper subset is a subset containing every element of the original set. A proper subset contains some but not all of the elements of the original set. The empty set is a proper subset of a given set.

Thus, if A is a proper subset of B, then there exists an element $x \in B$ such that $x \notin A$.

It follows immediately from this definition and the definition of equal sets that two sets A and B are equal iff $A \subseteq B$ and $B \subseteq A$.

Thus, whenever it is to be proved that two sets A and B are equal, we must prove that $A \subseteq B$ and $B \subseteq A$.

ILLUSTRATION 1 *Clearly* $\{1\} \subset \{1, 2, 3\}$, *but* $\{1, 4\} \not\subset \{1, 2, 3\}$.

ILLUSTRATION 2 *Clearly,* $N \subset Z \subset Q \subset R \subset C$, *where* N, Z, Q, R *and* C *have their usual meanings.*

ILLUSTRATION 3 *If* A *is the set of all divisors of 68 and* B *is the set of all prime divisors of 68, then* B *is the subset of* A *and we write* $B \subset A$.

ILLUSTRATION 4 *Consider the sets* ϕ, $A = \{1, 3\}$, $B = \{1, 5, 9\}$, $C = \{1, 3, 5, 7, 9\}$. *We find that*

(i) $\phi \subset B$ *as* ϕ *is subset of every set.* (ii) $A \not\subset B$ *as* $3 \in A$ *but* $3 \notin B$.

(iii) $A \subset C$ *because each element of* A *belong to* C. *Also,* $B \subset C$.

ILLUSTRATION 5 *If* $A = \{a, e, i, o, u\}$ *and* $B = \{a, b, c, d\}$, *then neither* $A \subset B$ *nor* $B \subset A$, *because* $i \in A$ *but* $i \notin B$ *and* $d \in B$ *but* $d \notin A$.

ILLUSTRATION 6 *Given* $X = \{1, 2, 3, \ldots, 100\}$. *Then, write the subset*

(i) A *of* X, *whose elements are odd numbers.* (ii) B *of* X, *whose elements are even numbers.*

(iii) C *of* X, *whose elements are represented by* $x + 2$, *where* $x \in X$.

(iv) D *of* X, *whose elements are perfect squares* [NCERT EXEMPLAR]

SOLUTION (i) We have, $A = \{x : x \in X \text{ and } x \text{ is odd}\} = \{1, 3, 5, 7, \ldots, 97, 99\}$

(ii) We have, $B = \{x \in X : x \text{ is even}\} = \{2, 4, 6, 8, \ldots, 98, 100\}$

(iii) We have, $C = \{y \in X : y = x + 2, x \in X\}$.

Now, $x = 1$ and $y = x + 2 \Rightarrow y = 3$; $x = 2$ and $y = x + 2 \Rightarrow y = 4, \ldots\ldots\ldots$

$x = 97$ and $y = x + 2 \Rightarrow y = 99$; $x = 98$ and $y = x + 2 \Rightarrow y = 100$

Hence, $C = \{3, 4, 5, \ldots, 99, 100\}$

(iv) We have, $D = \{x \in X : x \text{ is a perfect square}\} \Rightarrow D = \{1, 4, 9, 16, \ldots\ldots, 81, 100\}$

1.4.1 SOME RESULTS ON SUBSETS

THEOREM 1 *Every set is a subset of itself.*

PROOF Let A be any set. Then, each element of A is clearly in A itself. Hence, $A \subseteq A$.

THEOREM 2 *The empty set is a subset of every set.*

PROOF Let A be any set and ϕ be the empty set. In order to show that $\phi \subseteq A$, we must show that every element of ϕ is an element of A also. But, ϕ contains no element. So, every element of ϕ is in A. Hence, $\phi \subset A$.

THEOREM 3 *The total number of subsets of a finite set containing* n *elements is* 2^n.

PROOF Let A be a finite set containing n elements. Let $0 \le r \le n$. Consider those subsets of A that have r elements each. We know that the number of ways in which r elements can be chosen out of n elements is ${}^nC_r$. Therefore, the number of subsets of A having r elements each is ${}^nC_r$. Hence, the total number of subsets of A is ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = (1 + 1)^n = 2^n$.

ILLUSTRATION 1 *Two finite sets have* m *and* n *elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. Find the values of* m *and* n.

[NCERT EXEMPLAR]

SOLUTION Let A and B be two sets having m and n elements respectively. Then,

Number of subsets of set $A = 2^m$, Number of subsets of set $B = 2^n$.

It is given that, $2^m = 56 + 2^n$

i.e. $2^m - 2^n = 56 \Rightarrow 2^n (2^{m-n} - 1) = 2^3 (2^3 - 1) \Rightarrow n = 3$ and $m - n = 3 \Rightarrow n = 3$ and $m = 6$.

ILLUSTRATION 2 *If* $X = \{4^n - 3n - 1 : n \in N\}$ *and* $Y = \{9(n - 1) : n \in N\}$, *prove that* $X \subset Y$.

SOLUTION Let $x_n = 4^n - 3n - 1, n \in N$. Then, $x_1 = 4 - 3 - 1 = 0$. For any $n \ge 2$, we find that

$$x_n = 4^n - 3n - 1 = (1 + 3)^n - 3n - 1$$

$\Rightarrow \quad x_n = {}^nC_0 + {}^nC_1(3) + {}^nC_2(3^2) + {}^nC_3(3^3) + \ldots + {}^nC_n(3^n) - 3n - 1$ [Using Binomial Theorem]

$\Rightarrow \quad x_n = 1 + 3n + {}^nC_2(3^2) + {}^nC_3(3^3) + \ldots + {}^nC_n(3^n) - 3n - 1 \qquad [\because \ {}^nC_0 = 1, {}^nC_1 = n]$

$$\Rightarrow \quad x_n = 3^2\left\{{}^nC_2 + {}^nC_3(3) + {}^nC_4(3^2) + \ldots + {}^nC_n(3^{n-2})\right\}$$

$$\Rightarrow \quad x_n = 9\left\{{}^nC_2 + {}^nC_3(3) + {}^nC_4(3^2) + \ldots + {}^nC_n(3^{n-2})\right\}$$

$\Rightarrow \quad x_n$ is some positive integral multiple of 9 for all $n \geq 2$.

Thus, X consists of all those positive integral multiples of 9 which are of the form

$$9\left\{{}^nC_2 + 3 \times {}^nC_3 + 3^2 \times {}^nC_4 + \ldots + 3^{n-2} \times {}^nC_n\right\} \text{ together with } 0.$$

Clearly, $Y = \{9(n-1) : n \in N\}$ consists of all integral multiples of 9 together with 0. Hence, $X \subset Y$.

1.4.2 SUBSETS OF THE SET R OF REAL NUMBERS

Following sets are important subsets of the set R of all real numbers:

(i) The set of all natural numbers $N = \{1, 2, 3, 4, 5, 6, \ldots\}$

(ii) The set of all integers $Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

(iii) The set of all rational numbers $Q = \left\{x : x = \frac{m}{n}, m, n \in Z, n \neq 0\right\}$.

(iv) The set of all irrational numbers. It is denoted by T.

Thus, $T = \{x : x \in R \text{ and } x \notin Q\}$.

Clearly, $N \subset Z \subset Q \subset R, T \subset R$ and $N \not\subset T$.

1.4.3 INTERVALS AS SUBSETS OF R

On real line various types of infinite subsets are designated as intervals as defined below:

CLOSED INTERVAL *Let a and b be two given real numbers such that $a < b$. Then, the set of all real numbers x such that $a \leq x \leq b$ is called a closed interval and is denoted by $[a, b]$.*

Thus, $[a, b] = \{x \in R : a \leq x \leq b\}$. On the real line, $[a, b]$ may be graphed as shown in Fig. 1.1.

Fig. 1.1 Closed interval

For example, $[-1, 2] = \{x \in R : -1 \leq x \leq 2\}$ is the set of all real numbers lying between -1 and 2 including the end points. Clearly, it is an infinite subset of R.

OPEN INTERVAL *If a and b are two real numbers such that $a < b$, then the set of all real numbers x satisfying $a < x < b$ is called an open interval and is denoted by (a, b) or $]a, b[$.*

Thus, $(a, b) = \{x \in R : a < x < b\}$. On the real line, (a, b) may be graphed as shown in Fig. 1.2.

Fig. 1.2 Open interval

Here, encircling a and b means that a and b are not included in the set.

For example, $(1, 2) = \{x \in R : 1 < x < 2\}$ is the set of all real numbers lying between 1 and 2 excluding the end-points 1 and 2. This is an infinite subset of R.

SEMI-OPEN OR SEMI-CLOSED INTERVAL *If a and b are two real numbers such that $a < b$, then the sets $(a, b] = \{x \in R : a < x \le b\}$ and $[a, b) = \{x \in R : a \le x < b\}$ are known as semi-open or semi-closed intervals. $(a, b]$ and $[a, b)$ are also denoted by $]a, b]$ and $[a, b[$ respectively.*

On real line these sets may be graphed as shown in Figs. 1.3 and 1.4 respectively.

Fig. 1.3 Semi-open interval

Fig. 1.4 Semi open interval

The number $b - a$ is called the length of any of the intervals (a, b), $[a, b]$, $[a, b)$ and $(a, b]$.

These notations provide an alternative way of designating the subsets of the set R of all real numbers. For example, the interval $[0, \infty)$ denotes the set R^+ of all non-negative real numbers, while the interval $(-\infty, 0)$ denotes the set R^- of all negative real numbers. The interval $(-\infty, \infty)$ denotes the set R of all real numbers.

1.5 UNIVERSAL SET

In any discussion in set theory, there always happens to be a set that contains all sets under consideration i.e. it is a super set of each of the given sets. Such a set is called the universal set and is denoted by U.

Thus, a set that contains all sets in a given context is called the universal set.

ILLUSTRATION 1 *When we study two dimensional coordinate geometry, then the set of all points in xy-plane is the universal set.*

ILLUSTRATION 2 *When we are using sets containing natural numbers, then N is the universal set.*

ILLUSTRATION 3 *If $A = \{1, 2, 3\}$, $B = \{2, 4, 5, 6\}$ and $C = \{1, 3, 5, 7\}$, then $U = \{1, 2, 3, 4, 5, 6, 7\}$ can be taken as the universal set.*

ILLUSTRATION 4 *When we are using intervals on real line, the set R of real numbers is taken as the universal set.*

1.6 POWER SET

POWER SET *Let A be a set. Then the collection or family of all subsets of A is called the power set of A and is denoted by P (A).*

That is, $P(A) = \{S : S \subset A\}$.

Since the empty set and the set A itself are subsets of A and are therefore elements of $P(A)$. Thus, the power set of a given set is always non-empty.

ILLUSTRATION 1 *Let $A = \{1, 2, 3\}$. Then, the subsets of A are : ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$* and $\{1, 2, 3\}$. *Hence, $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.*

ILLUSTRATION 2 *If A is the void set ϕ, then P(A) has just one element ϕ i.e. $P(\phi) = \{\phi\}$.*

ILLUSTRATION 3 *Show that $n\{P[P(P(\phi)]\} = 4$.*

SOLUTION We have, $P(\phi) = \{\phi\}$

$\therefore \quad P(P(\phi)) = \{\phi, \{\phi\}\} \Rightarrow P[P(P(\phi))] = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$.

Hence, $P[P(P(\phi))]$ consists of 4 elements i.e. $n\{P[P(P(\phi))]\} = 4$.

We know that a set having n elements has 2^n subsets. Therefore, if A is a finite set having n elements, then P(A) has 2^n elements.

ILLUSTRATION 4 *If $A = \{a, \{b\}\}$, find P (A).*

SOLUTION Let $B = \{b\}$. Then, $A = \{a, B\}$.

Therefore, $P(A) = \{\phi, \{a\}, \{B\}, \{a, B\}\} = \{\phi, \{a\}, \{\{b\}\}, \{a, \{b\}\}\}$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Consider the following sets:* ϕ, $A = \{1, 2\}$, $B = \{1, 4, 8\}$, $C = \{1, 2, 4, 6, 8\}$.

Insert the correct symbol $\subset$ *or* $\not\subset$ *between each of the following pair of sets:*

(i) $\phi \ldots B$ (ii) $A \ldots B$ (iii) $A \ldots C$ (iv) $B \ldots C$

SOLUTION (i) Since null set is subset of every set. Therefore, $\phi \subset B$.

(ii) Clearly, $2 \in A$ but $2 \notin B$. So, $A \not\subset B$.

(iii) Since all elements of set A are in C. So, $A \subset C$.

(iv) Clearly, all elements of set B are in set C. So, $B \subset C$.

EXAMPLE 2 *Let* $A = \{a, b, c, d\}$, $B = \{a, b, c\}$ *and* $C = \{b, d\}$. *Find all sets X such that:*

(i) $X \subset B$ *and* $X \subset C$ (ii) $X \subset A$ *and* $X \not\subset B$.

SOLUTION (i) We have,

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\} \ldots\}$, $P(B) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

and, $P(C) = \{\phi, \{b\}, \{d\}, \{b, d\}\}$

Now, $X \subset B$ and $X \subset C \Rightarrow X \in P(B)$ and $X \in P(C) \Rightarrow X \in \{\phi, \{b\}\} \Rightarrow X = \phi, \{b\}$

(ii) We have,

$X \subset A$ and $X \not\subset B$

$\Rightarrow$ X is a subset of A but X is not a subset of B

$\Rightarrow$ $X \in P(A)$ but $X \notin P(B)$

$\Rightarrow$ $X = \{d\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c, d\}$.

EXAMPLE 3 *Write the following subsets of R as intervals:*

(i) $\{x : x \in R, -4 < x \leq 6\}$ (ii) $\{x : x \in R, -12 < x < -10\}$

(iii) $\{x : x \in R, 0 \leq x < 7\}$ (iv) $\{x : x \in R, 3 \leq x \leq 4\}$.

Also, find the length of each interval.

SOLUTION (i) $\{x : x \in R, -4 < x \leq 6\} = (-4, 6]$. Length $= 6 - (-4) = 10$

(ii) $\{x : x \in R, -12 < x < -10\} = (-12, -10)$. Length $= -10 - (-12) = 2$

(iii) $\{x : x \in R, 0 \leq x < 7\} = [0, 7)$. Length $= 7 - 0 = 7$

(iv) $\{x : x \in R, 3 \leq x \leq 4\} = [3, 4]$. Length $= 4 - 3 = 1$

EXAMPLE 4 *Write the following intervals in the set-builder form:*

(i) $(-7, 0)$ (ii) $[6, 12]$ (iii) $(6, 12]$ (iv) $[-20, 3)$

SOLUTION (i) $(-7, 0) = \{x : x \in R$ and $-7 < x < 0\}$ (ii) $[6, 12] = \{x : x \in R$ and $6 \leq x \leq 12\}$

(iii) $(6, 12] = \{x : x \in R$ and $6 < x \leq 12\}$ (iv) $[-20, 3) = \{x : x \in R$ and $-20 \leq x < 3\}$

EXAMPLE 5 *Let* $A = \{1, 2, \{3, 4\}, 5\}$. *Which of the following statements are incorrect and why?*

(i) $\{3, 4\} \subset A$ (ii) $\{3, 4\} \in A$ (iii) $\{\{3, 4\}\} \subset A$ (iv) $1 \in A$

(v) $1 \subset A$ (vi) $\{1, 2, 5\} \subset A$ (vii) $\{1, 2, 5\} \in A$ (viii) $\{1, 2, 3\} \subset A$

(ix) $\phi \in A$ (x) $\phi \subset A$ (xi) $\{\phi\} \subset A$

SOLUTION $\{3, 4\}$ is an element of set A. Therefore, $\{3, 4\} \in A$ is correct and $\{3, 4\} \subset A$ is incorrect. So, (i) is incorrect and (ii) is correct. As $\{3, 4\}$ is an element of set A. Therefore, $\{\{3, 4\}\}$ is a set containing element $\{3, 4\}$ which belongs to A. So, $\{\{3, 4\}\} \subset A$. Hence, (iii) is correct.

Since 1 is an element of set A. So, $1 \in A$ is correct and $1 \subset A$ is incorrect. So, (iv) is correct and (v) is incorrect.

Since 1, 2, 5 are elements of set A. Therefore, $\{1, 2, 5\}$ is a subset of set A. Hence, (vi) is correct and (vii) is incorrect.

As 3 is not an element of set A. So, $\{1, 2, 3\} \subset A$ is incorrect. The null set is subset of every set. So, $\phi \subset A$ is correct and $\phi \in A$ is incorrect. Hence, (ix) is incorrect and (x) is correct.

As $\phi \subset A$ but $\{\phi\}$ is not a subset of A. So, (xi) is incorrect.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 6 *Prove that $A \subset \phi$ implies $A = \phi$.*

SOLUTION We know that two sets A and B are equal iff $A \subset B$ and $B \subset A$. Also, we know that $\phi \subset A$. It is given that $A \subset \phi$. Thus, we have $A \subset \phi$ and $\phi \subset A \Rightarrow A = \phi$

EXAMPLE 7 *Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3\}$ and $C = \{2, 4\}$. Find all sets X satisfying each pair of conditions:*

(i) $X \subset B$ and $X \not\subset C$ (ii) $X \subset B, X \neq B$ and $X \not\subset C$ (iii) $X \subset A, X \subset B$ and $X \subset C$.

SOLUTION (i) We have,

$X \subset B$ and $X \not\subset C$

$\Rightarrow$ X is a subset of B but X is not a subset of C

$\Rightarrow$ $X \in P(B)$ but $X \notin P(C) \Rightarrow X = \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

(ii) We have,

$X \subset B, X \neq B$ and $X \not\subset C$

$\Rightarrow$ X is a subset of B other than B itself and X is not a subset of C

$\Rightarrow$ $X \in P(B), X \notin P(C)$ and $X \neq B \Rightarrow X = \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$

(iii) We have,

$X \subset A, X \subset B$ and $X \subset C$

$\Rightarrow$ $X \in P(A), X \in P(B)$ and $X \in P(C) \Rightarrow X$ is a subset of A, B and C $\Rightarrow X = \phi, \{2\}$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 8 *Let A, B and C be three sets. If $A \in B$ and $B \subset C$, is it true that $A \subset C$? If not give an example.* [NCERT]

SOLUTION Consider the following sets: $A = \{a\}$, $B = \{\{a\}, b\}$ and $C = \{\{a\}, b, c\}$.

Clearly, $A \in B$ and $B \subset C$. But, $A \not\subset C$ as $a \in A$ but $a \notin C$. Thus, the given statement is not true.

EXAMPLE 9 *Let B be a subset of a set A and let $P(A : B) = \{X \in P(A) : X \supset B\}$.*

(i) *Show that:* $P(A : \phi) = P(A)$ (ii) *If $A = \{a, b, c, d\}$ and $B = \{a, b\}$.*

List all the members of the set $P(A : B)$.

SOLUTION (i) We have,

$P(A : B) = \{X \in P(A) : X \supset B\} = \{X \in P(A) : B \subset X\}$ = Set of all those subsets of A which contain B

$\therefore$ $P(A : \phi)$ = Set of all those subsets of A which contain ϕ = Set of all subsets of set $A = P(A)$.

(ii) If $A = \{a, b, c, d\}$ and $B = \{a, b\}$. Then,

$P(A : B)$ = Set of all those subsets of set A which contain $B = \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}$

EXAMPLE 10 *In each of the following, determine whether the statement is true or false. If it is true, prove it. If it is false, give an example.*

(i) If $x \in A$ and $A \in B$, then $x \in B$ (ii) *If $A \subset B$ and $B \in C$, then $A \in C$*

(iii) *If $A \subset B$ and $B \subset C$, then $A \subset C$* (iv) *If $A \not\subset B$ and $B \not\subset C$, then $A \not\subset C$*

(v) *If $x \in A$ and $A \not\subset B$, then $x \in B$* (vi) *If $A \subset B$ and $x \notin B$, then $x \notin A$*

SOLUTION (i) False: Consider sets $A = \{1\}$ and, $B = \{\{1\}, 2\}$.

Clearly $1 \in A$ and $A \in B$, but $1 \notin B$. So, $x \in A$ and $A \in B$ need not imply that $x \in B$.

(ii) False: Let $A = \{1\}$, $B = \{1, 2\}$ and $C = \{\{1,2\}, 3\}$. Then, we observe that $A \subset B$ and $B \in C$ but $A \notin C$. Thus, $A \subset B$ and $B \in C$ need not imply that $A \in C$.

(iii) True: Let $x \in A$. Then, $A \subset B \Rightarrow x \in B \Rightarrow x \in C$ $[\because B \subset C]$
Thus, $x \in A \Rightarrow x \in C$ for all $x \in A \Rightarrow A \subset C$. Hence, $A \subset B$ and $B \subset C \Rightarrow A \subset C$.

(iv) False: Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 2, 5\}$. Then, $A \not\subset B$ and $B \not\subset C$. But, $A \subset C$. Thus, $A \not\subset B$ and $B \not\subset C$ need not imply that $A \not\subset C$.

(v) False: Let $A = \{1, 2\}$ and $B = \{2, 3, 4, 5\}$. Then, we observe that $1 \in A$ and $A \not\subset B$, but $1 \notin B$. Thus, $x \in A$ and $A \not\subset B$ need not imply that $x \in B$.

(vi) True: Let $A \subset B$. Then, we observe that: $x \in A \Rightarrow x \in B \Leftrightarrow x \notin B \Rightarrow x \notin A$.

EXAMPLE 11 *Let P be the set of prime numbers and let $S = \{t : 2^t - 1$ is a prime$\}$. Prove that $S \subset P$.*

[NCERT EXEMPLAR]

SOLUTION In order to prove that $S \subset P$, it is sufficient to show that $x \in S \Rightarrow x \in P$.
We know that $x \in S \Rightarrow x \in P \Leftrightarrow x \notin P \Rightarrow x \notin S$. Therefore, to prove that $S \subset P$, it is sufficient to show that $x \notin P \Rightarrow x \notin S$.
So, let $x \notin P$. Then, $x \notin P \Rightarrow x$ is a composite number.
We have to prove that $x \notin S$.
If possible, let us assume that $x \in S$. Then,

$$x \in S \Rightarrow 2^x - 1 \text{ is a prime number}$$
$$\Rightarrow 2^x - 1 = m, \text{ where } m \text{ is a prime number } \Rightarrow 2^x = m + 1$$

This is not true for every composite number. Because, for $x = 4$, $2^x = 2^4 = 16$ cannot be written as the sum of a prime number m and 1. Therefore, our supposition is not correct. Consequently, $x \notin S$. Thus, $x \notin P \Rightarrow x \notin S \Leftrightarrow x \in S \Rightarrow x \in P$. Hence, $S \subset P$.

EXERCISE 1.4

BASIC

1. Which of the following statements are true ? Give reason to support your answer.
   (i) For any two sets A and B either $A \subseteq B$ or $B \subseteq A$.
   (ii) Every subset of an infinite set is infinite.
   (iii) Every subset of a finite set is finite.
   (iv) Every set has a proper subset.
   (v) $\{a, b, a, b, a, b, \ldots\}$ is an infinite set.
   (vi) $\{a, b, c\}$ and $\{1, 2, 3\}$ are equivalent sets.
   (vii) A set can have infinitely many subsets.
   (viii) The set of all integers is contained in the set of all rational numbers.
   (ix) The set of all crows is contained in the set of all birds.
   (x) The set of all rectangles is contained in the set of all squares.
   (xi) The set of all real numbers is contained in the set of all complex numbers.
   (xii) The sets $P = \{a\}$ and $B = \{\{a\}\}$ are equal.
   (xiii) The sets $A = \{x : x$ is a letter of the word "LITTLE"$\}$ and, $B = \{x : x$ is a letter of the word "TITLE"$\}$ are equal.
2. State whether the following statements are true or false:
   (i) $1 \in \{1, 2, 3\}$ (ii) $a \subset \{b, c, a\}$ (iii) $\{a\} \in \{a, b, c\}$
   (iv) $\{a, b\} = \{a, a, b, b, a\}$ (v) The set $\{x : x + 8 = 8\}$ is the null set.
3. Decide among the following sets, which are subsets of which:
   $A = \{x : x$ satisfies $x^2 - 8x + 12 = 0\}$, $B = \{2, 4, 6\}$, $C = \{2, 4, 6, 8, \ldots\}$, $D = \{6\}$.
4. Which of the following statements are correct? Write a correct form of each of the incorrect statements.
   (i) $a \subset \{a, b, c\}$ (ii) $\{a\} \in \{a, b, c\}$ (iii) $a \in \{\{a\}, b\}$ (iv) $\{a\} \subset \{\{a\}, b\}$

(v) $\{b, c\} \subset \{a, \{b, c\}\}$ (vi) $\{a, b\} \subset \{a, \{b, c\}\}$ (vii) $\phi \in \{a, b\}$ (viii) $\phi \subset \{a, b, c\}$
(ix) $\{x : x + 3 = 3\} = \phi$

5. Let $A = \{a, b, \{c, d\}, e\}$. Which of the following statements are false and why?
(i) $\{c, d\} \subset A$ (ii) $\{c, d\} \in A$ (iii) $\{\{c, d\}\} \subset A$ (iv) $a \in A$
(v) $a \subset A$ (vi) $\{a, b, e\} \subset A$ (vii) $\{a, b, e\} \in A$ (viii) $\{a, b, c\} \subset A$
(ix) $\phi \in A$ (x) $\{\phi\} \subset A$

6. Write down all possible proper subsets each of the following sets:
(i) $\{1, 2\}$ (ii) $\{1, 2, 3\}$ (iii) $\{1\}$

7. What is the total number of proper subsets of a set consisting of n elements?

BASED ON LOTS

8. Let $A = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$. Determine which of the following is true or false:
(i) $1 \in A$ (ii) $\{1, 2, 3\} \subset A$ (iii) $\{6, 7, 8\} \in A$ (iv) $\{\{4, 5\}\} \subset A$
(v) $\phi \in A$ (vi) $\phi \subset A$

9. Let $A = \{\phi, \{\phi\}, 1, \{1, \phi\}, 2\}$. Which of the following are true?
(i) $\phi \in A$ (ii) $\{\phi\} \in A$ (iii) $\{1\} \in A$ (iv) $\{2, \phi\} \subset A$
(v) $2 \subset A$ (vi) $\{2, \{1\}\} \not\subset A$ (vii) $\{\{2\}, \{1\}\} \not\subset A$
(viii) $\{\phi, \{\phi\}, \{1, \phi\}\} \subset A$ (ix) $\{\{\phi\}\} \subset A$.

10. Write down all possible subsets of each of the following sets:
(i) $\{a\}$ (ii) $\{0, 1\}$ (iii) $\{a, b, c\}$ (iv) $\{1, \{1\}\}$ (v) $\{\phi\}$

11. What universal set (s) would you propose for each of the following:
(i) The set of right triangles. (ii) The set of isosceles triangles.

BASED ON HOTS

12. If A is any set, prove that: $A \subseteq \phi \Leftrightarrow A = \phi$.

13. Prove that: $A \subseteq B$, $B \subseteq C$ and $C \subseteq A \Rightarrow A = C$.

14. How many elements has $P(A)$, if $A = \phi$?

15. If $X = \{8^n - 7n - 1 : n \in N\}$ and $Y = \{49(n-1) : n \in N\}$, then prove that $X \subseteq Y$.

ANSWERS

1. (i) F, $A = \{1, 2, 3\}$, $B = \{a, b\}$ (ii) F, $A = \{1, 2\}$ is a finite subset of N.
(iii) T (iv) F, ϕ does not have a proper subset
(v) F, Given set $= \{a, b\}$ (vi) T (vii) F

2. (i) T (ii) F (iii) F (iv) T (v) F (viii) T (ix) T (x) F (xi) T
(xii) F (xiii) T

3. $D \subset A \subset B \subset C$

4. (i) $a \in \{a, b, c\}$ (ii) $\{a\} \subset \{a, b, c\}$ (iii) $\{a\} \in \{\{a\}, b\}$
(iv) $\{\{a\}\} \subset \{\{a\}, b\}$ (v) $\{b, c\} \in \{a, \{b, c\}\}$ (vi) $\{a, b\} \not\subset \{a, \{b, c\}\}$
(vii) $\phi \subset \{a, b\}$ (viii) $\phi \subset \{a, b, c\}$ (ix) $\{x : x + 3 = 3\} \neq \phi$

5. (i) F (ii) T (iii) T (iv) T (v) F (vi) T (vii) F
(viii) F (ix) F (x) F

7. $2^n - 1$

6. (i) $\phi, \{1\}, \{2\}$ (ii) $\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}$ (iii) ϕ

8. (i) F (ii) F (iii) T (iv) T (v) F (vi) T

9. (i) T (ii) T (iii) F (iv) T (v) F (vi) T (vii) T
(viii) T (ix) T

10. (i) $\phi, \{a\}$ (ii) $\phi, \{0\}, \{1\}, \{0, 1\}$ (iii) $\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}$
(iv) $\phi, \{1\}, \{\{1\}\}, \{1, \{1\}\}$ (v) $\phi, \{\phi\}$.

11. (i) The set of all triangles in a plane. (ii) The set of all triangles in a plane.

14. 1

15. Let $x_n = 8^n - 7n - 1 = (1+7)^n - 7n - 1 = {}^nC_2\, 7^2 + {}^nC_3\, 7^3 + \ldots + {}^nC_n\, 7^n$

$$= 49\,({}^nC_2 + {}^nC_3\, 7 + \ldots + {}^nC_n\, 7^{n-2}) \text{ for } n \geq 2.$$

For $n = 1$, $x_n = 0$. Thus, X contains all positive integral multiples of 49 of the form $49\, k_n$, where $k_n = {}^nC_2 + {}^nC_3\,(7) + {}^nC_4\,(7^2) + \ldots + {}^nC_n\,(7^{n-2})$.

Also, Y contains all positive integral multiples of 49 including zero. Thus, $X \subseteq Y$.

1.7 VENN DIAGRAMS

Sometimes pictures are very helpful in our thinking. First of all a Swiss mathematician Euler gave an idea to represent a set by the points in a closed curve. Later on British mathematician John-Venn (1834-1883) brought this idea to practice. That is why the diagrams drawn to represent sets are called *Venn-Euler diagrams* or simply Venn-diagrams. In Venn-diagrams the universal set U is represented by points within a rectangle and its subsets are represented by points in closed curves (usually circles) within the rectangle. If a set A is a subset of a set B, then the circle representing A is drawn inside the circle representing B as shown in Fig. 1.5 (i). If A and B are not equal but they have some common elements, then to represent A and B we draw two intersecting circles. (See Fig. 1.5 (ii)). Two disjoint sets are represented by two non-intersecting circles. (See Fig. 1.5 (iii)).

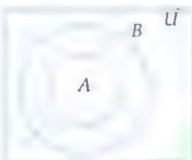


Fig. 1.5 (i)

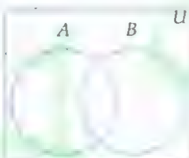


Fig. 1.5 (ii)

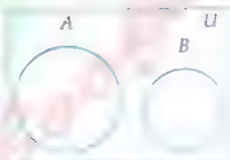


Fig. 1.5 (iii)

1.8 OPERATIONS ON SETS

In this section, we shall introduce some operations on sets to construct new sets from given ones.

UNION OF SETS *Let A and B be two sets. The union of A and B is the set of all those elements which belong either to A or to B or to both A and B.*

We shall use the notation $A \cup B$ (read as "A union B") to denote the union of A and B.

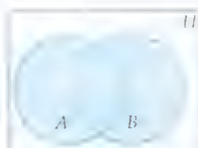


Fig. 1.6 Union of two sets

Thus, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Clearly, $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$.

And, $x \notin A \cup B \Leftrightarrow x \notin A$ and $x \notin B$.

In Fig. 1.6 the shaded part represents $A \cup B$. It is evident from the definition that $A \subseteq A \cup B$, $B \subseteq A \cup B$.

If A and B are two sets such that $A \subset B$, then $A \cup B = B$. Also, $A \cup B = A$, if $B \subset A$.

ILLUSTRATION 1 *If $A = \{1, 2, 3\}$ and $B = \{1, 3, 5, 7\}$, then $A \cup B = \{1, 2, 3, 5, 7\}$.*

ILLUSTRATION 2 *If $A = \{x : x = 2n + 1, n \in Z\}$ and $B = \{x : x = 2n, n \in Z\}$, then*

$A \cup B = \{x : x \text{ is an odd integer}\} \cup \{x : x \text{ is an even integer}\} = \{x : x \text{ is an integer}\} = Z$.

NOTE *If $A_1, A_2, \ldots, A_n$ is a finite family of sets, then their union is denoted by $\bigcup_{i=1}^{n} A_i$ or, by $A_1 \cup A_2 \cup \ldots \cup A_n$.*

ILLUSTRATION 3 *Let $A = \{1, 2, 3\}$, $B = \{3, 5\}$, $C = \{4, 7, 8\}$. Then, $A \cup B \cup C = \{1, 2, 3, 4, 5, 7, 8\}$.*

INTERSECTION OF SETS *Let A and B be two sets. The intersection of A and B is the set of all those elements that belong to both A and B.*

The intersection of A and B is denoted by $A \cap B$ (read as "A intersection B").

Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Clearly, $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$.

In Fig. 1.7 the shaded region represents $A \cap B$.

Evidently, $A \cap B \subset A$, $A \cap B \subset B$.

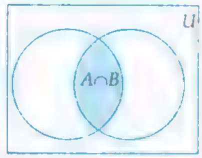


Fig. 1.7 Intersection of two sets

If A and B are two sets, then $A \cap B = A$, if $A \subset B$ and $A \cap B = B$, if $B \subset A$.

NOTE *If $A_1, A_2, ..., A_n$ is a finite family of sets, then their intersection is denoted by $\bigcap_{i=1}^{n} A_i$ or, by $A_1 \cap A_2 \cap ... \cap A_n$.*

ILLUSTRATION 4 If $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 9, 12\}$, then $A \cap B = \{1, 3\}$.

ILLUSTRATION 5 If $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{2, 4, 6, 8, 10\}$ *and* $C = \{4, 6, 7, 8, 9, 10, 11\}$, *then* $A \cap B = \{2, 4, 6\}$. *Therefore,* $A \cap B \cap C = \{4, 6\}$.

ILLUSTRATION 6 *If* $A = \{x : x = 2n, n \in Z\}$ *and* $B = \{x : x = 3n, n \in Z\}$, *then*

$$A \cap B = \{x : x = 2n, n \in Z\} \cap \{x : x = 3n, n \in Z\}$$
$$= \{..., -4, -2, 0, 2, 4, 6, ...\} \cap \{..., -9, -6, -3, 0, 3, 6, 9, ...\}$$
$$= \{..., -6, 0, 6, 12, ...\} = \{x : x = 6n, n \in Z\}.$$

ILLUSTRATION 7 *If* $A = \{x : x = 3n, n \in Z\}$ *and* $B = \{x : x = 4n, n \in Z\}$, *then find* $A \cap B$.

SOLUTION Clearly,

$x \in A \cap B$

$\Leftrightarrow$ $x = 3n$ and $x = 4n, n \in Z$

$\Leftrightarrow$ x is a multiple of 3 and x is a multiple of 4

$\Leftrightarrow$ x is a multiple of 3 and 4 both $\Leftrightarrow$ x is a multiple of 12 $\Leftrightarrow$ $x = 12n$, $n \in Z$

Hence, $A \cap B = \{x : x = 12n, n \in Z\}$.

DISJOINT SETS *Two sets A and B are said to be disjoint, if $A \cap B = \phi$*

If $A \cap B \neq \phi$, then A and B are said to be intersecting or overlapping sets.

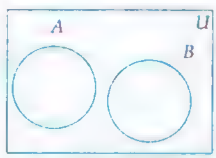


Fig. 1.8 Disjoint sets

ILLUSTRATION 8 *If* $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{7, 8, 9, 10, 11\}$ *and* $C = \{6, 8, 10, 12, 14\}$, *then A and B are disjoint sets, while A and C are intersecting sets.*

DIFFERENCE OF SETS *Let A and B be two sets. The difference of A and B, written as $A - B$, is the set of all those elements of A which do not belong to B.*

Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$ or, $A - B = \{x \in A : x \notin B\}$.

Clearly, $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$. In Fig. 1.9, the shaded part represents $A - B$.

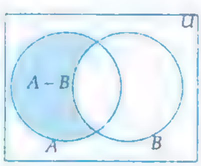


Fig. 1.9 Difference $A - B$

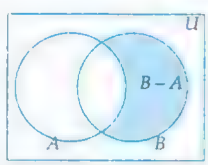


Fig. 1.10 Difference $B - A$

Similarly, the difference $B - A$ is the set of all those elements of B that do not belong to A i.e.

$$B - A = \{x \in B : x \notin A\}.$$

In Fig. 1.10, the shaded part represents $B - A$.

ILLUSTRATION 9 *If $A = \{2, 3, 4, 5, 6, 7\}$ and $B = \{3, 5, 7, 9, 11, 13\}$, then $A - B = \{2, 4, 6\}$ and $B - A = \{9, 11, 13\}$.*

SYMMETRIC DIFFERENCE OF TWO SETS *Let A and B be two sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$.*

Thus, $A \Delta B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$.

The shaded part in Fig. 1.11 represents $A \Delta B$.

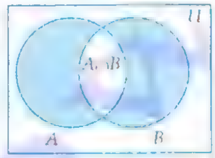


Fig. 1.11 Symmetric difference

ILLUSTRATION 10 *If $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{1, 3, 5, 6, 7, 8, 9\}$, then $A - B = \{2, 4\}$, $B - A = \{9\}$. Therefore, $A \Delta B = \{2, 4, 9\}$.*

ILLUSTRATION 11 *If $A = \{x \in R : 0 < x < 3\}$ and $B = \{x \in R : 1 \le x \le 5\}$, then*

$$A - B = \{x \in R : 0 < x < 1\}, \text{ and, } B - A = \{x \in R : 3 \le x \le 5\}$$

$$\therefore \quad A \Delta B = \{x \in R : 0 < x < 1\} \cup \{x \in R : 3 \le x \le 5\} = \{x \in R : 0 < x < 1 \text{ or } 3 \le x \le 5\}.$$

COMPLEMENT OF A SET *Let U be the universal set and let A be a set such that $A \subset U$. Then, the complement of A with respect to U is denoted by A' or A^c or $U - A$ and is defined as the set of all those elements of U which are not in A.*

Thus $A' = \{x \in U : x \notin A\}$. Clearly, $x \in A' \Leftrightarrow x \notin A$.

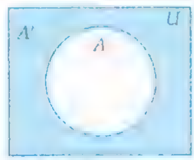


Fig. 1.12 Complement of a set

ILLUSTRATION 12 *Let the set of natural numbers $N = \{1, 2, 3, 4, \ldots\}$ be the universal set and let $A = \{2, 4, 6, 8, \ldots\}$. Then, $A' = \{1, 3, 5, \ldots\}$ = Set of odd natural numbers.*

ILLUSTRATION 13 *If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{1, 3, 5, 7, 9\}$, then $A' = \{2, 4, 6, 8\}$.*

Following results are direct consequences of the definition of the complement of a set:

(i) $U' = \{x \in U : x \notin U\} = \phi$ (ii) $\phi' = \{x \in U : x \notin \phi\} = U$

(iii) $(A')' = \{x \in U : x \notin A'\} = \{x \in U : x \in A\} = A$

(iv) $A \cup A' = \{x \in U : x \in A\} \cup \{x \in U : x \notin A\} = U$

(v) $A \cap A' = \{x \in U : x \in A\} \cap \{x \in U : x \notin A\} = \phi$.

EXERCISE 1.5

BASIC

1. If A and B are two sets such that $A \subset B$, then Find: (i) $A \cap B$ (ii) $A \cup B$
2. If $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, $C = \{7, 8, 9, 10, 11\}$ and $D = \{10, 11, 12, 13, 14\}$. Find:
   (i) $A \cup B$ (ii) $A \cup C$ (iii) $B \cup C$ (iv) $B \cup D$
   (v) $A \cup B \cup C$ (vi) $A \cup B \cup D$ (vii) $B \cup C \cup D$ (viii) $A \cap (B \cup C)$
   (ix) $(A \cap B) \cap (B \cap C)$ (x) $(A \cup D) \cap (B \cup C)$.
3. Let $A = \{x : x \in N\}$, $B = \{x : x = 2n, n \in N\}$, $C = \{x : x = 2n - 1, n \in N\}$ and, $D = \{x : x$ is a prime natural number $\}$. Find:
   (i) $A \cap B$ (ii) $A \cap C$ (iii) $A \cap D$ (iv) $B \cap C$
   (v) $B \cap D$ (vi) $C \cap D$

4. Let $A = \{3, 6, 12, 15, 18, 21\}$, $B = \{4, 8, 12, 16, 20\}$, $C = \{2, 4, 6, 8, 10, 12, 14, 16\}$ and $D = \{5, 10, 15, 20\}$. Find:
   (i) $A - B$ (ii) $A - C$ (iii) $A - D$ (iv) $B - A$
   (v) $C - A$ (vi) $D - A$ (vii) $B - C$ (viii) $B - D$
5. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. Find:
   (i) A' (ii) B' (iii) $(A \cap C)'$ (iv) $(A \cup B)'$
   (v) $(A')'$ (vi) $(B - C)'$
6. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$ and $B = \{2, 3, 5, 7\}$. Verify that:
   (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

ANSWERS

1. (i) A (ii) B
2. (i) $\{1, 2, 3, 4, 5, 6, 7, 8\}$ (ii) $\{1, 2, 3, 4, 5, 7, 8, 9, 10, 11\}$
   (iii) $\{4, 5, 6, 7, 8, 9, 10, 11\}$ (iv) $\{4, 5, 6, 7, 8, 10, 11, 12, 13, 14\}$
   (v) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ (vi) $\{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14\}$
   (vii) $\{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ (viii) $\{4, 5\}$ (ix) ϕ (x) $\{4, 5, 10, 11\}$
3. (i) B (ii) C (iii) D (iv) ϕ (v) $\{2\}$ (vi) $D - \{2\}$
4. (i) $\{3, 6, 15, 18, 21\}$ (ii) $\{3, 15, 18, 21\}$ (iii) $\{3, 6, 12, 18, 21\}$
   (iv) $\{4, 8, 16, 20\}$ (v) $\{2, 4, 8, 10, 14, 16\}$ (vi) $\{5, 10, 20\}$
   (vii) $\{20\}$ (viii) $\{4, 8, 12, 16\}$
5. (i) $\{5, 6, 7, 8, 9\}$ (ii) $\{1, 3, 5, 7, 9\}$ (iii) $\{1, 2, 5, 6, 7, 8, 9\}$
   (iv) $\{5, 7, 9\}$ (v) A (vi) $\{1, 3, 4, 5, 6, 7, 9\}$

1.9 LAWS OF ALGEBRA OF SETS

In this section, we shall state and prove some fundamental laws of algebra of sets.

THEOREM 1 *(Idempotent Laws) For any set A:* (i) $A \cup A = A$ (ii) $A \cap A = A$.

PROOF (i) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$
(ii) $A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A$.

THEOREM 2 *(identity Laws) For any set A:* (i) $A \cup \phi = A$ (ii) $A \cap U = A$.

i.e. ϕ and U are identity elements for union and intersection respectively.

PROOF (i) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$
(ii) $A \cap U = \{x : x \in A \text{ and } x \in U\} = \{x : x \in A\} = A$

THEOREM 3 *(Commutative Laws) For any two sets A and B.*
(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

i.e. union and intersection are commutative.

PROOF Recall that two sets X and Y are equal iff $X \subseteq Y$ and $Y \subseteq X$. Also, $X \subseteq Y$ if every element of X belongs to Y.

(i) Let x be an arbitrary element of $A \cup B$. Then,

$$x \in A \cup B \Rightarrow x \in A \text{ or } x \in B \Rightarrow x \in B \text{ or } x \in A \Rightarrow x \in B \cup A$$

$\therefore$ $A \cup B \subseteq B \cup A$.

Similarly, $B \cup A \subseteq A \cup B$. Hence, $A \cup B = B \cup A$.

THEOREM 4 *(Associative Laws) If A, B and C are any three sets, then*
(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$ [NCERT EXEMPLAR]
i.e. union and intersection are associative.

PROOF (i) Let x be an arbitrary element of $(A \cup B) \cup C$. Then,
$x \in (A \cup B) \cup C$
$\Rightarrow$ $x \in (A \cup B)$ or $x \in C$
$\Rightarrow$ $(x \in A$ or $x \in B)$ or $x \in C$
$\Rightarrow$ $x \in A$ or $(x \in B$ or $x \in C) \Rightarrow x \in A$ or $x \in (B \cup C) \Rightarrow x \in A \cup (B \cup C)$
$\therefore$ $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.
Similarly, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$. Hence, $(A \cup B) \cup C = A \cup (B \cup C)$.
(ii) Let x be an arbitrary element of $A \cap (B \cap C)$. Then,
$x \in A \cap (B \cap C)$
$\Rightarrow$ $x \in A$ and $x \in (B \cap C)$
$\Rightarrow$ $x \in A$ and $(x \in B$ and $x \in C)$
$\Rightarrow$ $(x \in A$ and $x \in B)$ and $x \in C \Rightarrow x \in (A \cap B)$ and $x \in C \Rightarrow x \in (A \cap B) \cap C$
$\therefore$ $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.
Similarly, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$. Hence, $A \cap (B \cap C) = (A \cap B) \cap C$.

THEOREM 5 *(Distributive Laws) If A, B and C are any three sets, then*
(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
i.e. union and intersection are distributive over intersection and union respectively.
[NCERT EXEMPLAR]

PROOF (i) Let x be an arbitrary element of $A \cup (B \cap C)$. Then,
$x \in (A \cup (B \cap C))$
$\Rightarrow$ $x \in A$ or $x \in (B \cap C)$
$\Rightarrow$ $x \in A$ or $(x \in B$ and $x \in C)$
$\Rightarrow$ $(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$ [$\because$ 'or' is distributive over 'and']
$\Rightarrow$ $x \in (A \cup B)$ and $x \in (A \cup C) \Rightarrow x \in ((A \cup B) \cap (A \cup C))$
$\therefore$ $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
Similarly, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(ii) Let x be an arbitrary element of $A \cap (B \cup C)$. Then,
$x \in A \cap (B \cup C)$
$\Rightarrow x \in A$ and $x \in (B \cup C)$
$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$
$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C) \Rightarrow x \in (A \cap B)$ or $x \in (A \cap C) \Rightarrow x \in (A \cap B) \cup (A \cap C)$
$\therefore$ $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$
Similarly, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

THEOREM 6 *(De-Morgan's Laws) If A and B are any two sets, then*
(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

PROOF (i) Let x be an arbitrary element of $(A \cup B)'$. Then,
$x \in (A \cup B)' \Rightarrow x \notin (A \cup B) \Rightarrow x \notin A$ and $\notin B \Rightarrow x \in A'$ and $x \in B' \Rightarrow x \in A' \cap B'$.
$\therefore$ $(A \cup B)' \subseteq A' \cap B'$.
Again, let y be an arbitrary element of $A' \cap B'$. Then,
$y \in A' \cap B' \Rightarrow y \in A'$ and $y \in B' \Rightarrow y \notin A$ and $y \notin B \Rightarrow y \notin A \cup B \Rightarrow y \in (A \cup B)'$
$\therefore$ $A' \cap B' \subseteq (A \cup B)'$

Hence, $(A \cup B)' = A' \cap B'$.

(ii) Let x be an arbitrary element of $(A \cap B)'$. Then,

$$x \in (A \cap B)' \Rightarrow x \notin (A \cap B) \Rightarrow x \notin A \text{ or } x \notin B \Rightarrow x \in A' \text{ or } x \in B' \Rightarrow x \in A' \cup B'$$

$\therefore \quad (A \cap B)' \subseteq A' \cup B'$.

Again, let y be an arbitrary element of $A' \cup B'$. Then ,

$$y \in A' \cup B' \Rightarrow y \in A' \text{ or } y \in B' \Rightarrow y \notin A \text{ or } y \notin B \Rightarrow y \notin (A \cap B) \Rightarrow y \in (A \cap B)'$$

$\therefore \quad A' \cup B' \subseteq (A \cap B)'$.

Hence, $(A \cap B)' = A' \cup B'$.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *If $a \in N$ such that $aN = \{ax : x \in N\}$. Describe the set $3N \cap 7N$.*

SOLUTION We have, $aN = \{ax : x \in N\}$

$\therefore \quad 3N = \{3x : x \in N\} = \{3, 6, 9, 12, ...,\}$ and, $7N = \{7x : x \in N\} = \{7, 14, 21, 28, ...\}$

Hence, $3N \cap 7N = \{21, 42, ...\} = \{21x : x \in N\} = 21N$.

EXAMPLE 2 *If $A = \{1, 3, 5, 7, 11, 13, 15, 17\}$, $B = \{2, 4, 6, ..., 18\}$ and N is the universal set, then find $A' \cup ((A \cup B) \cap B')$.*

SOLUTION Clearly, $(A \cup B) \cap B' = A$ [$\because$ A, B are disjoint sets]

$\therefore \quad A' \cup ((A \cup B) \cap B') = A' \cup A = N$.

EXAMPLE 3 *For any natural number a, we define $aN = \{ax : x \in N\}$. If $b, c, d \in N$ such that $bN \cap cN = dN$, then prove that d is the l.c.m. of b and c.*

SOLUTION We have,

$bN = \{bx : x \in N\}$ = The set of positive integral multiples of b

$cN = \{cx : x \in N\}$ = The set of positive integral multiples of c

$\therefore \quad bN \cap cN$ = The set of positive integral multiples of b and c both.

$\Rightarrow \quad bN \cap cN = \{kx : x \in N\}$, where k is the l.c.m. of b and c.

Hence, d = l.c.m. of b and c.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Suppose $A_1, A_2, \ldots A_{30}$ are thirty sets each with five elements and $B_1, B_2, \ldots, B_n$ are n sets each with three elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$. Assume that each element of S belongs to exactly ten of the A_i's and exactly 9 of B_j's. Find n.*

SOLUTION Since each A_i has 5 elements and each element of S belongs to exactly 10 of A_i's.

$$\therefore \quad S = \bigcup_{i=1}^{30} A_i \Rightarrow n(S) = \frac{1}{10} \sum_{i=1}^{30} n(A_i) = \frac{1}{10}(5 \times 30) = 15 \qquad \text{...(i)}$$

Again, each B_j has 3 elements and each element of S belongs to exactly 9 of B_j's

$$\therefore \quad S = \bigcup_{j=1}^{n} B_j \Rightarrow n(S) = \frac{1}{9} \sum_{j=1}^{n} n(B_j) = \frac{1}{9}(3n) = \frac{n}{3} \qquad \text{...(ii)}$$

From (i) and (ii), we get : $15 = \frac{n}{3} \Rightarrow n = 45$.

EXAMPLE 5 *For any two sets A and B, prove that $A \cup B = A \cap B \Leftrightarrow A = B$.*

SOLUTION First let $A = B$. Then, $A \cup B = A$ and $A \cap B = A \Rightarrow A \cup B = A \cap B$

Thus, $A = B \Rightarrow A \cup B = A \cap B$ ...(i)

Conversely, let $A \cup B = A \cap B$. Then, we have to prove that $A = B$. For this, let

$$\begin{aligned} x \in A &\Rightarrow x \in A \cup B \\ &\Rightarrow x \in A \cap B \qquad [\because A \cup B = A \cap B] \\ &\Rightarrow x \in A \text{ and } x \in B \Rightarrow x \in B \end{aligned}$$

$\therefore \quad A \subset B$ ...(ii)

Now, let

$$\begin{aligned} y \in B &\Rightarrow y \in A \cup B \\ &\Rightarrow y \in A \cap B \qquad [\because A \cup B = A \cap B] \\ &\Rightarrow y \in A \text{ and } y \in B \Rightarrow y \in A \end{aligned}$$

$\therefore \quad B \subset A$ ...(iii)

From (ii) and (iii), we get $A = B$.

Thus, $A \cup B = A \cap B \Rightarrow A = B$ ...(iv)

From (i) and (iv), we obtain : $A \cup B = A \cap B \Leftrightarrow A = B$.

EXAMPLE 6 *Let A, B and C be the sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that $B = C$.*

SOLUTION We have,

$$A \cup B = A \cup C$$

$\Rightarrow \quad (A \cup B) \cap C = (A \cup C) \cap C$

$\Rightarrow \quad (A \cap C) \cup (B \cap C) = C \qquad [\because (A \cup C) \cap C = C]$

$\Rightarrow \quad (A \cap B) \cup (B \cap C) = C \qquad [\because A \cap C = A \cap B]$ ...(i)

Again, $A \cup B = A \cup C$

$\Rightarrow \quad (A \cup B) \cap B = (A \cup C) \cap B$

$\Rightarrow \quad B = (A \cap B) \cup (C \cap B) \qquad [\because (A \cup B) \cap B = B]$

$\Rightarrow \quad B = (A \cap B) \cup (B \cap C)$ ...(ii)

From (i) and (ii), we get $B = C$.

EXAMPLE 7 *Let A and B be sets, if $A \cap X = B \cap X = \phi$ and $A \cup X = B \cup X$ for some set X, prove that $A = B$.*

SOLUTION We have,

$A \cup X = B \cup X$ for some set X

$\Rightarrow \quad A \cap (A \cup X) = A \cap (B \cup X)$

$\Rightarrow \quad A = (A \cap B) \cup (A \cap X) \qquad [\because A \cap (A \cup X) = A]$

$\Rightarrow \quad A = (A \cap B) \cup \phi \qquad [\therefore A \cap X = \phi \text{(given)}]$

$\Rightarrow \quad A = A \cap B \Rightarrow A \subset B$ ...(i)

Again, $A \cup X = B \cup X$

$\Rightarrow \quad B \cap (A \cup X) = B \cap (B \cup X)$

$\Rightarrow \quad (B \cap A) \cup (B \cap X) = B \qquad [\therefore B \cap (B \cup X) = B]$

$\Rightarrow \quad (B \cap A) \cup \phi = B \qquad [\because B \cap X = \phi \text{(given)}]$

$\Rightarrow \quad B \cap A = B \Rightarrow A \cap B = B \Rightarrow B \subset A$ ...(ii)

From (i) and (ii), we get $A = B$.

EXAMPLE 8 *For any two sets A and B, prove that: $P(A) = P(B) \Rightarrow A = B$.*

SOLUTION Let x be an arbitrary element of A. Then, there exists a subset, say X, of set A such that $x \in X$.

Now, $X \subset A$

$\Rightarrow \quad X \in P(A)$

$\Rightarrow \quad X \in P(B) \qquad [\because P(A) = P(B)]$

$\Rightarrow \quad X \subset B$

$\Rightarrow \quad x \in B$ $\quad [\because\ x \in X \text{ and } X \subset B\ \therefore\ x \in B]$

Thus, $x \in A \Rightarrow x \in B$ for all $x \in B$.

$\therefore \quad A \subset B$ ...(i)

Now, let y be an arbitrary element of B. Then, there exists a subset, say Y, of set B such that $y \in Y$.

Now, $Y \subset B$

$\Rightarrow \quad Y \in P(B)$

$\Rightarrow \quad Y \in P(A)$ $\quad [\because\ P(A) = P(B)]$

$\Rightarrow \quad Y \subset A \Rightarrow y \in A$

Thus, $y \in B \Rightarrow y \in A$ for all $y \in B$.

$\therefore \quad B \subset A$ ...(ii)

From (i) and (ii), we obtain $A = B$.

EXAMPLE 9 *For any two sets A and B prove that:* $P(A \cap B) = P(A) \cap P(B)$.

SOLUTION In order to prove that $P(A \cap B) = P(A) \cap P(B)$, it is sufficient to prove that

$P(A \cap B) \subset P(A) \cap P(B)$ and $P(A) \cap P(B) \subset P(A \cap B)$.

First let

$X \in P(A \cap B)$

$\Rightarrow \quad X \subset A \cap B \Rightarrow X \subset A$ and $X \subset B \Rightarrow X \in P(A)$ and $X \in P(B) \Rightarrow X \in P(A) \cap P(B)$

$\therefore \quad P(A \cap B) \subset P(A) \cap P(B)$ ...(i)

Now, let $Y \in P(A) \cap P(B)$. Then,

$Y \in P(A) \cap P(B) \Rightarrow Y \in P(A)$ and $Y \in P(B) \Rightarrow Y \subset A$ and $Y \subset B \Rightarrow Y \subset A \cap B$

$\Rightarrow \quad Y \in P(A \cap B)$

$\therefore \quad P(A) \cap P(B) \subset P(A \cap B)$ ...(ii)

From (i) and (ii), we get: $P(A \cap B) = P(A) \cap P(B)$.

EXAMPLE 10 *For any two sets A and B prove that* $P(A) \cup P(B) \subset P(A \cup B)$. *But,* $P(A \cup B)$ *is not necessarily a subset of* $P(A) \cup P(B)$.

SOLUTION Let $X \in P(A) \cup P(B)$. Then,

$X \in P(A) \cup P(B)$

$\Rightarrow \quad X \in P(A)$ or $X \in P(B) \Rightarrow X \subset A$ or $X \subset B \Rightarrow X \subset A \cup B \Rightarrow X \in P(A \cup B)$

$\therefore \quad P(A) \cup P(B) \subset P(A \cup B)$

Let $A = \{1, 2\}$ and $B = \{3, 4, 5\}$. Then, we find that $X = \{1, 2, 3, 4\} \subset (A \cup B)$. Therefore, $X \in P(A \cup B)$. But, $X \notin P(A)$, $X \notin P(B)$. So, $X \notin P(A) \cup P(B)$. Thus, $P(A \cup B)$ is not necessarily a subset of $P(A) \cup P(B)$.

EXERCISE 1.6

BASIC

1. Find the smallest set A such that $A \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$.
2. Let $A = \{1, 2, 4, 5\}$ $B = \{2, 3, 5, 6\}$ $C = \{4, 5, 6, 7\}$. Verify the following identities:
   (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (iii) $A \cap (B - C) = (A \cap B) - (A \cap C)$ (iv) $A - (B \cup C) = (A - B) \cap (A - C)$
   (v) $A - (B \cap C) = (A - B) \cup (A - C)$ (vi) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.
3. If $U = \{2, 3, 5, 7, 9\}$ is the universal set and $A = \{3, 7\}$, $B = \{2, 5, 7, 9\}$, then prove that:
   (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

BASED ON LOTS

4. For any two sets A and B, prove that
   (i) $B \subset A \cup B$ (ii) $A \cap B \subset A$ (iii) $A \subset B \Rightarrow A \cap B = A$ **[NCERT EXEMPLAR]**
5. For any two sets, prove that:
   (i) $A \cup (A \cap B) = A$ (ii) $A \cap (A \cup B) = A$
6. Each set X_r contains 5 elements and each set Y_r contains 2 elements and $\bigcup\limits_{r=1}^{20} X_r = S = \bigcup\limits_{r=1}^{n} Y_r$. If each element of S belongs to exactly 10 of the $X_r'^s$ and to exactly 4 of $Y_r'^s$, then find the value of n.

BASED ON HOTS

7. For any two sets A and B, show that the following statements are equivalent:
   (i) $A \subset B$ (ii) $A - B = \phi$ (iii) $A \cup B = B$ (iv) $A \cap B = A$.
8. For three sets A, B and C, show that
   (i) $A \cap B = A \cap C$ need not imply $B = C$. (ii) $A \subset B \Rightarrow C - B \subset C - A$
9. Find sets A, B and C such that $A \cap B, A \cap C$ and $B \cap C$ are non-empty sets and $A \cap B \cap C = \phi$.
10. For any two sets A and B, prove that: $A \cap B = \phi \Rightarrow A \subseteq B'$.
11. If A and B are sets, then prove that $A - B$, $A \cap B$ and $B - A$ are pair wise disjoint.
12. Using properties of sets, show that for any two sets A and B, $(A \cup B) \cap (A \cup B') = A$.
13. For any two sets of A and B, prove that:
    (i) $A' \cup B = U \Rightarrow A \subset B$ (ii) $B' \subset A' \Rightarrow A \subset B$
14. Is it true that for any sets A and B, $P(A) \cup P(B) = P(A \cup B)$? Justify your answer.
15. Show that for any sets A and B,
    (i) $A = (A \cap B) \cup (A - B)$ (ii) $A \cup (B - A) = A \cup B$

ANSWERS

1. $A = \{3, 5, 9\}$ 6. 20   9. $A = \{1, 2\}, B = \{1, 3\}, C = \{2, 3\}$ 14. False

HINTS TO SELECTED PROBLEMS

5. (i) $A \cup (A \cap B) = (A \cup A) \cap (A \cup B)$ [$\because$ $\cup$ is distributive over $\cap$]
   $= A \cap (A \cup B) = A$ [$\because A \subset A \cup B$]
   (ii) $A \cap (A \cup B) = (A \cap A) \cup (A \cap B) = A \cup (A \cap B) = A$
7. (i) $\Rightarrow$ (ii): We know that, $A - B = \{x \in A : x \notin B\}$
   Since $A \subset B$. Therefore, there is no element in A which does not belong to B. Therefore, $A - B = \phi$ Hence, (i) $\Rightarrow$ (ii).
   (ii) $\Rightarrow$ (iii): We have, $A - B = \phi \Rightarrow A \subset B \Rightarrow A \cup B = B$. Hence, (ii) $\Rightarrow$ (iii).
   (iii) $\Rightarrow$ (iv): We have, $A \cup B = B \Rightarrow A \subset B \Rightarrow A \cap B = A$. Hence, (iii) $\Rightarrow$ (iv).
   (iv) $\Rightarrow$ (i): We have, $A \cap B = A \Rightarrow A \subset B$. Hence, (iv) $\Rightarrow$ (i).
   Consequently, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).
8. (i) Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ and $C = \{1, 3, 4, 7, 8\}$. Then, $A \cap B = A \cap C$, but $B \neq C$.
   (ii) Let $x \in C - B$. Then,
   $x \in C - B \Rightarrow x \in C$ and $x \notin B \Rightarrow x \in C$ and $x \notin A \Rightarrow x \in C - A$
   $\therefore \quad C - B \subset C - A$
9. $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{2, 3\}$
10. $x \in A \Rightarrow x \notin B \Rightarrow x \in B'$. [$\because A \cap B = \phi$]
    $\therefore A \subset B'$

11. $A-B=\{x : x \in A \text{ and } x \notin B\}$, $B-A=\{x \in B \text{ and } x \notin A\} \Rightarrow A-B$ and $B-A$ are disjoint sets

Now, $x \in A-B \Rightarrow x \in A \text{ and } x \notin B \Rightarrow x \notin A \cap B$

$\therefore$ $(A-B)$ and $A \cap B$ are disjoint sets.

Similarly, $B-A$ and $A \cap B$ are disjoint sets.

12. $(A \cup B) \cap (A \cup B') = ((A \cup B) \cap A) \cup ((A \cup B) \cup B')$

$= A \cup ((A \cup B) \cap B') = A \cup (A \cap B') \cup (B \cap B') = A \cup (A \cap B') = A$

13. (i) Let $x \in A$. Then, $x \in A \Rightarrow x \in U \Rightarrow x \in A' \cup B \Rightarrow x \in B$ $[\because x \notin A']$

$\therefore$ $A \subset B$

(ii) Let $x \in A$. Then, $x \in A \Rightarrow x \notin A' \Rightarrow x \notin B' \Rightarrow x \in B$ $[\because B' \subset A']$

$\therefore$ $A \subset B$

1.10 MORE RESULTS ON OPERATIONS ON SETS

THEOREM 1 *If A and B are any two sets, then*

(i) $A-B=A \cap B'$ (ii) $B-A=B \cap A'$ (iii) $A-B=A \Leftrightarrow A \cap B=\phi$

(iv) $(A-B) \cup B=A \cup B$ (v) $(A-B) \cap B=\phi$ (vi) $A \subseteq B \Leftrightarrow B' \subseteq A'$

(vii) $A-(A \cap B)=A-B$ [NCERT EXEMPLAR]

(viii) $(A-B) \cup (B-A)=(A \cup B)-(A \cap B)$

PROOF (i) Let x be an arbitrary element of $A-B$. Then,

$x \in (A-B) \Rightarrow x \in A \text{ and } x \notin B \Rightarrow x \in A \text{ and } x \in B' \Rightarrow x \in A \cap B'$

$\therefore$ $A-B \subseteq A \cap B'$ ...(i)

Again, let y be an arbitrary element of $A \cap B'$. Then,

$y \in A \cap B' \Rightarrow y \in A \text{ and } y \in B' \Rightarrow y \in A \text{ and } y \notin B \Rightarrow y \in A-B$

$\therefore$ $A \cap B' \subseteq (A-B)$ ...(ii)

Hence, from (i) and (ii), we obtain $A-B=A \cap B'$.

(ii) Proceed as in (i).

(iii) In order to prove that $A-B=A \Leftrightarrow A \cap B=\phi$, we shall prove that :

(a) $A-B=A \Rightarrow A \cap B=\phi$ and, (b) $A \cap B=\phi \Rightarrow A-B=A$.

First, let $A-B=A$. Then we have to prove that $A \cap B=\phi$. If possible, let $A \cap B \neq \phi$. Then,

$A \cap B \neq \phi$

$\Rightarrow$ There exists $x \in A \cap B$

$\Rightarrow$ $x \in A$ and $x \in B$

$\Rightarrow$ $x \in A-B$ and $x \in B$ $[\because A-B=A]$

$\Rightarrow$ $(x \in A \text{ and } x \notin B)$ and $x \in B$ [Using definition of $A-B$]

$\Rightarrow$ $x \in A$ and $(x \notin B$ and $x \in B)$

But, $x \notin B$ and $x \in B$ both can never be possible simultaneously. Thus, we arrive at a contradiction. So, our supposition is wrong. Therefore, $A \cap B=\phi$.

Hence, $A-B=A \Rightarrow A \cap B=\phi$ ...(i)

Conversely, let $A \cap B=\phi$. Then we have to prove that $A-B=A$. For this we shall show that $A-B \subseteq A$ and $A \subseteq A-B$. Let x be an arbitrary element of $A-B$. Then,

$x \in A-B \Rightarrow x \in A \text{ and } x \notin B \Rightarrow x \in A$

$\therefore$ $A-B \subseteq A$.

Again let y be an arbitrary element of A. Then,

$y \in A \Rightarrow y \in A \text{ and } y \notin B$ $[\because A \cap B=\phi]$

$\Rightarrow$ $y \in A-B$ [Using definition of $A-B$]

$\therefore$ $A \subseteq A-B$.

So, we have $A-B \subseteq A$ and $A \subseteq A-B$. Therefore, $A-B=A$.

Thus, $A \cap B = \phi \Rightarrow A - B = A$ ...(ii)

Hence, from (i) and (ii), we obtain : $A - B = A \Leftrightarrow A \cap B = \phi$.

(iv) Let x be an arbitrary element of $(A - B) \cup B$. Then,

$x \in (A - B) \cup B$

$\Rightarrow$ $x \in A - B$ or $x \in B \Rightarrow$ ($x \in A$ and $x \notin B$) or $x \in B$

$\Rightarrow$ ($x \in A$ or $x \in B$) and ($x \notin B$ or $x \in B$) $\Rightarrow x \in A \cup B$

$\therefore$ $(A - B) \cup B \subseteq A \cup B$

Let y be an arbitrary element of $A \cup B$. Then,

$y \in A \cup B$

$\Rightarrow$ $y \in A$ or $y \in B \Rightarrow$ ($y \in A$ or $y \in B$) and ($y \notin B$ or $y \in B$) $\Rightarrow$ ($y \in A$ and $y \notin B$) or $y \in B$

$\Rightarrow$ $y \in (A - B) \cup B$

$\therefore$ $A \cup B \subseteq (A - B) \cup B$

Hence, $(A - B) \cup B = A \cup B$

(v) If possible let $(A - B) \cap B \neq \phi$ Then, there exists at least one element x, (say), in $(A - B) \cap B$.

Now, $x \in (A - B) \cap B$

$\Rightarrow$ $x \in (A - B)$ and $x \in B \Rightarrow$ ($x \in A$ and $x \notin B$) and $x \in B \Rightarrow x \in A$ and ($x \notin B$ and $x \in B$)

But, $x \notin B$ and $x \in B$ both can never be possible simultaneously. Thus, we arrive at a contradiction. So, our supposition is wrong. Hence, $(A - B) \cap B = \phi$.

(vi) First, let $A \subseteq B$. Then we have to prove that $B' \subseteq A'$. Let x be an arbitrary element of B'. Then,

$x \in B' \Rightarrow x \notin B \Rightarrow x \notin A \Rightarrow x \in A'$ [$\because A \subseteq B$]

$\therefore$ $B' \subseteq A'$.

Thus, $A \subseteq B \Rightarrow B' \subseteq A'$. ...(i)

Conversely, let $B' \subseteq A'$. Then, we have to prove that $A \subseteq B$. Let y be an arbitrary element of A. Then,

$y \in A \Rightarrow y \notin A' \Rightarrow y \notin B' \Rightarrow y \in B$ [$\because B' \subseteq A'$].

$\therefore$ $A \subseteq B$.

Thus, $B' \subseteq A' \Rightarrow A \subseteq B$ ...(ii)

From (i) and (ii), we obtain that $A \subseteq B \Leftrightarrow B' \subseteq A'$.

(vii) We have,

$A - (A \cap B) = A \cap (A \cap B)'$ [$\because A - B = A \cap B'$]

$= A \cap (A' \cup B')$ [By De Morgan's law]

$= (A \cap A') \cup (A \cap B')$ [By distributivity of $\cap$ over $\cup$]

$= \phi \cup (A \cap B') = A \cap B' = A - B$

(viii) Let x be an arbitrary element of $(A - B) \cup (B - A)$. Then ,

$x \in (A - B) \cup (B - A)$

$\Rightarrow$ $x \in A - B$ or, $x \in B - A$

$\Rightarrow$ ($x \in A$ and $x \notin B$) or ($x \in B$ and $x \notin A$)

$\Rightarrow$ ($x \in A$ or $x \notin B$) and ($x \notin B$ or $x \notin A$) $\Rightarrow x \in (A \cup B)$ and $x \notin (A \cap B) \Rightarrow x \in (A \cup B) - (A \cap B)$

$\therefore$ $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$. ...(i)

Again, let y be an arbitrary element of $(A \cup B) - (A \cap B)$. Then,

$y \in (A \cup B) - (A \cap B)$

$\Rightarrow$ $y \in A \cup B$ and $y \notin A \cap B$

$\Rightarrow$ ($y \in A$ or $y \in B$) and ($y \notin A$ or $y \notin B$)

$\Rightarrow \quad (y \in A \text{ and } y \notin B) \text{ or } (y \in B \text{ and } y \notin A)$

$\Rightarrow \quad y \in (A - B) \text{ or } y \in (B - A) \Rightarrow y \in (A - B) \cup (B - A)$

$\therefore \quad (A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A).$ ...(ii)

Hence, from (i) and (ii), we obtain : $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

THEOREM 2 *If A, B and C are any three sets, then prove that:*

(i) $A - (B \cap C) = (A - B) \cup (A - C)$ (ii) $A - (B \cup C) = (A - B) \cap (A - C)$

(iii) $A \cap (B - C) = (A \cap B) - (A \cap C)$ (iv) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

PROOF (i) Let x be any element of $A - (B \cap C)$. Then,

$x \in A - (B \cap C)$

$\Rightarrow \quad x \in A \text{ and } x \notin (B \cap C)$

$\Rightarrow \quad x \in A \text{ and } (x \notin B \text{ or } x \notin C)$

$\Rightarrow \quad (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C)$

$\Rightarrow \quad x \in (A - B) \text{ or } x \in (A - C) \Rightarrow x \in (A - B) \cup (A - C)$

$\therefore \quad A - (B \cap C) \subseteq (A - B) \cup (A - C)$

Similarly, $(A - B) \cup (A - C) \subseteq A - (B \cap C)$. Hence, $A - (B \cap C) = (A - B) \cup (A - C)$

(ii) Let x be an arbitrary element of $A - (B \cup C)$. Then

$x \in A - (B \cup C)$

$\Rightarrow \quad x \in A \text{ and } x \notin (B \cup C)$

$\Rightarrow \quad x \in A \text{ and } (x \notin B \text{ and } x \notin C)$

$\Rightarrow \quad (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C)$

$\Rightarrow \quad x \in (A - B) \text{ and } x \in (A - C) \Rightarrow x \in (A - B) \cap (A - C)$

$\therefore \quad A - (B \cup C) \subseteq (A - B) \cap (A - C)$

Similarly, $(A - B) \cap (A - C) \subseteq A - (B \cup C)$. Hence, $A - (B \cup C) = (A - B) \cap (A - C)$.

(iii) Let x be any arbitrary element of $A \cap (B - C)$. Then

$x \in A \cap (B - C)$

$\Rightarrow \quad x \in A \text{ and } x \in (B - C)$

$\Rightarrow \quad x \in A \text{ and } (x \in B \text{ and } x \notin C)$

$\Rightarrow \quad (x \in A \text{ and } x \in B) \text{ and } (x \in A \text{ and } x \notin C)$

$\Rightarrow \quad x \in (A \cap B) \text{ and } x \notin (A \cap C) \Rightarrow x \in (A \cap B) - (A \cap C)$

$\therefore \quad A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$

Similarly, $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$. Hence, $A \cap (B - C) = (A \cap B) - (A \cap C)$.

(iv) $A \cap (B \Delta C) = A \cap [(B - C) \cup (C - B)] = [A \cap (B - C)] \cup [A \cap (C - B)]$ [By distributive law]

$= [(A \cap B) - (A \cap C)] \cup [(A \cap C) - (A \cap B)]$ [Using (iii)]

$= (A \cap B) \Delta (A \cap C)$

ILLUSTRATIVE EXAMPLES

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 1 *Let A and B be two sets. Using properties of sets prove that:*

(i) $A \cap B' = \phi \Rightarrow A \subset B$ (ii) $A' \cup B = U \Rightarrow A \subset B$

SOLUTION (i) We have,

$A = (A \cap U)$

$\Rightarrow \quad A = A \cap (B \cup B')$ $[\because B \cup B' = U]$

$\Rightarrow \quad A = (A \cap B) \cup (A \cap B')$ [$\because$ $\cap$ is distributive over union]

$\Rightarrow \quad A = (A \cap B) \cup \phi$ $[\because A \cap B' = \phi]$

$\Rightarrow \quad A = A \cap B$

$\therefore \quad A \subset B$

(ii) From (i), we have

$A \cap B' = \phi$

$\Leftrightarrow \quad (A \cap B')' = \phi'$

$\Leftrightarrow \quad A' \cup (B')' = U$ $\quad [\because \phi' = U]$

$\Leftrightarrow \quad A' \cup B = U$ $\quad [\because (B')' = B]$

Thus, $A \cap B' = \phi \Leftrightarrow A' \cup B = U$ and, $A \cap B' = \phi \Rightarrow A \subset B$

$\therefore \quad A' \cup B = U \Rightarrow A \subset B$

ALITER We have,

$A' \cup B = U$

$\Rightarrow \quad A \cap (A' \cup B) = A \cap U$ $\quad$ [Taking intersection with A]

$\Rightarrow \quad (A \cap A') \cup (A \cap B) = A$ $\quad [\because A \cap U = A]$

$\Rightarrow \quad \phi \cup (A \cap B) = A \Rightarrow A \cap B = A \Rightarrow A \subset B$

EXAMPLE 2 *Let A and B be two sets. Prove that: $(A - B) \cup B = A$ if and only if $B \subset A$.*

SOLUTION First let, $(A - B) \cup B = A$. Then, we have to prove that $B \subset A$.

Now, $(A - B) \cup B = A$

$\Rightarrow \quad (A \cap B') \cup B = A$ $\quad [\because A - B = A \cap B']$

$\Rightarrow \quad (A \cup B) \cap (B' \cup B) = A \Rightarrow (A \cup B) \cap U = A \Rightarrow A \cup B = A \Rightarrow B \subset A.$

Conversely, let $B \subset A$. Then, we have to prove that $(A - B) \cup B = A$.

Now, $(A - B) \cup B = (A \cap B') \cup B = (A \cup B) \cap (B' \cup B) = (A \cup B) \cap U = A \cup B$

$= A$ $\quad [\because B \subset A \;\therefore A \cup B = A]$

EXAMPLE 3 *Let A, B and C be three sets such that $A \cup B = C$ and $A \cap B = \phi$ Then, prove that $A = C - B$.*

SOLUTION We have, $A \cup B = C$.

$\therefore \quad C - B = (A \cup B) - B$

$= (A \cup B) \cap B'$ $\quad [\because X - Y = X \cap Y']$

$= (A \cap B') \cup (B \cap B') = (A \cap B') \cup \phi = A \cap B' = A - B = A$ $\quad [\because A \cap B = \phi]$

EXAMPLE 4 *Let A and B be any two sets. Using properties of sets prove that:*

(i) $(A - B) \cup B = A \cup B$ (ii) $(A - B) \cup A = A$

(iii) $(A - B) \cap B = \phi$ (iv) $(A - B) \cap A = A \cap B'$

(v) $A \cup (B - A) = A \cup B$ [NCERT EXEMPLAR] (vi) $A - (A - B) = A \cap B$ [NCERT EXEMPLAR]

(vii) $A - (A \cap B) = A - B$ [NCERT EXEMPLAR] (viii) $(A \cup B) - B = A - B$ [NCERT EXEMPLAR]

SOLUTION (i) We find that

$(A - B) \cup B = (A \cap B') \cup B$ $\quad [\because A - B = A \cap B']$

$= (A \cup B) \cap (B' \cup B)$ $\quad [\because \cup$ is distributive over $\cap]$

$= (A \cup B) \cap U = A \cup B$ $\quad [\because B' \cup B = U]$

(ii) $(A - B) \cup A = A$ $\quad [\because A - B \subset A]$

(iii) $(A - B) \cap B = (A \cap B') \cap B = A \cap (B' \cap B) = A \cap \phi = \phi$

(iv) $(A - B) \cap A = A - B = A \cap B'$ $\quad [\because A - B \subset A]$

(v) $A \cup (B - A) = A \cup (B \cap A')$ $\quad [\because X - Y = X \cap Y']$

$= (A \cup B) \cap (A \cup A')$ $\quad$ [By distributing of $\cup$ over $\cap$]

$= (A \cup B) \cap U$

$= (A \cup B).$

(vi) $A - (A - B) = A - (A \cap B')$ $\quad [\because X - Y = X \cap Y']$

$= A \cap (A \cap B')'$ $\quad [\because X - Y = X \cap Y']$

$= A \cap (A' \cup B)$ [By De' Morgan's law]

$= (A \cap A') \cup (A \cap B)$

$= \phi \cup (A \cap B) = A \cap B$

(vii) $A - (A \cap B) = A \cap (A \cap B)'$ $[\because X - Y = X \cap Y']$

$= A \cap (A' \cup B')$

$= (A \cap A') \cup (A \cap B') = \phi \cup (A \cap B') = A \cap B' = A - B$

(viii) $(A \cup B) - B = (A \cup B) \cap B'$ $[\because X - Y = X \cap Y']$

$= (A \cap B') \cup (B \cap B') = (A \cap B') \cup \phi = A \cap B' = A - B.$

EXAMPLE 5 *For any two sets A and B prove by using properties of sets that:*

(i) $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ (ii) $(A \cap B) \cup (A - B) = A$ (iii) $(A \cup B) - A = B - A$

SOLUTION (i) We have,

$(A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)'$ $[\because X - Y = X \cap Y']$

$= (A \cup B) \cap (A' \cup B')$ $[\because (A \cap B)' = A' \cup B']$

$= X \cap (A' \cup B')$, where $X = A \cup B$

$= (X \cap A') \cup (X \cap B')$

$= (B \cap A') \cup (A \cap B')$ $\left[\begin{array}{l}\because X \cap A' = (A \cup B) \cap A' = (A \cap A') \cup (B \cap A') \\ = \phi \cup (B \cap A') = B \cap A' \text{ Similarly, } X \cap B' = A \cap B'\end{array}\right]$

$= (A \cap B') \cup (B \cap A')$

$= (A - B) \cup (B - A)$ $[\because A - B = A \cap B' \text{ and } B - A = B \cap A']$

(ii) $(A \cap B) \cup (A - B) = (A \cap B) \cup (A \cap B')$

$= X \cup (A \cap B')$, where $X = A \cap B$

$= (X \cup A) \cap (X \cup B')$

$= A \cap (A \cup B')$ $\left[\begin{array}{l}\because X \cup A = (A \cap B) \cup A = A \; [\because A \cap B \subset A] \\ X \cup B' = (A \cap B) \cup B' = (A \cup B') \cap (B \cup B') \\ = (A \cup B') \cap U = A \cup B'\end{array}\right]$

$= A$ $[\because A \subset A \cup B']$

(iii) $(A \cup B) - A = (A \cup B) \cap A'$ $[\because X - Y = X \cap Y']$

$= (A \cap A') \cup (B \cap A')$

$= \phi \cup (B \cap A')$ $[\because A \cap A' = \phi]$

$= B \cap A'$

$= B - A$ $[\because B - A = B \cap A']$

EXAMPLE 6 *For sets A, B and C using properties of sets, prove that:*

(i) $A - (B \cup C) = (A - B) \cap (A - C)$ [NCERT EXEMPLAR]

(ii) $A - (B \cap C) = (A - B) \cup (A - C)$

(iii) $(A \cup B) - C = (A - C) \cup (B - C)$

(iv) $(A \cap B) - C = (A - C) \cap (B - C)$ [NCERT EXEMPLAR]

SOLUTION (i) We have,

$A - (B \cup C) = A \cap (B \cup C)'$ $[\because X - Y = X \cap Y']$

$= A \cap (B' \cap C')$ $[\because (B \cup C)' = B' \cap C']$

$= (A \cap B') \cap (A \cap C') = (A - B) \cap (A - C)$

(ii) $A - (B \cap C) = A \cap (B \cap C)'$ $[\because X - Y = X \cap Y']$

$= A \cap (B' \cup C')$ $[\because (B \cap C)' = B' \cup C']$

$= (A \cap B') \cup (A \cap C') = (A - B) \cup (A - C)$ $[\because \cap \text{ is distributive over } \cup]$

(iii) $(A \cup B) - C = (A \cup B) \cap C'$ $[\because X - Y = X \cap Y']$

$= (A \cap C') \cup (B \cap C') = (A - C) \cup (B - C)$

(iv) $(A \cap B) - C = (A \cap B) \cap C' = (A \cap C') \cap (B \cap C') = (A - C) \cap (B - C)$

EXAMPLE 7 *For sets A, B and C using properties of sets, prove that:*

(i) $A - (B - C) = (A - B) \cup (A \cap C)$ (ii) $A \cap (B - C) = (A \cap B) - (A \cap C)$

SOLUTION (i) We have,

$A - (B - C) = A - (B \cap C')$ $[\because B - C = B \cap C']$

$= A \cap (B \cap C')'$ $[\because X - Y = X \cap Y']$

$= A \cap (B' \cup C)$ $[\because (B \cap C')' = B' \cup (C')' = B' \cup C]$

$= (A \cap B') \cup (A \cap C) = (A - B) \cup (A \cap C)$

(ii) $A \cap (B - C) = A \cap (B \cap C')$ $[\because B - C = B \cap C']$

$= (A \cap B) \cap C'$

$= \phi \cup ((A \cap B) \cap C')$

$= ((A \cap B) \cap A') \cup ((A \cap B) \cap C')$ $[\because (A \cap B) \cap A' = \phi]$

$= (A \cap B) \cap (A' \cup C') = (A \cap B) \cap (A \cap C)' = (A \cap B) - (A \cap C)$

EXERCISE 1.7

BASED ON HOTS

1. For any two sets A and B, prove that : $A' - B' = B - A$
2. For any two sets A and B, prove the following :
   (i) $A \cap (A' \cup B) = A \cap B$ (ii) $A - (A - B) = A \cap B$
   (iii) $A \cap (A \cup B)' = \phi$ (iv) $A - B = A \,\Delta\, (A \cap B)$.
3. If A, B, C are three sets such that $A \subset B$, then prove that $C - B \subset C - A$.
4. For any two sets A and B, prove that
   (i) $(A \cup B) - B = A - B$ (ii) $A - (A \cap B) = A - B$
   (iii) $A - (A - B) = A \cap B$ (iv) $A \cup (B - A) = A \cup B$ [NCERT EXEMPLAR]
   (v) $(A - B) \cup (A \cap B) = A$ [NCERT EXEMPLAR]
5. Determine whether each of the following statements is true or false. Justify your answer.
   (i) $(A - B) \cup (A \cap B) = A$ for all sets A and B
   (ii) $A - (B - C) = (A - B) - C$ for all sets A, B and C.
   (iii) If $A \subset B$, then (a) $A \cap C \subset B \cap C$ (b) $A \cup C \subset B \cup C$ for all sets A, B and C.
   (iv) If $A \subset C$ and $B \subset C$, then $A \cup B \subset C$ for all sets A, B and C. [NCERT EXEMPLAR]

ANSWERS

5. (i) True (ii) False (iii) (a) True (b) True (iv) True

HINTS TO SELECTED PROBLEMS

1. We know that $X - Y = X \cap Y'$. So $A' - B' = A' \cap (B')' = A' \cap B = B \cap A' = B - A$
2. (ii) $A - (A - B) = A - (A \cap B') = A \cap (A \cap B')' = A \cap (A' \cup (B')') = A \cap (A' \cup B) = A \cap B$
3. Let $x \in C - B$. Then,
   $x \in C - B \Rightarrow x \in C$ and $x \notin B \Rightarrow x \in C$ and $x \notin A \Rightarrow x \in C - A$ $[\because A \subset B]$
   $\therefore C - B \subset C - A$.

1.11 SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS

If A, B and C are finite sets, and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ or, $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets.

(iii) $n(A - B) = n(A) - n(A \cap B)$ i.e. $n(A - B) + n(A \cap B) = n(A)$

(iv) $n(B - A) = n(B) - n(A \cap B)$ or, $n(B - A) + n(A \cap B) = n(B)$

(v) $n(A \Delta B) =$ No. of elements which belong to exactly one of A or B

$= n((A - B) \cup (B - A))$

$= n(A - B) + n(B - A)$ [$\because$ $(A - B)$ and $(B - A)$ are disjoint]

$= n(A) - n(A \cap B) + n(B) - n(A \cap B)$

$= n(A) + n(B) - 2\,n(A \cap B)$

(vi) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(vii) Number of elements in exactly two of the sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3\,n(A \cap B \cap C)$.

(viii) Number of elements in exactly one of the sets A, B, C

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(ix) $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$

(x) $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *If S and T are two sets such that S has 21 elements, T has 32 elements, and $S \cap T$ has 11 elements, how many elements does $S \cup T$ have?* [NCERT EXEMPLAR]

SOLUTION We have, $n(S) = 21$, $n(T) = 32$ and $n(S \cap T) = 11$

$\therefore$ $n(S \cup T) = n(S) + n(T) - n(S \cap T) \Rightarrow n(S \cup T) = 21 + 32 - 11 = 42$

Hence, $S \cup T$ has 42 elements.

EXAMPLE 2 *If X and Y are two sets such that $X \cup Y$ has 18 elements X has 8 elements and Y has 15 elements, how many elements does $X \cap Y$ have?* [NCERT EXEMPLAR]

SOLUTION We have, $n(X \cup Y) = 18$, $n(X) = 8$, $n(Y) = 15$

$\therefore$ $n(X \cap Y) = n(X) + n(Y) - n(X \cup Y) \Rightarrow n(X \cap Y) = 8 + 15 - 18 = 5$

Hence, $X \cap Y$ has 5 elements.

EXAMPLE 3 *If X and Y are two sets such that X has 40 elements $X \cup Y$ has 60 elements and $X \cap Y$ has 10 elements, how many elements does Y have?* [NCERT EXEMPLAR]

SOLUTION We have, $n(X) = 40$, $n(X \cup Y) = 60$ and $n(X \cap Y) = 10$.

$\therefore$ $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$\Rightarrow$ $60 = 40 + n(Y) - 10 \Rightarrow 60 = n(Y) + 30 \Rightarrow n(Y) = 30$

Hence, Y has 30 elements

EXAMPLE 4 *If A and B are two sets such that $n(A) = 17$, $n(B) = 23$ and $n(A \cup B) = 38$, find*
(i) $n(A \cap B)$ (ii) $n(A - B)$ (iii) $n(B - A)$ (iv) *number of elements in exactly one of A and B.* [NCERT]

SOLUTION We have, $n(A) = 17$, $n(B) = 23$ and $n(A \cup B) = 38$.

(i) $n(A \cap B) = n(A) + n(B) - n(A \cup B) = 17 + 23 - 38 = 2$

(ii) $n(A-B)=n(A)-n(A\cap B) \Rightarrow n(A-B)=17-2=15$

(iii) $n(B-A)=n(B)-n(A\cap B) \Rightarrow n(B-A)=23-2=21$

(iv) Number of elements in exactly one of A and $B = n(A)+n(B)-2n(A\cap B)$

$$=17+23-2\times 2=36$$

EXAMPLE 5 *Let A and B be two sets such that $n(A)=35, n(A\cap B)=11$ and $n((A\cup B)')=17$. If $n(U)=57$, find :*

(i) $n(B)$ (ii) $n(A-B)$ (iii) $n(B-A)$

SOLUTION (i) We have,

$n((A\cup B)')=17$

$\Rightarrow$ $n(U)-n(A\cup B)=17 \Rightarrow 57-n(A\cup B)=17 \Rightarrow n(A\cup B)=57-17=40$

$\Rightarrow$ $n(A)+n(B)-n(A\cap B)=40 \Rightarrow 35+n(B)-11=40 \Rightarrow n(B)=16$

(ii) $n(A-B)=n(A)-n(A\cap B)=35-11=24$

(iii) $n(B-A)=n(B)-n(A\cap B)=16-11=5$

EXAMPLE 6 *In a group of 400 people, 250 can speak Hindi and 200 can speak English. How many can speak both Hindi and English?* **[NCERT]**

SOLUTION Let H denote the set of people speaking Hindi and E denote the set of people speaking English. We are given that: $n(H)=250$, $n(E)=200$ and $n(H\cup E)=400$ and we have to find $n(H\cap E)$.

Now, $n(H\cup E) = n(H)+n(E)-n(H\cap E)$

$\Rightarrow$ $n(H\cap E) = n(H)+n(E)-n(H\cup E) = 250+200-400 = 50.$

Hence, 50 persons can speak both Hindi and English.

EXAMPLE 7 *In a class of 35 students, 24 like to play cricket and 16 like to play football. Also, each student likes to play at least one of the two games. How many students like to play both cricket and football?* **[NCERT]**

SOLUTION Let C be the set of students who like to play cricket and F be the set of students who like to play football. Then, $C\cup F$ is the set of students who like to play at least one game and, $C\cap F$ is the set of all students who like to play both games. It is given that $n(C)=24, n(F)=16$, $n(C\cup F)=35$ and we have to find $n(C\cap F)$.

$$n(C\cap F)=n(C)+n(F)-n(C\cup F)=24+16-35=5.$$

EXAMPLE 8 *If A and B are two sets and U is the universal set such that $n(U)=700$, $n(A)=200$, $n(B)=300$ and $n(A\cap B)=100$. Find $n(A'\cap B')$.*

SOLUTION We have, $A'\cap B' = (A\cup B)'$

$\therefore$ $n(A'\cap B') = n((A\cup B)')=n(U)-n(A\cup B)$

$= n(U)-[n(A)+n(B)-n(A\cap B)]=700-(200+300-100)=300.$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 9 *Out of 500 car owners investigated, 400 owned car A and 200 owned car B, 50 owned both A and B cars. Is this data correct?* **[NCERT]**

SOLUTION Let U be the set of all car owners investigated, X be the set of persons who owned car A and B be the set of persons who owned car B. It is given that $n(U)=500$, $n(X)=400$, $n(Y)=200$ and $n(X\cap Y)=50$.

Now, $n(X\cup Y)=n(X)+n(Y)-n(X\cap Y) \Rightarrow n(X\cup Y)=400+200-50=550$

Clearly, $X \cup Y \subset U$. Therefore, $n(X \cup Y) \leq n(U)$. But, we find that as per the given data, $n(X \cup Y) > n(U)$. This is not possible. Hence, the given data is incorrect.

EXAMPLE 10 *In a group of 50 people, 35 speak Hindi, 25 speak both English and Hindi and all the people speak at least one of the two languages. How many people speak only English and not Hindi ? How many people speak English?*

SOLUTION Let H denote the set of people speaking Hindi and E the set of people speaking English. Then, it is given that: $n(H \cup E) = 50, n(H) = 35, n(H \cap E) = 25$.

Now, $\quad n(E - H) = n(H \cup E) - n(H) = 50 - 35 = 15$

Thus, the number of people speaking English but not Hindi is 15.

Now, $\quad n(H \cup E) = n(H) + n(E) - n(H \cap E) \Rightarrow 50 = 35 + n(E) - 25 \Rightarrow n(E) = 40$

Hence, the number of people who speak English is 40.

EXAMPLE 11 *There are 40 students in a Chemistry class and 60 students in a Physics class. Find the number of students which are either in Physics class or Chemistry class in the following cases:*

(i) *the two classes meet at the same hour.*

(ii) *the two classes meet at different hours and 20 students are enrolled in both the subjects.*

SOLUTION Let A be the set of students in Chemistry class and B be the set of students in Physics class. It is given that $n(A) = 40$ and $n(B) = 60$. We have to find $n(A \cup B)$ in both the cases.

(i) If two classes meet at the same hour, then there will not be a common student sitting in both the classes. Therefore, $n(A \cap B) = 0$.

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = 40 + 60 - 0 = 100$

(ii) If two classes meet at different timings then there can be some students attending both the classes. It is given that the number of such students is 20 i.e. $n(A \cap B) = 20$.

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = 40 + 60 - 20 = 80.$

EXAMPLE 12 *In a survey of 700 students in a college, 180 were listed as drinking Limca, 275 as drinking Miranda and 95 were listed as both drinking Limca as well as Miranda. Find how many students were drinking neither Limca nor Miranda.*

SOLUTION Let U be the set of all surveyed students, A denote the set of students drinking Limca and B be the set students drinking Miranda. It is given that $n(U) = 700, n(A) = 180, n(B) = 275$ and $n(A \cap B) = 95$. We have to find $n(A' \cap B')$.

Now, $\quad n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B) = n(U) - \{n(A) + n(B) - n(A \cap B)\}$

$= 700 - (180 + 275 - 95) = 700 - 360 = 340.$

EXAMPLE 13 *There are 200 individuals with a skin disorder, 120 has been exposed to chemical C_1, 50 to chemical C_2 and 30 to both the chemicals C_1 and C_2. Find the number of individuals exposed to (i) chemical C_1 or chemical C_2 (ii) chemical C_1 but not chemical C_2 (iii) chemical C_2 but not chemical C_1.* [NCERT]

SOLUTION Let U denote the universal set consisting of individuals suffering from the skin disorder, A denote the set of individuals exposed to chemical C_1 and B denote the set of individuals exposed to chemical C_2. It is given that: $n(U) = 200, n(A) = 120, n(B) = 50$ and $n(A \cap B) = 30$.

(i) The number of individuals exposed to chemical C_1 or chemical C_2 is given by $n(A \cup B)$.

Now, $\quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = 120 + 50 - 30 = 140$

Hence, required number of individuals is 140.

(ii) The number of individuals exposed to chemical C_1 but not chemical C_2 is given by $n(A \cap \overline{B})$.

Now, $\quad n(A \cap \overline{B}) = n(A) - n(A \cap B) = 120 - 30 = 90$

Hence, required number of individuals is 90.

(iii) The number of individuals exposed to chemical C_2 but not chemical C_1 is given by $n(\overline{A} \cap B)$.

Now, $n(\overline{A} \cap B) = n(B) - n(A \cap B) = 50 - 30 = 20$

Hence, required number is 20.

EXAMPLE 14 *In a class of 35 students, 17 have taken Mathematics, 10 have taken Mathematics but not Economics. Find the number of students who have taken both Mathematics and Economics and the number of students who have taken Economics but not Mathematics, if it is given that each student has taken either Mathematics or Economics or both.*

SOLUTION Let A denote the set of students who have taken Mathematics and B be the set of students who have taken Economics. It is given that $n(A \cup B) = 35$, $n(A) = 17$ and $n(A - B) = 10$.

Now, $n(A) = n(A - B) + n(A \cap B) \Rightarrow 17 = 10 + n(A \cap B) \Rightarrow n(A \cap B) = 7$

Thus, 7 students have taken both Mathematics and Economics.

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B) \Rightarrow 35 = 17 + n(B) - 7 \Rightarrow n(B) = 25$

$\therefore$ $n(B - A) = n(B) - n(A \cap B) = 25 - 7 = 18$

Thus, 18 students have taken Economics but not Mathematics.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 15 *If A and B are finite sets such that $n(A) = m_1$ and $n(B) = m_2$, then find the least and greatest values of $n(A \cup B)$.*

SOLUTION We know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow \quad n(A \cup B) \leq n(A) + n(B) \qquad [\because n(A \cap B) \geq 0]$$

$$\Rightarrow \quad n(A \cup B) \leq m_1 + m_2 \qquad \ldots\text{(i)}$$

So, the greatest value of $n(A \cup B)$ is $m_1 + m_2$. It may be noted that $n(A \cup B) = m_1 + m_2$ only when $n(A \cap B) = 0$ i.e. when $A \cap B = \phi$ i.e. when A and B are disjoint sets.

Also, we know that

$$A \subseteq A \cup B \text{ and } B \subseteq A \cup B$$

$$\Rightarrow \quad n(A) \leq n(A \cup B) \text{ and } n(B) \leq n(A \cup B)$$

$$\Rightarrow \quad m_1 \leq n(A \cup B) \text{ and } m_2 \leq n(A \cup B)$$

$$\Rightarrow \quad n(A \cup B) \geq m_1 \text{ and } n(A \cup B) \geq m_2$$

$$\Rightarrow \quad n(A \cup B) \geq \max\{m_1, m_2\} \qquad \ldots\text{(ii)}$$

So, the least value of $n(A \cup B)$ is $\max\{m_1, m_2\}$.

It may be noted that $n(A \cup B) = \max\{m_1, m_2\}$ only when either $A \subseteq B$ or $B \subseteq A$.

From (i) and (ii), we obtain

$$\max\{m_1, m_2\} \leq n(A \cup B) \leq m_1 + m_2 \text{ or, } \max\{n(A), n(B)\} \leq n(A \cup B) \leq n(A) + n(B)$$

EXAMPLE 16 *If A and B are two sets such that $n(A) = 35$, $n(B) = 30$ and $n(U) = 50$, then find*

(i) the greatest value of $n(A \cup B)$ *(ii) the least value of $(A \cap B)$*

SOLUTION (i) We know that

$$A \cup B \subseteq U \Rightarrow n(A \cup B) \leq n(U) \Rightarrow n(A \cup B) \leq 50$$

So, the greatest value of $n(A \cup B)$ is 50.

(ii) From (i), we have

$$n(A \cup B) \leq 50$$

$$\Rightarrow \quad n(A) + n(B) - n(A \cap B) \leq 50 \Rightarrow 35 + 30 - n(A \cap B) \leq 50 \Rightarrow 15 \leq (A \cap B) \Rightarrow n(A \cap B) \geq 15$$

So, the least value of $n(A \cap B)$ is 15.

EXAMPLE 17 *If A and B be two sets containing 3 and 6 elements respectively, what can be the minimum number of elements in $A \cup B$? Find also, the maximum number of elements in $A \cup B$.*

SOLUTION We have, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

This shows that $n(A \cup B)$ is minimum or maximum according as $n(A \cap B)$ is maximum or minimum respectively.

Case I *When $n(A \cap B)$ is minimum, i.e. $n(A \cap B) = 0$:* This is possible only when $A \cap B = \phi$. In this case, $n(A \cap B) = 0$.

$\therefore \quad n(A \cup B) = n(A) + n(B) - 0 = n(A) + n(B) = 3 + 6 = 9$

So, maximum number of elements in $A \cup B$ is 9.

Case II *When $n(A \cap B)$ is maximum:* This is possible only when $A \subseteq B$. In this case, $n(A \cap B) = 3$.

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = 3 + 6 - 3 = 6$

So, minimum number of elements in $A \cup B$ is 6.

ALITER We know that

$\max\{n(A), n(B)\} \leq n(A \cup B) \leq n(A) + n(B) \Rightarrow \max\{3, 6\} \leq n(A \cup B) \leq 3 + 6 \Rightarrow 6 \leq n(A \cup B) \leq 9$

Hence, the minimum and maximum number of elements in $A \cup B$ is 6 and 9 respectively.

EXAMPLE 18 *A market research group conducted a survey of 2000 consumers and reported that 1720 consumers liked product P_1 and 1450 consumers liked product P_2. What is the least number that must have liked both the products ?* [NCERT]

SOLUTION Let U be the set of all consumers who were questioned, A be the set of consumers who liked product P_1 and B be the set of consumers who liked the product P_2. It is given that $n(U) = 2000$, $n(A) = 1720$, $n(B) = 1450$.

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = 1720 + 1450 - n(A \cap B) = 3170 - n(A \cap B)$

Now, $\quad A \cup B \subset U$

$\Rightarrow n(A \cup B) \leq n(U) \Rightarrow 3170 - n(A \cap B) \leq 2000 \Rightarrow 3170 - 2000 \leq n(A \cap B) \Rightarrow n(A \cap B) \geq 1170$

Thus, the least value of $n(A \cap B)$ is 1170. Hence, the least number of consumer who liked both the products is 1170.

EXAMPLE 19 *A survey shows that 63% of the Americans like cheese whereas 76% like apples. If x% of the Americans like both cheese and apples, find the value of x.*

SOLUTION Let A denote the set of Americans who like cheese and let B denote those who like apples. Let the population of America be 100. Then, $n(A) = 63$, $n(B) = 76$.

Now, $n(A \cap B) = n(A) + n(B) - n(A \cup B) \Rightarrow n(A \cap B) = 63 + 76 - n(A \cup B) = 139 - n(A \cup B)$

But, $n(A \cup B) \leq 100$.

$\Rightarrow -n(A \cup B) \geq -100 \Rightarrow 139 - n(A \cup B) \geq 139 - 100 \Rightarrow 139 - n(A \cup B) \geq 39 \Rightarrow n(A \cap B) \geq 39$ ...(i)

Now, $A \cap B \subseteq A$ and $A \cap B \subseteq B$

$\Rightarrow n(A \cap B) \leq n(A)$ and $n(A \cap B) \leq n(B) \Rightarrow n(A \cap B) \leq 63$ and $n(A \cap B) \leq 76 \Rightarrow n(A \cap B) \leq 63$ ...(ii)

From (i) and (ii), we obtain : $39 \leq n(A \cap B) \leq 63 \Rightarrow 39 \leq x \leq 63$.

EXAMPLE 20 *In a town of 10,000 families it was found that 40% families buy newspaper A, 20% families buy newspaper B and 10% families buy newspaper C. 5% families buy A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all the three news papers, find the number of families which buy (i) A only (ii) B only (iii) none of A, B and C.* [NCERT EXEMPLAR]

SOLUTION Let P, Q and R denote the sets of families buying newspaper A, B and C respectively. Let U be the universal set. Then,

$n(P) = 40\%$ of $10{,}000 = 4000$, $n(Q) = 20\%$ of $10{,}000 = 2000$, $n(R) = 10\%$ of $10{,}000 = 1000$,

$n(P \cap Q) = 5\%$ of $10{,}000 = 500$, $n(Q \cap R) = 3\%$ of $10{,}000 = 300$, $n(R \cap P) = 4\%$ of $10{,}000 = 400$

$n(P \cap Q \cap R) = 2\%$ of $10{,}000 = 200$ and $n(U) = 10{,}000$.

(i) Required number $= n(P \cap Q' \cap R') = n(P \cap (Q \cup R)')$

$= n(P) - n[P \cap (Q \cup R)] \quad [\because n(A \cap B') = n(A) - n(A \cap B)]$

$= n(P) - n[(P \cap Q) \cup (P \cap R)]$

$= n(P) - [n(P \cap Q) + n(P \cap R) - n\{(P \cap Q) \cap (P \cap R)\}]$

$= n(P) - [n(P \cap Q) + n(P \cap R) - n(P \cap Q \cap R)]$

$= 4000 - (500 + 400 - 200) = 3300$

(ii) Required number $= n(P' \cap Q \cap R') = n(Q \cap P' \cap R') = n(Q \cap (P \cup R)')$

$= n(Q) - n(Q \cap (P \cup R))$ $\quad [\because n(A \cap B') = n(A) - n(A \cap B)]$

$= n(Q) - n[(Q \cap P) \cup (Q \cap R)]$

$= n(Q) - [n(Q \cap P) + n(Q \cap R) - n\{(Q \cap P) \cap (Q \cap R)\}]$

$= n(Q) - [n(P \cap Q) + n(Q \cap R) - n(P \cap Q \cap R)]$

$= 2000 - (500 + 300 - 200) = 1400$

(iii) Required number $= n(P' \cap Q' \cap R') = n[(P \cup Q \cup R)'] = n(U) - n(P \cup Q \cup R)$

$= n(U) - [n(P) + n(Q) + n(R) - n(P \cap Q) - n(Q \cap R) - n(R \cap P) + n(P \cap Q \cap R)]$

$= 10000 - [4000 + 2000 + 1000 - 500 - 300 - 400 + 200] = 4000.$

ALITER 1 It is given that $n(P \cap Q \cap R) = 200$ and $n(P \cap Q) = 500$. So, the number of families buying newspaper A and B only is $500 - 200 = 300$. Similarly, the number of families buying news papers B and C only is $300 - 200 = 100$ and news papers C and A only is $400 - 200 = 200$.

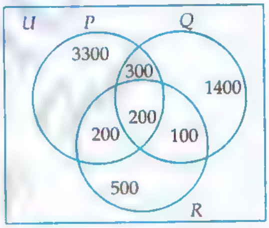


Fig. 1.13

(i) It is given that 4000 families buy news paper A. So, the number of families buying news paper A only $= 4000 - (300 + 200 + 200) = 3300$.

(ii) Similarly, the number of families buying news paper B only

$= 2000 - (300 + 200 + 100) = 1400$

The number of families buying news paper C only $= 1000 - (200 + 200 + 100) = 500$

(iii) The number of families buying none of the news papers

$= n(U) - n(P \cup Q \cup R) = 10000 - (3300 + 300 + 200 + 200 + 100 + 1400 + 500) = 4000$

ALITER 2 In the adjacent Venn diagram, let a, b, c, d, e, f and g denote the number of families buying news paper(s) represented by the respective regions.

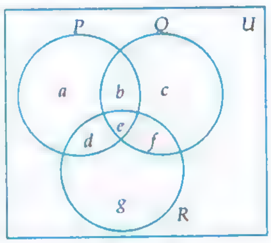


Fig. 1.14

It is given that

$a + b + d + e = 4000, b + c + e + f = 2000, d + e + f + g = 1000,$

$b + e = 500, e + f = 300, d + e = 400$ and $e = 200$.

Now, $\quad e = 200$ and $d + e = 400, e + f = 300, b + e = 500$

$\Rightarrow \quad d = 200, f = 100, b = 300.$

Substituting $b = 300, d = 200, e = 200, f = 100$ in $d + e + f + g = 1000$, $b + c + e + f = 2000$ and $a + b + d + e = 4000$, we obtain

$g = 500, \; a = 3300$ and $c = 1400$.

(i) Number of families buying news paper A only $= a = 3300$

(ii) Number of families buying news paper B only $= c = 1400$

(iii) Number of families not buying any news paper

$= 10{,}000 - (a + b + c + d + e + f + g)$

$= 10{,}000 - (3300 + 300 + 1400 + 200 + 200 + 100 + 500) = 4{,}000$

EXAMPLE 21 *A college awarded 38 medals in Football, 15 in Basketball and 20 to Cricket. If these medals went to a total of 58 men and only three men got medals in all the three sports, how many received medals in exactly two of the three sports?* **[NCERT]**

SOLUTION Let F denote the set of men who received medals in Football, B the set of men who received medals in Basketball and C the set of men who received medals in Cricket. It is given

that $n(F)=38, n(B)=15, n(C)=20,\ n(F\cup B\cup C)=58$ and $n(F\cap B\cap C)=3$.

Now,

$$n(F\cup B\cup C)=n(F)+n(B)+n(C)-n(F\cap B)-n(B\cap C)-n(F\cap C)+n(F\cap B\cap C)$$

$\Rightarrow\quad 58=38+15+20-n(F\cap B)-n(B\cap C)-n(F\cap C)+3$

$\Rightarrow\quad n(F\cap B)+n(B\cap C)+n(F\cap C)=76-58\ =\ 18$

Number of men who received medals in exactly two of the three sports

$$=n(F\cap B)+n(B\cap C)+n(F\cap C)-3\,n(F\cap B\cap C)=\ 18-3\times 3=\ 9$$

Thus, 9 men received medals in exactly two of the three sports.

ALITER In the adjacent Venn diagram let a, b, c, d, e, f and g denote the number of men who received medals in the game(s) represented by the respective regions. It is given that

Number of medals awarded in Football $=38 \Rightarrow a+b+d+e=38$

Number of medals awarded in Cricket $=20 \Rightarrow d+e+f+g=20$

Number of medals awarded in Basketball $=15 \Rightarrow b+c+e+f=15$

Number of medals awarded in all three sports $=3 \Rightarrow e=3$.

Total number of medals awarded $=58 \Rightarrow a+b+c+d+e+f+g=58$

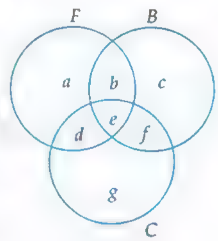


Fig. 1.15

We have to find a number of medals awarded in exactly two of the three sports which is equal to $b+d+f$.

Now, $\quad a+b+d+e=38, b+c+e+f=15$ and $d+e+f+g=20$

$\Rightarrow\quad a+b+d+e+b+c+e+f+d+e+f+g=38+15+20$

$\Rightarrow\quad (a+b+c+d+e+f+g)+(b+d+2e+f)=73$

$\Rightarrow\quad 58+(b+d+2\times 3+f)=73 \Rightarrow b+d+f=9$

EXAMPLE 22 *In a survey of 25 students, it was found that 15 had taken Mathematics, 12 had taken Physics and 11 had taken Chemistry, 5 had taken Mathematics and Chemistry, 9 had taken Mathematics and Physics, 4 had taken Physics and Chemistry and 3 had taken all the three subjects. Find the number of students that had taken*

(i) *only Chemistry.* (ii) *only Mathematics.*
(iii) *only Physics.* (iv) *Physics and Chemistry but not Mathematics.*
(v) *Mathematics and Physics but not Chemistry.* (vi) *only one of the subjects.*
(vii) *at least one of the three subjects.* (viii) *none of the subjects.*

SOLUTION Let M denote the set of students who had taken Mathematics, P the set of students who had taken Physics and C the set of students who had taken Chemistry. It is given that

$$n(U)=25, n(M)=15, n(P)=12, n(C)=11,\ n(M\cap C)=5,\ n(M\cap P)=9, n(P\cap C)=4$$

and, $\quad n(M\cap P\cap C)=3$

(i) Number of students who had opted Chemistry only.

$=\ n(M'\cap P'\cap C)\ =\ n((M\cup P)'\cap C)$

$=\ n(C)-n((M\cup P)\cap C)$ $\quad\quad [\because\ n(A\cap B')=n(A)-n(A\cap B)]$

$=\ n(C)-n((M\cap C)\cup(P\cap C))$

$=\ n(C)-\{n(M\cap C)+n(P\cap C)-n(M\cap P\cap C)\}=\ 11-(5+4-3)\ =\ 5$

(ii) The number of students who had opted Mathematics only.

$=\ n(M\cap P'\cap C')=n(M\cap(P\cap C)')=n(M)-n(M\cap(P\cup C))=n(M)-n((M\cap P)\cup(M\cap C))$

$= n(M) - \{n(M \cap P) + n(M \cap C) - n(M \cap P \cap C)\} = 15 - (9 + 5 - 3) = 4$

(iii) The number of students who had opted Physics only

$= n(P \cap M' \cap C') = n(P \cap (M \cup C)') = n(P) - n(P \cap (M \cup C)) = n(P) - n((P \cap M) \cup (P \cap C))$

$= n(P) - \{n(P \cap M) + n(P \cap C) - n(P \cap M \cap C)\} = 12 - (9 + 4 - 3) = 2.$

(iv) Required number of students $= n(P \cap C \cap M')$

$= n(P \cap C) - n(P \cap C \cap M)$ $[\because n(A \cap B') = n(A) - n(A \cap B)]$

$= 4 - 3 = 1$

(v) Required number of students $= n(M \cap P \cap C') = n(M \cap P) - n(M \cap P \cap C) = 9 - 3 = 6$

(vi) Required number of students

$= n(M) + n(P) + n(C) - 2\{n(M \cap P) + n(P \cap C) + n(M \cap C) + 3n(M \cap P \cap C)\}$

$= 15 + 12 + 11 - 2(9 + 4 + 5) + 3 \times 3 = 38 - 36 + 9 = 11$

(vii) Required number of students $= n(M \cup P \cup C)$

$= n(M) + n(P) + n(C) - n(M \cap P) - n(P \cap C) - n(M \cap C) + n(M \cap P \cap C)$

$= 15 + 12 + 11 - 9 - 4 - 5 + 3 = 23$

(viii) Required number of students $= n(M' \cap P' \cap C')$

$= n(M \cup P \cup C)' = n(U) - n(M \cup P \cup C) = 25 - 23 = 2.$

ALITER 1 Consider the Venn diagram shown in Fig. 1.16. Let a, b, c, d, e, f, g denote the number of students in the respective regions.

From the Venn-daigram, we find that

$n(M) = a + b + d + e, n(P) = b + c + e + f,$

$n(C) = d + e + f + g,$

$n(M \cap P) = b + e, n(P \cap C) = e + f,$

$n(M \cap C) = d + e$ and, $n(M \cap P \cap C) = e$

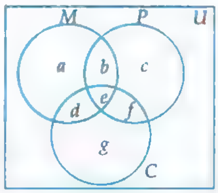


Fig. 1.16

In order to find the required values, we need to find the values of a, b, c, d, e, f and g. So, let us first determine these values.

It is given that : $n(M \cap P \cap C) = 3 \Rightarrow e = 3$

$n(M \cap P) = 9 \Rightarrow b + e = 9 \Rightarrow b + 3 = 9 \Rightarrow b = 6$

$n(P \cap C) = 4 \Rightarrow e + f = 4 \Rightarrow 3 + f = 4 \Rightarrow f = 1$

$n(M \cap C) = 5 \Rightarrow d + e = 5 \Rightarrow d + 3 = 5 \Rightarrow d = 2$

$n(M) = 15 \Rightarrow a + b + d + e = 15 \Rightarrow a + 6 + 2 + 3 = 15 \Rightarrow a = 4$

$n(P) = 12 \Rightarrow b + c + e + f = 12 \Rightarrow 6 + c + 3 + 1 = 12 \Rightarrow c = 2$

$n(C) = 11 \Rightarrow d + e + f + g = 11 \Rightarrow 2 + 3 + 1 + g = 11 \Rightarrow g = 5$

Thus, we have $a = 4, b = 6, c = 2, d = 2, e = 3, f = 1$ and $g = 5$.

(i) Number of students that had taken Chemistry only $= g = 5$

(ii) Number of students that had taken Mathematics only $= a = 4$

(iii) Number of students that had taken Physics only $= c = 2$.

(iv) Number of students that had taken Physics and Chemistry but not Mathematics $= f = 1$

(v) Number of students that had taken Mathematics and Physics but not Chemistry $= b = 6$

(vi) Number of students that had taken only one of the subjects $= a + c + g = 4 + 2 + 5 = 11$

(vii) Number of students that had taken at least one of the three subjects

$= a + b + c + d + e + f + g = 23$

(viii) Number of students that had taken none of the subjects

$= 25-(a+b+c+d+e+f+g) = 25-23 = 2.$

ALITER 2 It is given that 3 students had taken all three subjects and 9 had taken Mathematics and Physics. So, number of students who had taken Mathematics and Physics but not Chemistry is 6 as shown in the Venn diagram. 5 students had taken Mathematics and Chemistry and 3 students had taken all the three subjects. So, 2 students had taken Mathematics and Chemistry but not Physics. It is given that 15 students had taken Mathematics. So, number of students who had taken only Mathematics $= 15-(6+3+2) = 4$.

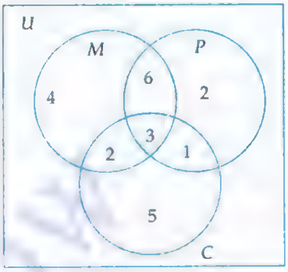


Fig. 1.17

Similarly, we compute the other values shown in the Venn diagram. It is evident from the Venn diagram that the number of students that had taken

(i) Chemistry only is 5.
(ii) only Mathematics is 4.
(iii) only Physics is 2.
(iv) Physics and Chemistry but not Mathematics is 1.
(v) Mathematics and Physics but not Chemistry is 6.
(vi) only one of the subjects is $4+2+5=11$
(vii) at least one of three subjects is $4+6+2+2+3+1+5=23$
(viii) none of the subjects $= 25-23 = 2.$

EXAMPLE 23 *In a survey of 100 students, the number of students studying the various languages were found to be : English only 18, English but not Hindi 23, English and German 8, English 26, German 48, German and Hindi 8, no language 24. Find the number of students who were studying (i) Hindi (ii) English and Hindi (iii) English, Hindi and German.*

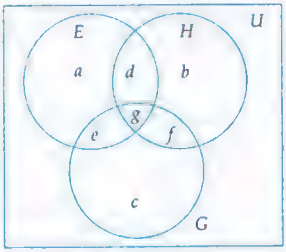


Fig. 1.18

SOLUTION Let E, H and G be the sets of students studying English, Hindi and German respectively. Let U be the set of students surveyed i.e. the universal set.

In the above Venn diagram, let a, b, c, d, e, f and g denote the number of students in the respective regions.

Clearly, $n(U) = 100$. It is given that:

Number of students studying English only $= 18 \Rightarrow a = 18$

Number of students studying English but not Hindi $= 23 \Rightarrow a+e = 23$

Number of students studying English and German $= 8 \Rightarrow e+g = 8$

Number of students studying English $= 26 \Rightarrow a+d+e+g = 26$

Number of students studying German $= 48 \Rightarrow c+e+g+f = 48$

Number of students studying German and Hindi $= 8 \Rightarrow g+f = 8$

Thus, we obtain : $a = 18$, $a+e = 23$, $e+g = 8$, $a+e+g+d = 26$, $e+g+f+c = 48$ and $g+f = 8$

$\Rightarrow$ $a = 18, e = 5, g = 3, d = 0, f = 5, c = 35$

It is given that 24 students study no language. Therefore, the number of students who study at least one language is $100-24 = 76$

i.e. $n(E \cup H \cup G) = 76 \Rightarrow a+b+c+d+e+f+g = 76 \Rightarrow 18+b+35+0+5+5+3 = 76 \Rightarrow b = 10$

(i) The number of students studying Hindi $= b+d+g+f = 10+0+3+5 = 18$

(ii) The number of students studying English and Hindi $= d + g = 0 + 3 = 3$

(iii) The number of students studying English, Hindi and German $= g = 3$.

EXAMPLE 24 *In an university, out of 100 students 15 offered Mathematics only; 12 offered Statistics only; 8 offered Physics only; 40 offered Physics and Mathematics ; 20 offered Physics and Statistics; 10 offered Mathematics and Statistics; 65 offered Physics. Find the number of students who*

(i) offered Mathematics (ii) offered statistics (iii) did not offer any of the above three subjects.

SOLUTION Let M, S and P be the sets of students who offered Mathematics, Statistics and Physics respectively. Let x be the number of students who offered all the three subjects. It is given that 10 students offered Mathematics and Statistics. Therefore, number of students who offered Mathematics and Statistics but not Physics is $10 - x$. Similarly, number of students in different regions are marked in Fig. 1.19. It is given that 65 students offered Physics.

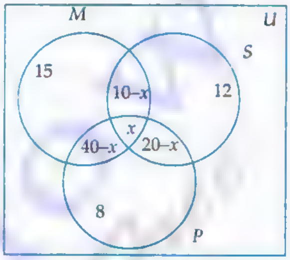


Fig. 1.19

$\therefore \quad (40 - x) + x + (20 - x) + 8 = 65 \Rightarrow 68 - x = 65 \Rightarrow x = 3$

(i) From Fig. 1.19, we find that

The number of students who offered Mathematics $= 15 + (10 - x) + x + 40 - x = 65 - x = 65 - 3 = 62$

(ii) The number of students who offered Statistics $= 12 + (10 - x) + x + (20 - x) = 42 - x = 42 - 3 = 39$

(iii) The number of students who offered any of three subjects

$$= 15 + 12 + 8 + (10 - x) + (40 - x) + (20 - x) + x$$
$$= 105 - 2x = 105 - 2 \times 3 = 99$$

$\therefore$ Number of students who did not offer any of the three subjects $= 100 - 99 = 1$.

EXAMPLE 25 *Out of 280 students in class XII of a school, 135 play Hockey, 110 play football, 80 play volleyball, 35 of these play hockey and football, 30 play volleyball and hockey, 20 play football and volleyball. Also, each students plays at least one of the three games. How many students play all the three games?*

SOLUTION Let H, F and V be the sets of students who play hockey, football and volleyball respectively. Let x be the number of students who play all the three games. It is given that 35 students play hockey and football. So, number of student who play hockey and football only is $(35 - x)$. Similarly, the number of students playing various games are written in the regions representing them in Fig. 1.20.

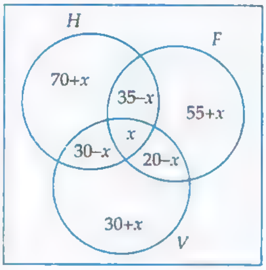


Fig.1.20

It is given that each student plays at least one of the three games.

$\therefore \quad n(H \cup F \cup V) = 280$

$\Rightarrow \quad (70 + x) + (35 - x) + (30 - x) + x + (20 - x) + (55 + x) + (30 + x) = 280$

$\Rightarrow \quad 240 + x = 280 \Rightarrow x = 40$

Hence, 40 students play all the three games.

EXAMPLE 26 *From 50 students taking examination in Mathematics, Physics and Chemistry, 37 passed Mathematic, 24 Physics and 43 Chemistry. At most 19 passed Mathematics and Physics; at most 29 passed Mathematics and Chemistry and at most 20 passed Physics and Chemistry. If each student has passed in at least one of the subjects, find the largest number of students who could have passed in all the three subjects.* [NCERT EXEMPLAR]

SOLUTION Let M, P and C be the sets of students passing in Mathematics, Physics and Chemistry respectively. It is given that : $n(M \cup P \cup C) = 50$, $n(M) = 37$, $n(P) = 24$, $n(C) = 43$, $n(M \cap P) \leq 19$, $n(M \cap C) \leq 29$ and $n(P \cap C) \leq 20$.

We know that

$$n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(M \cap P) - n(M \cap C) - n(P \cap C) + n(M \cap P \cap C)$$

$\Rightarrow \quad 50 = 37 + 24 + 43 - \{n(M \cap P) + n(M \cap C) + n(P \cap C)\} + n(M \cap P \cap C)$

$\Rightarrow \quad 50 = 104 - \{n(M \cap P) + n(M \cap C) + n(P \cap C)\} + n(M \cap P \cap C)$

$\Rightarrow \quad 54 + n(M \cap P \cap C) = n(M \cap P) + n(M \cap C) + n(P \cap C)$

$\Rightarrow \quad 54 + n(M \cap P \cap C) \leq 19 + 29 + 20$

$\Rightarrow \quad n(M \cap P \cap C) \leq 14 \qquad [\because n(M \cap P) \leq 19, n(M \cap C) \leq 29, n(P \cap C) \leq 20]$

Hence, the largest number of students that could have passed in all the three subjects is 14.

EXAMPLE 27 *A school awarded 58 medals for Honesty, 20 for Punctuality and 25 for Obedience. If these medals were bagged by a total of 78 students and only 5 students got medals for all the three values, find the number of students who received medals for exactly two of the three values.*

SOLUTION Let H, P and O be the sets of students who bagged medals in Honesty, Punctuality and Obedience respectively. It is given that $n(H) = 58$, $n(P) = 20$, $n(O) = 25$, $n(H \cup P \cup O) = 78$ and $n(H \cap P \cap O) = 5$.

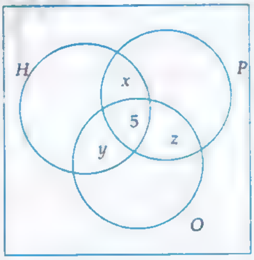


Fig. 1.21

Let x denote the number of students who got medals in Honesty and Punctuality only, y denote the number of students who got medals in Honesty and Obedience only and z denote the number of students who got medals in Punctuality and Obedience only.

Now, $\quad n(H \cup P \cup O) = 78$

$\Rightarrow \quad n(H) + n(P) + n(O) - n(H \cap P) - n(P \cap O) - n(H \cap O) + n(H \cap P \cap O) = 78$

$\Rightarrow \quad 58 + 20 + 25 - n(H \cap P) - n(P \cap O) - n(H \cap O) + 5 = 78$

$\Rightarrow \quad n(H \cap P) + n(P \cap O) + n(H \cap O) = 30 \Rightarrow (x+5) + (y+5) + (z+5) = 30 \Rightarrow x + y + z = 15$

Hence, required number of students $= x + y + z = 15$

ALITER Number of students who bagged medals in Honesty only $= 58 - (x + y + 5) = 53 - x - y$
Number of students who bagged medals in Punctuality only $= 20 - (x + 5 + z) = 15 - x - z$
Number of students who bagged medals in Obedience only $= 25 - (y + 5 + z) = 20 - y - z$
It is given that medals were bagged by a total of 78 students.

$\therefore \quad (53 - x - y) + x + (15 - x - z) + y + 5 + z + 20 - y - z = 78$

$\Rightarrow \quad 93 - (x + y + z) = 78 \Rightarrow x + y + z = 15$

Hence, number of students who bagged medals in exactly two of the three values $= x + y + z = 15$.

EXERCISE 1.8

BASIC

1. If A and B are two sets such that $n(A \cup B) = 50$, $n(A) = 28$ and $n(B) = 32$, find $n(A \cap B)$. [NCERT]
2. If P and Q are two sets such that P has 40 elements, $P \cup Q$ has 60 elements and $P \cap Q$ has 10 elements, how many elements does Q have?
3. In a school there are 20 teachers who teach mathematics or physics. Of these, 12 teach mathematics and 4 teach physics and mathematics. How many teach physics? [NCERT]

4. In a group of 70 people, 37 like coffee, 52 like tea and each person likes at least one of the two drinks. How many like both coffee and tea? [NCERT]
5. Let A and B be two sets such that : $n(A) = 20$, $n(A \cup B) = 42$ and $n(A \cap B) = 4$. Find
   (i) $n(B)$ (ii) $n(A - B)$ (iii) $n(B - A)$
6. A survey shows that 76% of the Indians like oranges, whereas 62% like bananas. What percentage of the Indians like both oranges and bananas?
7. In a group of 950 persons, 750 can speak Hindi and 460 can speak English. Find:
   (i) how many can speak both Hindi and English
   (ii) how many can speak Hindi only (iii) how many can speak English only.
8. In a group of 50 persons, 14 drink tea but not coffee and 30 drink tea. Find:
   (i) how may drink tea and coffee both (ii) how many drink coffee but not tea.
9. In a group of 1000 people, there are 750 who can speak Hindi and 400 who can speak Bengali. How many can speak Hindi only? How many can speak Bengali ? How many can speak both Hindi and Bengali?
10. In a class of 60 students, 25 students play cricket, 20 students play tennis, and 10 students play both the games. Find the number of students who play neither? [NCERT EXEMPLAR]
11. In a survey of 400 students in a school, 100 were listed as taking apple juice, 150 as taking orange juice and 75 were listed as taking both apple as well as orange juice. Find how many students were taking neither apple juice nor orange juice. [NCERT]
12. In a committee, 50 people speak French, 20 speak Spanish and 10 speak both Spanish and French. How many speak at least one of these two languages? [NCERT]
13. In a group of 65 people, 40 like cricket, 10 like both cricket and tennis. How many like tennis only and not cricket? How many like tennis? [NCERT]

BASED ON LOTS

14. In a survey of 60 people, it was found that 25 people read newspaper H, 26 read newspaper T, 26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, 3 read all three newspapers. Find:
    (i) the numbers of people who read at least one of the newspapers.
    (ii) the number of people who read exactly one newspaper. [NCERT]
15. Of the members of three athletic teams in a certain school, 21 are in the basketball team, 26 in hockey team and 29 in the football team. 14 play hockey and basket ball, 15 play hockey and football, 12 play football and basketball and 8 play all the three games. How many members are there in all?
16. A survey of 500 television viewers produced the following information; 285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basketball, 50 do not watch any of the three games. How many watch all the three games? How many watch exactly one of the three games?
17. In a survey of 100 persons it was found that 28 read magazine A, 30 read magazine B, 42 read magazine C, 8 read magazines A and B, 10 read magazines A and C, 5 read magazines B and C and 3 read all the three magazines. Find:
    (i) How many read none of three magazines? (ii) How many read magazine C only?
18. In a survey of 100 students, the number of students studying the various languages were found to be : English only 18, English but not Hindi 23, English and Sanskrit 8, English 26, Sanskrit 48, Sanskrit and Hindi 8, no language 24. Find:

(i) How many students were studying Hindi?

(ii) How many students were studying English and Hindi?

19. In a survey it was found that 21 persons liked product P_1, 26 liked product P_2 and 29 liked product P_3. If 14 persons liked products P_1 and P_2 ; 12 persons liked product P_3 and P_1 ; 14 persons liked products P_2 and P_3 and 8 liked all the three products. Find how many liked product P_3 only. [NCERT]

20. In a survey of 200 students of a school, it was found that 120 study Mathematics, 90 study Physics and 70 study Chemistry, 40 study Mathematics and Physics, 30 study Physics and Chemistry, 50 study Chemistry and Mathematics and 20 none of these subjects. Find the number of students who study all the three subjects. [NCERT EXEMPLAR]

21. In a group of 50 students, the number of students studying French, English, Sanskrit were found to be as follows : French = 17, English = 13, Sanskrit = 15, French and English = 9, English and Sanskrit = 4, French and Sanskrit = 5, English, French and Sanskrit = 3. Find the number of students who study

(i) French only (ii) English only (iii) Sanskrit only

(iv) English and Sanskrit but not French (v) French and Sanskrit but not English

(vi) French and English but not Sanskrit (vii) at least one of the three languages

(viii) None of three languages.

22. Out of 100 students; 15 passed in English, 12 passed in Mathematics, 8 in Science, 6 in English and Mathematics, 7 in Mathematics and Science; 4 in English and Science; 4 in all the three. Find how many passed :

(i) in English and Mathematics but not in Science.

(ii) in Mathematics but not in Science. (iii) in Mathematics only.

(iv) in more than one subject only. [NCERT EXEMPLAR]

ANSWERS

1. 10 2. 30 3. 12 4. 19 5. (i) 26 (ii) 16 (iii) 22 6. 38%
7. (i) 260 (ii) 490 (iii) 200 8. (i) 16 (ii) 20 9. (i) 600 (ii) 250 (iii) 150
10. 25 11. 225 12. 60 13. 25, 35 14. (i) 52 (ii) 30 15. 43 16. 20, 325
17. (i) 20 (ii) 30 18. (i) 18 (ii) 3 19. 11 20. 20
21. (i) 6 (ii) 3 (iii) 9 (iv) 1 (v) 2 (vi) 6 (vii) 30 (viii) 20
22. (i) 2 (ii) 3 (iii) 3 (iv) 9

HINTS TO SELECTED PROBLEMS

5. (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B) \Rightarrow n(B) = 26$

(ii) $n(A - B) = n(A) - n(A \cap B) \Rightarrow n(A - B) = 16$

(iii) $n(B - A) = n(B) - n(A \cap B) \Rightarrow 22$

7. (iii) Let A and B denote the sets of persons who can speak Hindi and English respectively. Then, $n(A \cup B) = 950$, $n(A) = 750$ and $n(B) = 460$.

(i) $n(A \cap B) = n(A) + n(B) - n(A \cup B) = 750 + 460 - 950 = 260$

(ii) Required number $= n(A - B) = n(A) - n(A \cap B)$

(iii) Required number $= n(B - A) = n(B) - n(A \cap B)$.

8. (ii) Let A and B be sets of persons who drink tea and coffee respectively. Then, $n(A \cup B) = 50$, $n(A - B) = 14$, $n(A) = 30$.

(i) $n(A - B) = 14 \Rightarrow n(A) - n(A \cap B) = 14 \Rightarrow n(A \cap B) = n(A) - 14 = 30 - 14 = 16$

(ii) Required number $= n(B-A) = n(B) - n(A \cap B)$.

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B) \Rightarrow 50 = 30 + n(B) - 16 \Rightarrow n(B) = 36$.

$\therefore\ n(B-A) = n(B) - n(A \cap B) \Rightarrow n(B-A) = 36 - 16 = 20$

9. Let A and B be the sets of persons who can speak Hindi and Bengali respectively. Then, $n(A \cup B) = 1000$, $n(A) = 750$ and $n(B) = 400$.

No. of persons who can speak Hindi only $= n(A-B) = n(A) - n(A \cap B)$

No. of persons who can speak Bengali only $= n(B-A) = n(B) - n(B \cap A)$

No. of persons who can speak both Hindi and Bengali $= n(A \cap B) = n(A) + n(B) - n(A \cup B)$.

15. Let A, B and C be the sets of members of basketball, hockey and football teams respectively. Then, $n(A) = 21$, $n(B) = 26$, $n(C) = 29$, $n(A \cap B) = 14$, $n(B \cap C) = 15$, $n(A \cap C) = 12$ and $n(A \cap B \cap C) = 8$.

Required number $= n(A \cup B \cup C)$

$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.

16. N = Total number of television viewers = 500, $n(F) = 285$, $n(H) = 195$, $n(F \cap B) = 45$, $n(F \cap H) = 70$, $n(H \cap B) = 50$, $n(F' \cap H' \cap B') = 50$.

Now, $n(F' \cap H' \cap B') = 50$

$\Rightarrow\ n[(F \cup H \cup B)'] = 50$

$\Rightarrow\ N - n(F \cup H \cup B) = 50$

$\Rightarrow\ 500 - [n(F) + n(H) + n(B) - n(F \cap H) - n(F \cap B) - n(H \cap B) + n(F \cap H \cap B)] = 50$

$\Rightarrow\ n(F \cap H \cap B) = 500 - 285 - 195 - 115 + 70 + 50 + 45 - 50 = 20$.

$\therefore$ Required number $= n(F \cap H \cap B) = 20$

Required number $= n(F \cap H' \cap B') + n(F' \cap H' \cap B) + n(F' \cap H \cap B')]$

$= n(F) + n(H) + n(B) - 2\{n(F \cap H) + n(H \cap B) + n(B \cap F)\} + 3\,n(F \cap H \cap B)$

18. We have, $a = 18$, $a + b = 23$, $d + e = 8$, $a + b + d + e = 26$,

$d + e + f + g = 48$,

and, $a + b + c + d + e + f + g = 100 - 24 = 76$

$\therefore\ a = 18, b = 0, c = 10, d = 5, e = 3, f = 5$ and, $g = 35$

(i) $n(H) = b + c + e + f = 18$ (ii) $n(H \cap E) = b + e = 3$

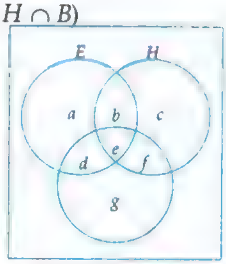


Fig. 1.22

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If A and B are two finite sets, then $n(A) + n(B)$ is equal to ....
2. If A is a finite set containing n elements, then the number of subsets of A is ....
3. The set $\{x \in R : 1 \le x < 2\}$ can be written as ....
4. If A and B are finite sets such that $A \subset B$, then $n(A \cup B) =$ ....
5. If A and B are any two sets, then $A - B$ is equal to ....
6. When $A = \phi$, then the number of elements in $P(A)$ is ....
7. When $A = \phi$, then the number of elements in $P(P(A))$ is ....
8. The power set of set $A = \{1, 2\}$ is ....
9. For all sets A and B, $A - (A \cap B)$ is equal to ....
10. For all sets A and B, $B - (A \cap B)$ is equal to ....

11. Given the sets $A=\{1, 3, 5\}$, $B=\{2, 4, 6\}$ and $C=\{0, 2, 4, 6, 8\}$. Then, the universal set of all the three sets A, B and C can be ....
12. If $U=\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A=\{1, 2, 3, 5\}$, $B=\{2, 4, 6, 7\}$ and $C=\{2, 3, 4, 8\}$. Then,
(i) $(B \cup C)' =$ .... (ii) $(C-A)' =$ ....
13. If A and B are two sets, then $A \cap (A \cup B)'$ is equal to ....
14. If A and B are two sets, then $((A' \cup B')-A)'$ is equal to ....
15. For any two sets A and B, $[B' \cup (B'-A)]'$ is equal to ....
16. For any three sets A, B and C, $(A-B)-(B-C)$ is equal to ....
17. For any three sets A, B and C, $(A-B) \cap (C-B)$ is equal to ....
18. If A and B are two sets, then $(A \cap B')' \cup (B \cap C)$ is equal to ....
19. For any three sets A, B and C, $(A \cup B \cup C) \cap (A \cap B' \cap C') \cap C'$ is equal to ....
20. Let $S=\{x : x$ is a positive multiple of 3 less than $100\}$ $P=\{x : x$ is a prime number less than $20\}$ Then, $n(S)+n(P) =$ ....
21. If $n(A \cap B)=10$, $n(B \cap C)=20$ and $n(A \cap C)=30$, then the greatest possible value of $n(A \cap B \cap C)$ is ....
22. If A, B and C are any three non-empty sets such that any two of them are disjoint, then $(A \cup B \cup C) \cap (A \cap B \cap C) =$ ....
23. If $n(A \cap B)=5$, $n(A \cap C)=7$ and $n(A \cap B \cap C)=3$, then the minimum possible value of $n(B \cap C)$ is ....
24. A and B are any two non-empty sets and A is proper subset of B. If $n(A)=5$, then the minimum possible value of $n(A \Delta B)$ is ....
25. For any two sets A and B, if $n(A)=15$, $n(B)=12$, $A \cap B \neq \phi$ and $B \not\subset A$, then the maximum and minimum possible values of $n(A \Delta B)$ are .... and .... respectively.
26. If A and B are two finite sets such that $n(A) > n(B)$ and the difference of the number of elements of the power sets of A and B is 96, then $n(A)-n(B) =$ ....

ANSWERS

1. $n(A \cap B)+n(A \cap B)$ 2. 2^n 3. $[1, 2)$ 4. $n(B)$ 5. $A \cap \overline{B}$ 6. 1
7. 2 8. $P(A)=\{\phi, \{1\}, \{2\}, \{1, 2\}\}$ 9. $A \cap B'$ 10. $B \cap A'$ 11. $\{0, 1, 2, 3, 4, 5, 6, 8\}$
12. (i) $\{5, 9, 10\}$ (ii) $\{1, 2, 3, 5, 6, 7, 9, 10\}$ 13. ϕ 14. A
15. B 16. $A-B$ 17. $(A \cap C)-B$ 18. $A' \cup B$ 19. $B \cap C'$
20. 41 21. 10 22. ϕ 23. 3 24. 1
25. 25, 5 26. 2

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If a set contains n elements, then write the number of elements in its power set.
2. Write the number of elements in the power set of null set.
3. Let $A = \{x : x \in N$, x is a multiple of $3\}$ and $B=\{x : x \in N$ and x is a multiple of $5\}$. Write $A \cap B$.
4. Let A and B be two sets having 3 and 6 elements respectively. Write the minimum number of elements that $A \cup B$ can have.

5. If $A = \{x \in C : x^2 = 1\}$ and $B = \{x \in C : x^4 = 1\}$, then write $A - B$ and $B - A$.
6. If A and B are two sets such that $A \subset B$, then write $B' - A'$ in terms of A and B.
7. Let A and B be two sets having 4 and 7 elements respectively. Then write the maximum number of elements that $A \cup B$ can have.
8. If $A = \{(x, y) : y = \frac{1}{x}, 0 \neq x \in R\}$ and $B = \{(x, y) : y = -x, x \in R\}$, then write $A \cap B$.
9. If $A = \{(x, y) : y = e^x, x \in R\}$ and $B = \{(x, y) : y = e^{-x}, x \in R\}$, then, write $A \cap B$.
10. If A and B are two sets such that $n(A) = 20, n(B) = 25$ and $n(A \cup B) = 40$, then write $n(A \cap B)$.
11. If A and B are two sets such that $n(A) = 115, n(B) = 326, n(A - B) = 47$, then write $n(A \cup B)$.

ANSWERS

1. 2^n 2. 1 3. $\{x : x \in N, x$ is a multiple of $15\}$ 4. 6
5. $A - B = \phi, B - A = \{i, -i\}$ 6. ϕ 7. 11 8. ϕ 9. $\{(0, 1)\}$
10. 5 11. 373

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. For any set A, $(A')'$ is equal to
   (a) A' (b) A (c) ϕ (d) none of these
2. Let A and B be two sets in the same universal set. Then, $A - B =$
   (a) $A \cap B$ (b) $A' \cap B$ (c) $A \cap B'$ (d) none of these
3. The number of subsets of a set containing n elements is
   (a) n (b) $2^n - 1$ (c) n^2 (d) 2^n
4. For any two sets A and B, $A \cap (A \cup B) =$
   (a) A (b) B (c) ϕ (d) none of these
5. If $A = \{1, 3, 5, B\}$ and $B = \{2, 4\}$, then
   (a) $4 \in A$ (b) $\{4\} \subset A$ (c) $B \subset A$ (d) none of these
6. The symmetric difference of A and B is not equal to
   (a) $(A - B) \cap (B - A)$ (b) $(A - B) \cup (B - A)$
   (c) $(A \cup B) - (A \cap B)$ (d) $\{(A \cup B) - A\} \cup \{(A \cup B) - B\}$
7. The symmetric difference of $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ is
   (a) $\{1, 2\}$ (b) $\{1, 2, 4, 5\}$ (c) $\{4, 3\}$ (d) $\{2, 5, 1, 4, 3\}$
8. For any two sets A and B, $(A - B) \cup (B - A) =$
   (a) $(A - B) \cup A$ (b) $(B - A) \cup B$
   (c) $(A \cup B) - (A \cap B)$ (d) $(A \cup B) \cap (A \cap B)$
9. Which of the following statement is false :
   (a) $A - B = A \cap B'$ (b) $A - B = A - (A \cap B)$

(c) $A - B = A - B'$ (d) $A - B = (A \cup B) - B$

10. For any three sets A, B and C
(a) $A \cap (B - C) = (A \cap B) - (A \cap C)$ (b) $A \cap (B - C) = (A \cap B) - C$
(c) $A \cup (B - C) = (A \cup B) \cap (A \cup C')$ (d) $A \cup (B - C) = (A \cup B) - (A \cup C)$.

11. Let $A = \{x : x \in R, x > 4\}$ and $B = \{x \in R : x < 5\}$. Then, $A \cap B =$
(a) $(4, 5]$ (b) $(4, 5)$ (c) $[4, 5)$ (d) $[4, 5]$

12. Let U be the universal set containing 700 elements. If A, B are sub-sets of U such that $n(A) = 200, n(B) = 300$ and $n(A \cap B) = 100$. Then, $n(A' \cap B') =$
(a) 400 (b) 600 (c) 300 (d) none of these.

13. Let A and B be two sets such that $n(A) = 16, n(B) = 14, n(A \cup B) = 25$. Then, $n(A \cap B)$ is equal to
(a) 30 (b) 50 (c) 5 (d) none of these

14. If $A = \{1, 2, 3, 4, 5\}$, then the number of proper subsets of A is
(a) 120 (b) 30 (c) 31 (d) 32

15. In set-builder method the null set is represented by
(a) $\{ \}$ (b) Φ (c) $\{x : x \neq x\}$ (d) $\{x : x = x\}$

16. If A and B are two disjoint sets, then $n(A \cup B)$ is equal to
(a) $n(A) + n(B)$ (b) $n(A) + n(B) - n(A \cap B)$
(c) $n(A) + n(B) + n(A \cap B)$ (d) $n(A)\, n(B)$

17. For two sets $A \cup B = A$ iff
(a) $B \subseteq A$ (b) $A \subseteq B$ (c) $A \neq B$ (d) $A = B$

18. If A and B are two sets such that $n(A) = 70, n(B) = 60, n(A \cup B) = 110$, then $n(A \cap B)$ is equal to
(a) 240 (b) 50 (c) 40 (d) 20

19. If A and B are two given sets, then $A \cap (A \cap B)^c$ is equal to
(a) A (b) B (c) Φ (d) $A \cap B^c$

20. If $A = \{x : x$ is a multiple of 3$\}$ and, $B = \{x : x$ is a multiple of 5$\}$, then $A - B$ is
(a) $A \cap B$ (b) $A \cap \overline{B}$ (c) $\overline{A} \cap \overline{B}$ (d) $\overline{A \cap B}$

21. In a city 20% of the population travels by car, 50% travels by bus and 10% travels by both car and bus. Then, persons travelling by car or bus is
(a) 80% (b) 40% (c) 60% (d) 70%

22. If $A \cap B = B$, then
(a) $A \subseteq B$ (b) $B \subseteq A$ (c) $A = \Phi$ (d) $B = \Phi$

23. An investigator interviewed 100 students to determine the performance of three drinks: milk, coffee and tea. The investigator reported that 10 students take all three drinks milk, coffee and tea; 20 students take milk and coffee; 25 students take milk and tea; 20 students take coffee and tea; 12 students take milk only; 5 students take coffee only and 8 students take tea only. Then the number of students who did not take any of three drinks is
(a) 10 (b) 20 (c) 25 (d) 30

24. Two finite sets have m and n elements. The number of elements in the power set of first set is 48 more than the total number of elements in power set of the second set. Then, the values of m and n are:
(a) 7, 6 (b) 6, 3 (c) 6, 4 (d) 7, 4
[NCERT EXEMPLAR]

25. In a class of 175 students the following data shows the number of students opting one or more subjects. Mathematics 100; Physics 70; Chemistry 40; Mathematics and Physics 30; Mathematics and Chemistry 28; Physics and Chemistry 23; Mathematics, Physics and Chemistry 18. How many students have offered Mathematics alone?
(a) 35 (b) 48 (c) 60 (d) 22

26. Suppose $A_1, A_2, \ldots, A_{30}$ are thirty sets each having 5 elements and $B_1, B_2, \ldots, B_n$ are n sets each with 3 elements, let $\bigcup_{i=1}^{30} = \bigcup_{j=1}^{n} B_j = S$ and each element of S belongs to exactly 10 of the $A_i^{'s}$ and exactly 9 of the $B_j^{'s}$, then n is equal to
(a) 15 (b) 3 (c) 45 (d) 35
[NCERT EXEMPLAR]

27. Two finite sets have m and n elements. The number of subsets of the first set is 112 more than that of the second. The values of m and n are respectively
(a) 4, 7 (b) 7, 4 (c) 4, 4 (d) 7, 7
[NCERT EXEMPLAR]

28. For any two sets A and B, $A \cap (A \cup B)'$ is equal to
(a) A (b) B (c) ϕ (d) $A \cap B$

29. The set $(A \cup B') \cup (B \cap C)$ is equal to
(a) $A' \cup B \cup C$ (b) $A' \cup B$ (c) $A' \cup C'$ (d) $A' \cap B$
[NCERT EXEMPLAR]

30. Let F_1 be the set of all parallelograms, F_2 the set of all rectangles, F_3 the set of all rhombuses, F_4 the set of all squares and F_5 the set of trapeziums in a plane. Then F_1 may be equal to
(a) $F_2 \cap F_3$ (b) $F_3 \cap F_4$ (c) $F_2 \cup F_3$ (d) $F_2 \cup F_3 \cup F_4 \cup F_1$
[NCERT EXEMPLAR]

31. If $X = \{8^n - 7n - 1 : n \in N\}$ and $Y = \{49n - 49 : n \in N\}$. Then,
(a) $X \subset Y$ (b) $Y \subset X$ (c) $X = Y$ (d) $X \cap Y = \phi$
[NCERT EXEMPLAR]

32. A survey shows that 63% of the people watch a News channel whereas 76% watch another channel. If x % of the people watch both channel, then
(a) $x = 35$ (b) $x = 63$ (c) $39 \leq x \leq 63$ (d) $x = 39$
[NCERT EXEMPLAR]

33. If sets A and B are defined as $A = \left\{(x, y) : y = \dfrac{1}{x}, 0 \neq x \in R\right\}$, $B = \{(x, y) : y = -x, x \in R\}$, then
(a) $A \cap B = A$ (b) $A \cap B = B$ (c) $A \cap B = \phi$ (d) $A \cup B = A$
[NCERT EXEMPLAR]

34. Each set X_r contains 5 elements and each set Y_r contains 2 elements and $\bigcup_{r=1}^{20} X_r = S = \bigcup_{r=1}^{n} Y_r$. If each element of S belongs to exactly 10 of the $X_r^{'s}$ and to exactly 4 of the $Y_r^{'s}$, then n is
(a) 10 (b) 20 (c) 100 (d) 50
[NCERT EXEMPLAR]

35. Two finite sets have m and n elements respectively. The total number of subsets of first set is 56 more than the total number of subsets of the second set. The value of m and n respectively are:
(a) 7, 6 (b) 5, 1 (c) 6, 3 (d) 8, 7
[NCERT EXEMPLAR]

36. The set $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cup C'$ is equal to
(a) $B \cap C'$ (b) $A \cap C$ (c) $B \cup C'$ (d) $A \cap C'$
[NCERT EXEMPLAR]

37. If A and B are two sets, then $A \cap (A \cup B)$ equals
(a) A (b) B (c) ϕ (d) $A \cap B$
[NCERT EXEMPLAR]

38. Let $S = \{x : x$ is a positive multiple of 3 less than 100$\}$, $P = \{x : x$ is a prime less than 20$\}$. Then, $n(S) + n(P)$ is
(a) 34 (b) 41 (c) 33 (d) 30
[NCERT EXEMPLAR]

39. In a town of 840 persons, 450 persons read Hindi, 300 read English and 200 both. Then the number of persons who read neither is
(a) 210 (b) 290 (c) 180 (d) 260
[NCERT EXEMPLAR]

40. In a class of 60 students, 25 students play cricket and 20 students play tennis and 10 students play both the games. Then the number of students who play neither is
(a) 0 (b) 25 (c) 35 (d) 45
[NCERT EXEMPLAR]

41. Let S = the set of points inside the square, T = the set of points inside the triangle and C = the set of points inside the circle. If the triangle and circle intersect each other and are contained in a square. Then,
(a) $S \cap T \cap C = \phi$ (b) $S \cup T \cup C = C$ (c) $S \cup T \cup C = S$ (d) $S \cup T = S \cap C$
[NCERT EXEMPLAR]

ANSWERS

1. (b)	2. (c)	3. (d)	4. (a)	5. (d)	6. (b)	7. (b)	8. (c)	9. (c)
10. (a), (b), (c)		11. (b)	12. (c)	13. (c)	14. (c)	15. (c)	16. (a)	17. (a)
18. (d)	19. (d)	20. (b)	21. (c)	22. (b)	23. (d)	24. (c)	25. (c)	26. (c)
27. (b)	28. (c)	29. (b)	30. (d)	30. (d)	31. (a)	32. (c)	33. (c)	34. (b)
35. (c)	36. (a)	37. (a)	38. (b)	39. (b)	40. (b)	41. (c)		

SUMMARY

1. A set is a well defined collection of objects.
2. A set is described either in set builder form or tabular form.
3. A set consisting of no element is called the null set and is denoted by ϕ
4. A set consisting of a single element is called a singleton set.
5. A set consisting of a definite number of elements is called a finite set, otherwise the set is called an infinite set.
6. The number of elements in a finite set A is called its cardinal number or order and is denoted by $n(A)$.

7. Two sets A and B are equal if they have exactly the same elements.
8. A set A is said to be a subset of a set B, if every element of A is also an element of B.
9. If a, b are real numbers such that $a < b$, then the set
   (i) $\{x : x \in R$ and $a \le x \le b\}$ is called the closed interval $[a, b]$
   (ii) $\{x : x \in R$ and $a < x < b\}$ is called the open interval (a, b)
   (iii) $\{x : x \in R$ and $a \le x < b\}$ is called the semi-open or semi-closed interval $[a, b)$.
   (iv) $\{x : x \in R$ and $a < x \le b\}$ is called the semi-open or semi-closed interval $(a, b]$.
10. The total number of subsets of a finite set consisting of n elements is 2^n.
11. The collection of all subsets of a set A is called the power set of A and is denoted by $P(A)$.
12. The union of two sets A and B is the set of all those elements which are either in A or in B or in both and is denoted by $A \cup B$. Thus, $A \cup B = \{x : x \in A \text{ or } x \in B\}$
13. The intersection of two sets A and B is the set of all those elements which are common to both A and B and is denoted by $A \cap B$. Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
14. The difference $A - B$ of two sets A and B is the set of all those elements of A which do not belong to B i.e. $A - B = \{x : x \in A \text{ and } x \notin B\}$. Similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$.
15. The symmetric difference of two sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$.
16. The complement of a subset A of universal set U is the set of all those elements of U which are not 'he elements of A. The complement of A is denoted by A' or A^c.
17. For any three sets A, B and C, we have
   (i) $A \cup A = A$ and $A \cap A = A$ (Idempotent laws)
   (ii) $A \cup \phi = A$ and $A \cap U = A$ (Identity laws)
   (iii) $A \cup B = B \cup A$ and $A \cap B = B \cap A$ (Commutative laws)
   (iv) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative laws)
   (v) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive laws)
   (vi) $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$ (De' Morgan's laws)
18. If A, B and C are finite sets and U be the finite universal set, then
   (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
   (ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets
   (iii) $n(A - B) = n(A) - n(A \cap B)$ i.e., $n(A - B) + n(A \cap B) = n(A)$
   (iv) $n(A \Delta B) = n(A - B) + n(B - A) = n(A) + n(B) - 2n(A \cap B)$
   (v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
   (vi) Number of elements in exactly two of sets A, B and C
   $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3\,n(A \cap B \cap C)$
   (vii) Number of elements in exactly one of sets A, B and C
   $= n(A) + n(B) + n(C) - 2\,n(A \cap B) - 2\,n(B \cap C) - 2\,n(A \cap C) + 3\,n(A \cap B \cap C)$.

CHAPTER 2

RELATIONS

2.1 INTRODUCTION

In previous chapter, we have discussed various operations on sets to create more sets out of given sets. In this chapter, we shall study one more operation which is known as the cartesian product of sets. This will finally enable us to introduce the concept of relation.

2.2 ORDERED PAIRS

ORDERED PAIR *An ordered pair consists of two objects or elements in a given fixed order.*

For example, if A and B are any two sets, then by an ordered pair of elements we mean a pair (a, b) in that order, where $a \in A, b \in B$.

NOTE *An ordered pair is not a set consisting of two elements. The ordering of the two elements in an ordered pair is important and the two elements need not be distinct.*

ILLUSTRATION 1 *The position of a point in a two dimensional plane in cartesian coordinates is represented by an ordered pair. Accordingly, the ordered pairs* (1, 3), (2, 4), (2, 3) *and* (3, 2) *represent different points in a plane.*

EQUALITY OF ORDERED PAIRS *Two ordered pairs (a_1, b_1) and (a_2, b_2) are equal iff $a_1 = a_2$* and $b_1 = b_2$.

i.e. $\quad (a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$

It is evident from this definition that $(1, 2) \neq (2, 1)$ and $(1, 1) \neq (2, 2)$.

ILLUSTRATION 2 *Find the values of a and b, if $(3a-2, b+3) = (2a-1, 3)$.*

SOLUTION By the definition of equality of ordered pairs, we obtain

$$(3a-2, b+3) = (2a-1, 3) \Leftrightarrow 3a-2 = 2a-1 \text{ and } b+3 = 3 \Leftrightarrow a = 1 \text{ and } b = 0$$

2.3 CARTESIAN PRODUCT OF SETS

CARTESIAN PRODUCT OF SETS *Let A and B be any two non-empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of sets A and B and is denoted by $A \times B$.*

Thus, $\quad A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

If $\quad A = \phi$ or $B = \phi$, then we define $A \times B = \phi$.

ILLUSTRATION 1 *If $A = \{2, 4, 6\}$ and $B = \{1, 2\}$, then*

$$A \times B = \{2, 4, 6\} \times \{1, 2\} = \{(2, 1), (2, 2), (4, 1), (4, 2), (6, 1), (6, 2)\}$$

and, $$B \times A = \{1, 2\} \times \{2, 4, 6\} = \{(1, 2), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6)\}$$

It is evident from the above illustration that to write $A \times B$, we take an element from set A and form all ordered pairs with this element as first element and elements of B as second elements. Next we choose another element from A and corresponding to each element in B we form ordered pairs with this element as first element and elements of B as second elements. This process is continued till all elements of A are exhausted.

ILLUSTRATION 2 *If* $A=\{a, b\}$ *and* $B=\{1, 2, 3\}$, *find* $A\times B$, $B\times A$, $A\times A$, $B\times B$, *and* $(A\times B)\cap(B\times A)$.

SOLUTION We have, $A=\{a, b\}$ and $B=\{1, 2, 3\}$

$$\therefore \quad A\times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$
$$B\times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$
$$A\times A = \{(a, a), (a, b), (b, a), (b, b)\}$$
$$B\times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

Clearly, $(A\times B)\cap(B\times A)=\phi$.

CARTESIAN PRODUCT OF THREE SETS *Let A, B and C be three sets. Then, $A\times B\times C$ is the set of all ordered triplets having first element from A, second element from B and third element from C.*

i.e. $$A\times B\times C = \{(a, b, c) : a\in A, b\in B, c\in C\}$$

ILLUSTRATION 3 *If* $A=\{1, 2\}$, $B=\{3, 4\}$ *and* $C=\{4, 5, 6\}$. *Then,*

$$A\times B = \{1, 2\}\times\{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

and, $$A\times B\times C = \{(1, 3, 4), (1, 3, 5), (1, 3, 6), (1, 4, 4), (1, 4, 5), (1, 4, 6), (2, 3, 4), (2, 3, 5), (2, 3, 6), (2, 4, 4), (2, 4, 5), (2, 4, 6)\}$$

NOTE *It should be noted that* $A\times B\times C \neq (A\times B)\times C \neq A\times(B\times C)$.

If $A_1, A_2, A_3, \ldots, A_n$ are n sets, then the cartesian product $A_1\times A_2\times\ldots\times A_n$ of these n sets is the set of all n-tuples of the form $(a_1, a_2, a_3, \ldots, a_n)$, where $a_1\in A$, $a_2\in A_2, \ldots, a_n\in A_n$.

i.e. $$A_1\times A_2\times\ldots\times A_n = \{(a_1, a_2, a_3, \ldots, a_n) : a_1\in A_1, a_2\in A_2, a_3\in A_3, \ldots, a_n\in A_n\}$$

2.3.1 NUMBER OF ELEMENTS IN THE CARTESIAN PRODUCT OF TWO SETS

THEOREM *If A and B are two finite sets, then $n(A\times B) = n(A)\times n(B)$.*

PROOF Let $A=\{a_1, a_2, a_3, \ldots, a_m\}$ and $B=\{b_1, b_2, b_3, \ldots, b_n\}$ be two sets having m and n elements respectively. Then,

$$\begin{aligned} A\times B = \{&(a_1, b_1), (a_1, b_2), (a_1, b_3), \ldots, (a_1, b_n)\\ &(a_2, b_1), (a_2, b_2), (a_2, b_3)\ldots, (a_2, b_n)\\ &\quad\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad \vdots\\ &(a_m, b_1), (a_m, b_2), (a_m, b_3)\ldots, (a_m, b_n)\}\end{aligned}$$

Clearly, in the tabular representation of $A\times B$ there are m rows of ordered pairs and each row has n distinct ordered pairs. So, $A\times B$ has mn elements.

Hence, $n(A\times B) = mn = n(A)\times n(B)$ Q.E.D.

REMARK (i) *If either A or B is an infinite set, then $A\times B$ is an infinite set.*

(ii) *If A, B, C are finite sets, then* $n(A\times B\times C)=n(A)\times n(B)\times n(C)$

2.3.2 GRAPHICAL REPRESENTATION OF CARTESIAN PRODUCT OF SETS

Let A and B be any two non-empty sets. To represent $A\times B$ graphically, we draw two mutually perpendicular lines, one horizontal and other vertical. On the horizontal line, we represent the elements of set A and on the vertical line, the elements of B. If $a\in A$, $b\in B$, we draw a vertical line through a and a horizontal line through b. These two lines will meet in a point which will

denote the ordered pair (a, b). In this manner we mark points corresponding to each ordered pair in $A \times B$. The set of points so obtained represents $A \times B$ graphically as illustrated below.

ILLUSTRATION *If $A = \{1, 2, 3\}$ and $B = \{2, 4\}$, find $A \times B$ and show it graphically.*

SOLUTION Clearly, $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$.

In order to represent $A \times B$ graphically, we draw two perpendicular lines OX and OY as shown in Fig. 2.1. Now, we represent the set A by three points on OX and the set B by two points on OY. The set $A \times B$ is represented by the six points as shown in Fig. 2.1.

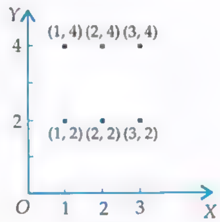


Fig. 2.1

2.3.3 DIAGRAMATIC REPRESENTATION OF CARTESIAN PRODUCT OF TWO SETS

In order to represent $A \times B$ by an arrow diagram, we first draw Venn diagrams representing sets A and B one opposite to the other as shown in Fig. 2.2. Now, we draw line segments starting from each element of A and terminating to each element of set B.

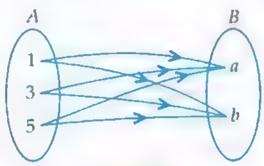


Fig. 2.2

If $A = \{1, 3, 5\}$ and $B = \{a, b\}$, then following figure gives the arrow diagram of $A \times B$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **ON EQUALITY OF ORDERED PAIRS**

EXAMPLE 1 *Find x and y, if $(x + 3, 5) = (6, 2x + y)$.*

SOLUTION By the definition of equality of ordered pairs

$$(x + 3, 5) = (6, 2x + y)$$

$\Rightarrow \quad x + 3 = 6$ and $5 = 2x + y$

$\Rightarrow \quad x = 3$ and $5 = 2x + y \Rightarrow x = 3, 5 = 6 + y \Rightarrow x = 3$ and $y = -1$

Type II **ON FINDING THE CARTESIAN PRODUCT OF TWO SETS**

EXAMPLE 2 *If $A = \{1, 3, 5, 6\}$ and $B = \{2, 4\}$, find $A \times B$ and $B \times A$.*

SOLUTION We have, $A = \{1, 3, 5, 6\}$ and $B = \{2, 4\}$. Therefore,

$A \times B = \{(1, 2), (1, 4), (3, 2), (3, 4), (5, 2), (5, 4), (6, 2), (6, 4)\}$

and, $B \times A = \{(2, 1), (2, 3), (2, 5), (2, 6), (4, 1), (4, 3), (4, 5), (4, 6)\}$

EXAMPLE 3 *If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{1, 3, 5\}$, find*

(i) $A \times (B \cup C)$ (ii) $A \times (B \cap C)$ (iii) $(A \times B) \cap (A \times C)$

SOLUTION (i) Clearly, $B \cup C = \{1, 3, 4, 5\}$

$\therefore \quad A \times (B \cup C) = \{1, 2, 3\} \times \{1, 3, 4, 5\}$

$= \{(1, 1), (1, 3), (1, 4), (1, 5), (2, 1), (2, 3), (2, 4), (2, 5), (3, 1), (3, 3), (3, 4), (3, 5)\}$

(ii) Clearly, $B \cap C = \{3\}$.

$\therefore \quad A \times (B \cap C) = \{1, 2, 3\} \times \{3\} = \{(1, 3), (2, 3), (3, 3)\}$

(iii) $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$,

and, $A \times C = \{(1, 1), (1, 3), (1, 5), (2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5)\}$

$\therefore$ $(A \times B) \cap (A \times C) = \{(1, 3), (2, 3), (3, 3)\}$.

EXAMPLE 4 *Let $A = \{1, 2, 3\}$ and $B = \{x : x \in N, x$ is prime less than $5\}$. Find $A \times B$ and $B \times A$.*

SOLUTION We have, $A = \{1, 2, 3\}$ and, $B = \{x : x \in N, x$ is prime less than $5\} = \{2, 3\}$

$\therefore$ $A \times B = \{1, 2, 3\} \times \{2, 3\} = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$

and, $B \times A = \{2, 3\} \times \{1, 2, 3\} = \{(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

EXAMPLE 5 *If $A \times B = \{(a, 1), (a, 5), (a, 2), (b, 2), (b, 5), (b, 1)\}$, find $B \times A$.*

SOLUTION Clearly, $B \times A$ can be obtained from $A \times B$ by interchanging the entries (or components) of ordered pair in $A \times B$.

$\therefore$ $B \times A = \{(1, a), (5, a), (2, a), (2, b), (5, b), (1, b)\}$

EXAMPLE 6 *If $A = \{1, 2\}$, form the set $A \times A \times A$.*

SOLUTION We have, $A = \{1, 2\}$.

$\therefore$ $A \times A = \{(1,1), (1, 2), (2, 1), (2, 2)\}$

and, $A \times A \times A = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

EXAMPLE 7 *If R is the set of all real numbers, what do the cartesian products $R \times R$ and $R \times R \times R$ represent?*

SOLUTION The cartesian product of the set R of all real numbers with itself i.e. $R \times R$ is the set of all ordered pairs (x, y) where $x, y \in R$. In other words, $R \times R = \{(x, y) : x, y \in R\}$.

Clearly, $R \times R$ is the set of all points in XY-plane. The set $R \times R$ is also denoted by R^2.

Similarly, we obtain: $R \times R \times R = \{(x, y, z) : x, y, z \in R\}$

Clearly, it represents the set of all points in space. The set $R \times R \times R$ is also denoted by R^3.

EXAMPLE 8 *Express $A = \{(a, b) : 2a + b = 5, a, b \in W\}$ as the set of ordered pairs.*

SOLUTION Here, W denotes the set of whole numbers (non-negative integers).

We have, $2a + b = 5$, where $a, b \in W$.

$\therefore$ $a = 0 \Rightarrow b = 5, a = 1 \Rightarrow b = 3$ and, $a = 2 \Rightarrow b = 1$

For $a > 3$, the values of b given by the above relation are not whole numbers.

$\therefore$ $A = \{(0, 5), (1, 3), (2, 1)\}$

Type III ON FINDING SETS A AND B WHEN $A \times B$ OR SOME ELEMENTS OF $A \times B$ ARE GIVEN

EXAMPLE 9 *If $A \times B = \{(a, 1), (b, 3), (a, 3), (b, 1), (a, 2), (b, 2)\}$, find A and B.*

SOLUTION Clearly, A is the set of all first components in ordered pairs in $A \times B$ and B is the set of all second components in ordered pairs in $A \times B$.

$\therefore$ $A = \{a, b\}$ and $B = \{1, 2, 3\}$

EXAMPLE 10 *Let A and B be two sets such that $A \times B$ consists of 6 elements. If three elements of $A \times B$ are : $(1, 4), (2, 6), (3, 6)$. Find $A \times B$ and $B \times A$.*

SOLUTION Since $(1, 4), (2, 6)$ and $(3, 6)$ are elements of $A \times B$. It follows that $1, 2, 3$ are elements of A and $4, 6$ are elements of B. It is given that $A \times B$ has 6 elements. So, $A = \{1, 2, 3\}$ and $B = \{4, 6\}$.

Hence, $A \times B = \{1, 2, 3\} \times \{4, 6\} = \{(1, 4), (1, 6), (2, 4), (2, 6), (3, 4), (3, 6)\}$

and, $B \times A = \{4, 6\} \times \{1, 2, 3\} = \{(4, 1), (4, 2), (4, 3), (6, 1), (6, 2), (6, 3)\}$

EXAMPLE 11 *The cartesian product $A \times A$ has 9 elements among which are found $(-1, 0)$ and $(0, 1)$. Find the set A and the remaining elements of $A \times A$.*

SOLUTION Given that $(-1, 0) \in A \times A$ and $(0, 1) \in A \times A$.

Now, $(-1, 0) \in A \times A \Rightarrow -1, 0 \in A$ and, $(0, 1) \in A \times A \Rightarrow 0, 1 \in A$

$\therefore \quad -1, 0, 1 \in A$

It is given that $A \times A$ has 9 elements. Therefore, A has exactly three elements.

Hence, $A = \{-1, 0, 1\}$ and remaining elements of $A \times A$ are $(-1, -1), (-1, 1), (0, -1), (0, 0), (1, -1), (1, 0), (1, 1)$.

EXAMPLE 12 *Let A and B be two sets such that $n(A) = 5$ and $n(B) = 2$. If a, b, c, d, e are distinct and $(a, 2), (b, 3), (c, 2), (d, 3), (e, 2)$ are in $A \times B$, find A and B.*

SOLUTION It is given that $(a, 2), (b, 3), (c, 2), (d, 3), (e, 2)$ are elements of $A \times B$. Therefore, $a, b, c, d, e \in A$ and $2, 3 \in B$.

It is also given that $n(A) = 5$ and $n(B) = 2$.

$\therefore \quad a, b, c, d, e \in A$ and $n(A) = 5 \Rightarrow A = \{a, b, c, d, e\}$

$2, 3 \in B$ and $n(B) = 2 \Rightarrow B = \{2, 3\}$

Type IV ON GRAPHICAL AND DIAGRAMATIC REPRESENTATION OF $A \times B$

EXAMPLE 13 *Let $A = \{-1, 3, 4\}$ and $B = \{2, 3\}$. Represent the following products graphically i.e. by lattices:* (i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$

SOLUTION (i) We have, $A = \{-1, 3, 4\}$ and $B = \{2, 3\}$.

$\therefore \; A \times B = \{(-1, 2), (-1, 3), (3, 2), (3, 3), (4, 2), (4, 3)\}$

In order to represent $A \times B$ graphically, we follow the following steps:

Step I Draw two mutually perpendicular lines one horizontal and other vertical.

Step II On the horizontal line represent the elements of set A and on the vertical line represent the elements of set B.

Step III Draw vertical dotted lines through points representing elements of A on horizontal line and horizontal lines through points representing elements of B on the vertical line. Points of intersection of these lines will represent $A \times B$ graphically as shown in Fig. 2.3.

Fig. 2.3 Graphical representation of $A \times B$

(ii) Clearly, $B \times A = \{2, 3\} \times \{-1, 3, 4\} = \{(2, -1), (2, 3), (2, 4), (3, -1), (3, 3), (3, 4)\}$

Here, we represent B on the horizontal line and A on vertical line. Graphical representation of $B \times A$ is as shown in Fig. 2.4.

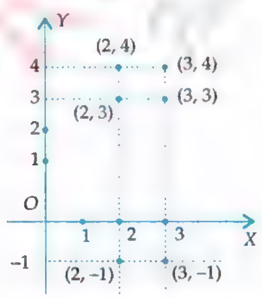


Fig. 2.4 Graphical representation of $B \times A$

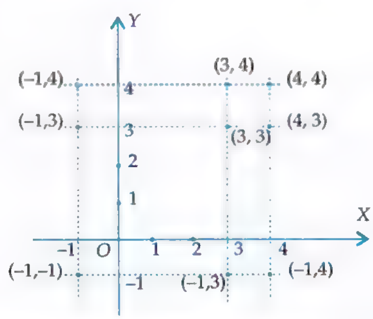


Fig. 2.5 Graphical representation of $A \times A$

(iii) We have, $A = \{1, 3, 4\}$

$\therefore \quad A \times A = \{-1, 3, 4\} \times \{-1, 3, 4\}$

$= \{(-1, -1), (-1, 3), (-1, 4), (3, -1), (3, 3), (3, 4), (4, -1), (4, 3), (4, 4)\}$

Graphical representation of $A \times A$ is shown in Fig. 2.5.

EXAMPLE 14 If $A = \{1, 3, 5\}$, $B = \{x, y\}$ represent the following products by arrow diagrams:

(i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

SOLUTION (i) We have, $A = \{1, 3, 5\}$ and $B = \{x, y\}$

$\therefore \quad A \times B = \{1, 3, 5\} \times \{x, y\} = \{(1, x), (1, y), (3, x), (3, y), (5, x), (5, y)\}$

Following arrow diagram represents $A \times B$. (see Fig. 2.6).

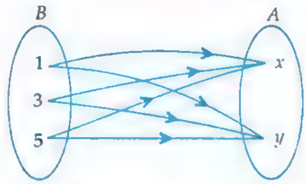


Fig. 2.6 Arrow diagram of $A \times B$

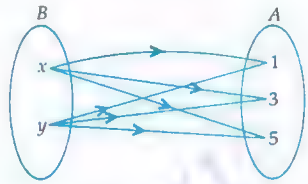


Fig. 2.7 Arrow diagram of $B \times A$

(ii) We have, $B = \{x, y\}$ and $A = \{1, 3, 5\}$.

$\therefore \quad B \times A = \{x, y\} \times \{1, 3, 5\} = \{(x, 1), (x, 3), (x, 5), (y, 1), (y, 3), (y, 5)\}$

It has been represented by the arrow diagram shown in Fig. 2.7.

(iii) We have, $A = [1, 3, 5\}$

$\therefore \quad A \times A = \{1, 3, 5\} \times \{1, 3, 5\} = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$

It has been represented by the arrow diagram shown in Fig. 2.8.

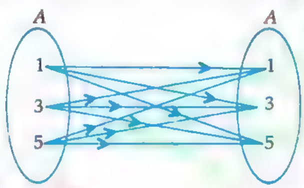


Fig. 2.8 Arrow diagram of $A \times A$

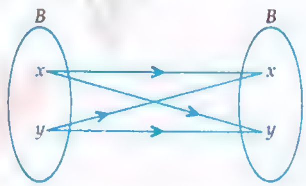


Fig. 2.9 Arrow diagram of $B \times B$

(iv) We have, $B = \{x, y\}$

$\therefore \quad B \times B = \{x, y\} \times \{x, y\} = \{(x, x), (x, y), (y, x), (y, y)\}$

It has been represented by the arrow diagram shown in Fig. 2.9.

EXERCISE 2.1

BASIC

1. (i) If $\left(\frac{a}{3} + 1, b - \frac{2}{3}\right) = \left(\frac{5}{3}, \frac{1}{3}\right)$, find the values of a and b.

   (ii) If $(x + 1, 1) = (3, y - 2)$, find the values of x and y.
2. If the ordered pairs $(x, -1)$ and $(5, y)$ belong to the set $\{(a, b) : b = 2a - 3\}$, find the values of x and y.
3. If $a \in \{-1, 2, 3, 4, 5\}$ and $b \in \{0, 3, 6\}$, write the set of all ordered pairs (a, b) such that $a + b = 5$.
4. If $a \in \{2, 4, 6, 9\}$ and $b \in \{4, 6, 18, 27\}$, then form the set of all ordered pairs (a, b) such that a divides b and $a < b$.
5. If $A = \{1, 2\}$ and $B = \{1, 3\}$, find $A \times B$ and $B \times A$.
6. Let $A = \{1, 2, 3\}$ and $B = \{3, 4\}$. Find $A \times B$ and show it graphically.
7. If $A = \{1, 2, 3\}$ and $B = \{2, 4\}$, what are $A \times B$, $B \times A$, $A \times A$, $B \times B$, and $(A \times B) \cap (B \times A)$?
8. If A and B are two sets having 3 elements in common. If $n(A) = 5$, $n(B) = 4$, find $n(A \times B)$ and $n[(A \times B) \cap (B \times A)]$.

9. Let A and B be two sets. Show that the sets $A \times B$ and $B \times A$ have an element in common *iff* the sets A and B have an element in common.
10. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$.

    If $(x, 1), (y, 2), (z, 1)$ are in $A \times B$, find A and B, where x, y, z are distinct elements.
11. Let $A = \{1, 2, 3, 4\}$ and $R = \{(a, b) : a \in A, b \in A, a \text{ divides } b\}$. Write R explicitly.
12. If $A = \{-1, 1\}$, find $A \times A \times A$.
13. State whether each of the following statements are true or false. If the statement is false, re-write the given statement correctly:
    (i) If $P = \{m, n\}$ and $Q = \{n, m\}$, then $P \times Q = \{(m, n), (n, m)\}$
    (ii) If A and B are non-empty sets, then $A \times B$ is a non-empty set of ordered pairs (x, y) such that $x \in B$ and $y \in A$.
    (iii) If $A = \{1, 2\}$, $B = \{3, 4\}$, then $A \times (B \cap \phi) = \phi$.
14. If $A = \{1, 2\}$, form the set $A \times A \times A$.
15. If $A = \{1, 2, 4\}$ and $B = \{1, 2, 3\}$, represent following sets graphically:
    (i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

ANSWERS

1. (i) $a = 2, b = 1$ (ii) $x = 2, y = 3$ 2. $x = 1, y = 7$ 3. $\{(-1, 6), (2, 3), (5, 0)\}$
4. $\{(2, 4), (2, 6), (2, 18), (6, 18), (9, 18), (9, 27)\}$
5. $A \times B = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$ and $B \times A = \{(1, 1), (1, 2), (3, 1), (3, 2)\}$
6. $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$.
7. $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$
   $B \times A = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2), (4, 3)\}$
   $A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
   $B \times B = \{(2, 2), (2, 4), (4, 2), (4, 4)\}$.
   $(A \times B) \cap (B \times A) = \{(2, 2)\}$
8. $n(A \times B) = 20$, $n[(A \times B) \cap (B \times A)] = 9$ 10. $A = \{x, y, z\}$, $B = \{1, 2\}$
11. $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
12. $A \times A \times A = \{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1), (1, 1, 1)\}$
13. (i) F (ii) F (iii) T
14. $A \times A \times A = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

HINTS TO SELECTED PROBLEMS

8. $n(A \times B) = n(A) \times n(B) = 5 \times 4 = 20$. From theorem 9 on page 2.10 *if* A and B have n elements in common, then $(A \times B)$ and $B \times A$ have n^2 elements in common. Therefore,

$$n[(A \times B) \cap (B \times A)] = 3^2 = 9.$$

*2.4 SOME USEFUL RESULTS

In this section, we intend to study some results on cartesian product of sets which are given as theorems.

---

* May be skipped. Not from examination point of view.

THEOREM 1 *For any three sets A, B, C, prove that:*

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

PROOF (i) Let (a, b) be an arbitrary element of $A \times (B \cup C)$. Then,

$(a, b) \in A \times (B \cup C)$

$\Rightarrow$ $a \in A$ and $b \in B \cup C$ [By definition]

$\Rightarrow$ $a \in A$ and $(b \in B$ or $b \in C)$ [By definition of union]

$\Rightarrow$ $(a \in A$ and $b \in B)$ or $(a \in A$ and $b \in C)$

$\Rightarrow$ $(a, b) \in A \times B$ or $(a, b) \in A \times C \Rightarrow (a, b) \in (A \times B) \cup (A \times C)$

$\therefore$ $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ ...(i)

Now, let (x, y) be an arbitrary element of $(A \times B) \cup (A \times C)$. Then,

$(x, y) \in (A \times B) \cup (A \times C)$

$\Rightarrow$ $(x, y) \in A \times B$ or, $(x, y) \in A \times C$

$\Rightarrow$ $(x \in A$ and $y \in B)$ or, $(x \in A$ and $y \in C)$

$\Rightarrow$ $x \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow$ $x \in A$ and $y \in (B \cup C) \Rightarrow (x, y) \in A \times (B \cup C)$

$\therefore$ $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ ...(ii)

Hence, from (i) and (ii), we obtain

$A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(ii) Let (a, b) be an arbitrary element of $A \times (B \cap C)$. Then,

$(a, b) \in A \times (B \cap C)$

$\Rightarrow$ $a \in A$ and $b \in (B \cap C)$ [By definition]

$\Rightarrow$ $a \in A$ and $(b \in B$ and $b \in C)$

$\Rightarrow$ $(a \in A$ and $b \in B)$ and $(a \in A$ and $b \in C)$

$\Rightarrow$ $(a, b) \in A \times B$ and $(a, b) \in A \times C$ [By definition]

$\Rightarrow$ $(a, b) \in (A \times B) \cap (A \times C)$

$\therefore$ $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ ...(i)

Now, let (x, y) be an arbitrary element of $(A \times B) \cap (A \times C)$. Then,

$(x, y) \in (A \times B) \cap (A \times C)$

$\Rightarrow$ $(x, y) \in (A \times B)$ and $(x, y) \in A \times C$

$\Rightarrow$ $(x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Rightarrow$ $x \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow$ $x \in A$ and $y \in (B \cap C) \Rightarrow (x, y) \in A \times (B \cap C)$

$\therefore$ $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ ...(ii)

Hence, from (i) and (ii), we obtain

$A \times (B \cap C) = (A \times B) \cap (A \times C)$ Q.E.D.

THEOREM 2 *For any three sets A, B, C, prove that:* $A \times (B - C) = (A \times B) - (A \times C)$.

PROOF Let (a, b) be an arbitrary element of $A \times (B - C)$. Then,

$(a, b) \in A \times (B - C)$

$\Rightarrow$ $a \in A$ and $b \in (B - C)$

$\Rightarrow$ $a \in A$ and $(b \in B$ and $b \notin C)$

$\Rightarrow$ $(a \in A$ and $b \in B)$ and $(a \in A$ and $b \notin C)$

$\Rightarrow$ $(a, b) \in (A \times B)$ and $(a, b) \notin (A \times C) \Rightarrow (a, b) \in (A \times B) - (A \times C)$

$\therefore$ $A \times (B - C) \subseteq (A \times B) - (A \times C)$ ...(i)

Now, let (x, y) be an arbitrary element of $(A \times B) - (A \times C)$. Then,

$(x, y) \in (A \times B) - (A \times C)$

$\Rightarrow$ $(x, y) \in A \times B$ and $(x, y) \notin A \times C$

$\Rightarrow$ $(x \in A$ and $y \in B)$ and $(x \in A$ and $y \notin C)$

$\Rightarrow \quad x \in A$ and $(y \in B$ and $y \notin C)$
$\Rightarrow \quad x \in A$ and $y \in (B - C) \Rightarrow (x, y) \in A \times (B - C)$
$\therefore \quad (A \times B) - (A \times C) \subseteq A \times (B - C)$ ...(ii)

Hence, from (i) and (ii), we get
$A \times (B - C) = (A \times B) - (A \times C)$ Q.E.D.

THEOREM 3 *If A and B are any two non-empty sets, then prove that:* $A \times B = B \times A \Leftrightarrow A = B$.

PROOF *First, let* $A = B$. *Then we have to prove that* $A \times B = B \times A$.

Now, $\quad A = B$
$\Rightarrow \quad A \times B = A \times A$ and $B \times A = A \times A$ $\quad [\because B = A]$
$\Rightarrow \quad A \times B = B \times A$

Conversely, let $A \times B = B \times A$. Then we have to prove that $A = B$.

Let x be an arbitrary element of A. Then,
$x \in A$
$\Rightarrow \quad (x, b) \in A \times B$ for all $b \in B$
$\Rightarrow \quad (x, b) \in B \times A$ $\quad [\because A \times B = B \times A]$
$\Rightarrow \quad x \in B$ $\quad$ [By definition]
$\therefore \quad A \subseteq B$

Now, let y be an arbitrary element of B. Then,
$y \in B$
$\Rightarrow \quad (a, y) \in A \times B$ for all $a \in A$
$\Rightarrow \quad (a, y) \in B \times A$ $\quad [\because A \times B = B \times A]$
$\Rightarrow \quad y \in A$ $\quad$ [By definition]
$\therefore \quad B \subseteq A$

Hence, A = B. Q.E.D.

THEOREM 4 *If* $A \subseteq B$, *show that* $A \times A \subseteq (A \times B) \cap (B \times A)$.

PROOF *Let* (a, b) *be an arbitrary element of* $A \times A$. *Then,*
$(a, b) \in A \times A$
$\Rightarrow \quad a \in A$ and $b \in A$
$\Rightarrow \quad (a \in A, b \in A)$ and $(a \in A, b \in A)$
$\Rightarrow \quad (a \in A, b \in B)$ and $(a \in B, b \in A)$ $\quad [\because A \subseteq B \therefore a, b \in A \Rightarrow a, b \in B]$
$\Rightarrow \quad (a, b) \in (A \times B)$ and $(a, b) \in (B \times A) \Rightarrow (a, b) \in (A \times B) \cap (B \times A)$
$\therefore \quad A \times A \subseteq (A \times B) \cap (B \times A)$

Hence, $A \subseteq B \Rightarrow A \times A \subseteq (A \times B) \cap (B \times A)$. Q.E.D.

THEOREM 5 *If* $A \subseteq B$, *prove that* $A \times C \subseteq B \times C$ *for any set C.*

PROOF Let (a, b) be an arbitrary element of $A \times C$. Then,
$(a, b) \in A \times C$
$\Rightarrow \quad a \in A$ and $b \in C$
$\Rightarrow \quad a \in B$ and $b \in C$ $\quad [\because A \subseteq B \therefore a \in A \Rightarrow a \in B]$
$\Rightarrow \quad (a, b) \in B \times C$

Thus, $(a, b) \in A \times C \Rightarrow (a, b) \in B \times C$ for all $(a, b) \in (A \times C)$.
$\therefore \quad A \times C \subseteq B \times C$. Q.E.D.

THEOREM 6 *If* $A \subseteq B$ *and* $C \subseteq D$, *prove that* $A \times C \subseteq B \times D$.

PROOF Let (a, b) be an arbitrary element of $A \times C$. Then,
$(a, b) \in A \times C$
$\Rightarrow \quad a \in A$ and $b \in C$
$\Rightarrow \quad a \in B$ and $b \in D$ $\quad [\because A \subseteq B$ and $C \subseteq D]$

$\Rightarrow \quad (a, b) \in B \times D$

Thus, $(a, b) \in A \times C \Rightarrow (a, b) \in B \times D$ for all $(a, b) \in (A \times C)$.

$\therefore \quad A \times C \subseteq B \times D$ Q.E.D.

THEOREM 7 *For any sets A, B, C, D prove that:* $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

PROOF Let (a, b) be an arbitrary element of $(A \times B) \cap (C \times D)$. Then,

$(a, b) \in (A \times B) \cap (C \times D)$

$\Rightarrow \quad (a, b) \in A \times B$ and $(a, b) \in C \times D$

$\Rightarrow \quad (a \in A$ and $b \in B)$ and $(a \in C$ and $b \in D)$

$\Rightarrow \quad (a \in A$ and $a \in C)$ and $(b \in B$ and $b \in D)$

$\Rightarrow \quad a \in (A \cap C)$ and $b \in B \cap D \Rightarrow (a, b) \in (A \cap C) \times (B \cap D)$

$\therefore \quad (A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$

Similarly, $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$

Hence, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ Q.E.D.

COROLLARY *For any sets A and B, prove that* $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$.

THEOREM 8 *For any three sets A, B, C prove that:*

(i) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$ (ii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$.

PROOF (i) We have,

$A \times (B' \cup C')' = A \times ((B')' \cap (C')')$ [By De-Morgan's law]

$= A \times (B \cap C) = (A \times B) \cap (A \times C)$ [See Theorem 1]

(ii) $A \times (B' \cap C')' = A \times ((B')' \cup (C')')$ [By De-Morgan's Law]

$= A \times (B \cup C) = (A \times B) \cup (A \times C)$ [See Theorem 1]

Q.E.D.

THEOREM 9 *Let A and B be two non-empty sets having n elements in common, then prove that* $A \times B$ *and* $B \times A$ *have* n^2 *elements in common.*

PROOF We have,

$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ [See Theorem 7]

$\Rightarrow \quad (A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$ [On replacing C by B and D by A]

$\Rightarrow \quad (A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$

It is given that $A \cap B$ has n elements, so $(A \cap B) \times (B \cap A)$ has n^2 elements.

But, $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$ [Proved above]

$\therefore \quad (A \times B) \cap (B \times A)$ has n^2 elements.

Hence, $A \times B$ and $B \times A$ have n^2 elements in common. Q.E.D.

THEOREM 10 *Let A be a non-empty set such that* $A \times B = A \times C$. *Show that* $B = C$.

PROOF Let b be an arbitrary element of B. Then,

$(a, b) \in A \times B$ for all $a \in A$

$\Rightarrow \quad (a, b) \in A \times C$ for all $a \in A$ [$\because A \times B = A \times C$]

$\Rightarrow \quad b \in C$

Thus, $b \in B \Rightarrow b \in C$

$\therefore \quad B \subset C$ ...(i)

Now, let c be an arbitrary element of C. Then,

$(a, c) \in A \times C$ for all $a \in A$

$\Rightarrow \quad (a, c) \in A \times B$ for all $a \in A$ [$\because A \times B = A \times C$]

$\Rightarrow \quad c \in B$

Thus, $\quad c \in C \Rightarrow c \in B$

$\therefore \quad C \subset B$ ...(ii)

From (i) and (ii), we get $B = C$.

EXERCISE 2.2

BASIC

1. Given $A = \{1, 2, 3\}, B = \{3, 4\}, C = \{4, 5, 6\}$, find $(A \times B) \cap (B \times C)$.
2. If $A = \{2, 3\}, B = \{4, 5\}, C = \{5, 6\}$, find $A \times (B \cup C), A \times (B \cap C), (A \times B) \cup (A \times C)$.
3. If $A = \{1, 2, 3\}, B = \{4\}, C = \{5\}$, then verify that:
   (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   (iii) $A \times (B - C) = (A \times B) - (A \times C)$.
4. Let $A = \{1, 2\}, B = \{1, 2, 3, 4\}, C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$. Verify that:
   (i) $A \times C \subset B \times D$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
5. If $A = \{1, 2, 3\}, B = \{3, 4\}$ and $C = \{4, 5, 6\}$, find
   (i) $A \times (B \cap C)$ (ii) $(A \times B) \cap (A \times C)$ (iii) $A \times (B \cup C)$ (iv) $(A \times B) \cup (A \times C)$

BASED ON HOTS

6. Prove that: (i) $(A \cup B) \times C = (A \times C) \cup (B \times C)$ (ii) $(A \cap B) \times C = (A \times C) \cap (B \times C)$
7. If $A \times B \subseteq C \times D$ and $A \times B \neq \phi$, prove that $A \subseteq C$ and $B \subseteq D$.

ANSWERS

1. $\{3, 4\}$.
2. $A \times (B \cup C) = \{(2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$
   $A \times (B \cap C) = \{(2, 5), (3, 5)\}$,
   $(A \times B) \cup (A \times C) = \{(2, 4), (2, 5), (3, 4), (3, 5), (2, 6), (3, 6)\}$.
4. $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
5. (i) $\{(1, 4), (2, 4), (3, 4)\}$ (ii) $\{(1, 4), (2, 4), (3, 4)\}$
   (iii) $\{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$
   (iv) $\{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$

2.5 RELATIONS

Let A and B denote the sets of all male and female members in the royal family of Dasrath's kingdom. Clearly, A = { Dasrath, Ram, Laxman, Shatrughan, Bharat } and B = { Kaushalya, Kaikai, Sumitra, Sita, Urmila, Shrutkirti, Mandvi }.

If we write R for the relation "was husband of" then the fact that Dasrath was husband of Kaushalya, Kaikai and Sumitra, Ram was husband of Sita, Laxman was husband of Urmila, Bharat was husband of Mandvi and Shatrughan was husband of Shrutkirti can be represented as:

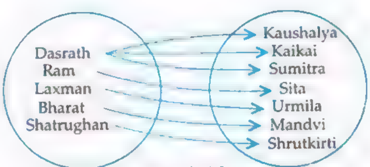


Fig. 2.10

Dasrath R Kaushlya, Dasrath R Kaikai, Dasrath R Sumitra, Ram R Sita, Laxman R Urmila, Bharat R Mandvi and Shatrughan R Shrutkirti.

Now, if we omit the letter R between the pairs of names and write them as ordered pairs, then the above fact can also be written as a set R of ordered pairs as given below:

R = { (Dasrath, Kaushalya), (Dasrath, Kaikai), (Dasrath, Sumitra), (Ram, Sita), (Laxman, Urmila), (Bharat, Mandvi), (Shatrughan, Shrutkirti) }.

Clearly, $R \subseteq A \times B$.

A visual representation of this relation R in the form of an arrow diagram is shown in Fig. 2.10 Thus, we see that the relation "was husband of" from set A to set B gives rise to a subset R of $A \times B$ such that $(x, y) \in R$ *iff* xRy.

Keeping this example in mind, we may define a relation as follows.

RELATION *Let A and B be two sets. Then a relation R from A to B is a subset of* $A \times B$.

Thus, R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

If R is a relation from a non-void set A to a non-void set B and *if* $(a, b) \in R$, then we write aRb which is read as `a is related to b by the relation R. *If* $(a, b) \notin R$, then we write aRb and we say that a is not related to b by the relation R.

ILLUSTRATION 1 *If* $A = \{1, 2, 3\}$ *and* $B = \{a, b, c\}$, *then* $R = \{(1, b), (2, c), (1, a), (3, a)\}$, *being a subset of* $A \times B$, *is a relation from A to B. Here,* $(1, b), (2, c), (1, a)$ *and* $(3, a) \in R$, *so we write* $1Rb$, $2Rc$, $1Ra$ *and* $3Ra$. *But,* $(2, b) \notin R$, *so we write* $2Rb$

ILLUSTRATION 2 *If* $A = \{a, b, c, d\}$, $B = \{p, q, r, s\}$, *then which of the following are relations from A to B? Give reasons for your answer.*

(i) $R_1 = \{(a, p), (b, r), (c, s)\}$ (ii) $R_2 = \{(q, b), (c, s), (d, r)\}$

(iii) $R_3 = \{(a, p), (a, q), (d, p), (c, r), (b, r)\}$ (iv) $R_4 = \{(a, p), (q, a), (b, s), (s, b)\}$.

SOLUTION (i) Clearly, $R_1 \subseteq A \times B$. So, R_1 is a relation from A to B.

(ii) Since $(q, b) \in R_2$ but $(q, b) \notin A \times B$. So, $R_2 \not\subseteq A \times B$. Thus, R_2 is not a relation from A to B.

(iii) Clearly, $R_3 \subseteq A \times B$. So it is a relation from A to B.

(iv) R_4 is not a relation from A to B, because (q, a) and (s, b) are elements of R_4 but (q, a) and (s, b) are not in $A \times B$. As such $R_4 \not\subseteq A \times B$.

TOTAL NUMBER OF RELATIONS Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then, $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn}. Since each subset of $A \times B$ defines a relation from A to B, so total number of relations from A to B is 2^{mn}. Among these 2^{mn} relations the void relation ϕ and the universal relation $A \times B$ are trivial relations from A to B.

2.5.1 REPRESENTATION OF A RELATION

A relation from a set A to a set B can be represented in any one of the following forms:

(i) ROSTER FORM In this form a relation is represented by the set of all ordered pairs belonging to R.

For example, if R is a relation from set $A = \{-2, -1, 0, 1, 2\}$ to set $B = \{0, 1, 4, 9, 10\}$ by the rule $a\,R\,b \Leftrightarrow a^2 = b$. Then, $0\,R\,0, -2\,R\,4, -1\,R\,1, 1\,R\,1$ and $2\,R\,4$.

So, R can be described in Roster form as $R = \{(0, 0), (-1, 1), (-2, 4), (1, 1), (2, 4)\}$

(ii) SET-BUILDER FORM In this form the relation R from set A to set B is represented as $R = \{(a, b) : a \in A, b \in B$ and a, b satisfy the rule which associates a and $b\}$.

For example, if $A = \{1, 2, 3, 4, 5\}$, $B = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\right\}$ and R is a relation from A to B given by $R = \left\{(1, 1), \left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right), \left(4, \frac{1}{4}\right), \left(5, \frac{1}{5}\right)\right\}$.

Then, R in set-builder form can be described as: $R = \left\{(a, b) : a \in A, b \in B \text{ and } b = \frac{1}{a}\right\}$.

It should be noted that it is not possible to express every relation from set A to set B in set-builder form. For example, the relation $R = \{(1, a), (1, c), (3, b)\}$ from set $A = \{1, 2, 3, 4\}$ to set $B = \{a, b, c\}$ cannot be described in set-builder form.

(iii) BY ARROW DIAGRAM In order to represent a relation from set A to a set B by an arrow diagram, we draw arrows from first components to the second components of all ordered pairs belonging to R.

For example, relation $R = \{(1, 2), (2, 4), (3, 2), (1, 3), (3, 4)\}$ from set $A = \{1, 2, 3, 4, 5\}$ to set $B = \{2, 3, 4, 5, 6, 7\}$ can be represented by the arrow diagram shown in Fig. 2.11.

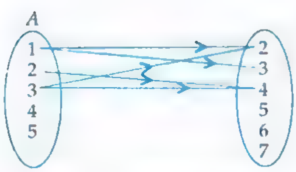


Fig. 2.11 Arrow diagram

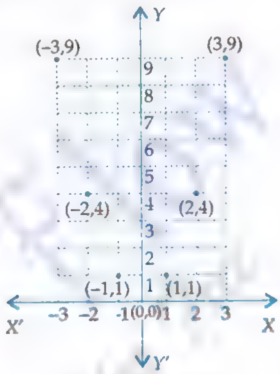


Fig. 2.12 Lattice

(iv) BY LATTICE In this form, the relation R from set A to set B is represented by darking the dots in the lattice for $A \times B$ which represent the ordered pairs in R.

For example, if $R = \{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$ is a relation from set $A = \{-3, -2, -1, 0, 1, 2, 3\}$ to set $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then R can be represented by the lattice shown in Fig. 2.12.

2.5.2 DOMAIN AND RANGE OF A RELATION

Let R be a relation from a set A to a set B. Then, the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom $(R) = \{a : (a, b) \therefore \in R\}$ and Range $(R) = \{b : (a, b) \in R\}$.

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B. The set B is called the co-domain of relation R.

ILLUSTRATION 1 *If $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8, 10\}$ and let $R = \{(1, 8), (3, 6), (5, 2), (1, 4)\}$ be a relation from A to B. Then,*

Domain $(R) = \{1, 3, 5\}$ and Range $(R) = \{8, 6, 2, 4\}$

ILLUSTRATION 2 *Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$ be two sets and let R be a relation from A to B defined by the phrase "$(x, y) \in R \Leftrightarrow x > y$". Under this relation R, we obtain $3R2, 5R2, 5R4, 7R2, 7R4$ and $7R6$*

i.e. $R = \{(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)\}$.

$\therefore$ Domain $(R) = \{3, 5, 7\}$ and Range $(R) = \{2, 4, 6\}$

ILLUSTRATION 3 *If R is a relation from set $A = \{2, 4, 5\}$ to set $B = \{1, 2, 3, 4, 6, 8\}$ defined by $xRy \Leftrightarrow x$ divides y.*

(i) Write R as a set of ordered pairs, *(ii) Find the domain and the range of R.*

SOLUTION (i) Clearly, $2R2, 2R4, 2R6, 2R8, 4R4$, and $4R8$.

$\therefore$ $R = \{(2, 2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8)\}$

(ii) Clearly, Domain $(R) = \{2, 4\}$ and Range $(R) = \{2, 4, 6, 8\}$

RELATION ON A SET *Let A be a non-void set. Then, a relation from A to itself i.e. a subset of $A \times A$, is called a relation on set A.*

2.5.3 INVERSE OF A RELATION

INVERSE RELATION *Let A, B be two sets and let R be a relation from a set A to a set B. Then, the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.*

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, Dom $(R) =$ Range (R^{-1}) and, Range $(R) =$ Dom (R^{-1}).

ILLUSTRATION 1 *Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ be two sets and let $R = \{(1, a), (1, c), (2, d), (2, c)\}$ be a relation from A to B. Then, $R^{-1} = \{(a, 1), (c, 1), (d, 2), (c, 2)\}$ is a relation from B to A.*

Also, Dom $(R) = \{1, 2\} =$ Range (R^{-1}), and Range $(R) = \{a, c, d\} =$ Dom (R^{-1}).

ILLUSTRATION 2 *Let A be the set of first ten natural numbers and let R be a relation on A defined by $(x, y) \in R \Leftrightarrow x + 2y = 10$ i.e. $R = \{(x, y) : x \in A, y \in A$ and $x + 2y = 10\}$. Express R and R^{-1} as sets of ordered pairs. Also, determine (i) domains of R and R^{-1} (ii) ranges of R and R^{-1}.*

SOLUTION We have,

$$(x, y) \in R \Leftrightarrow x + 2y = 10 \Leftrightarrow y = \frac{10 - x}{2}, x, y \in A \text{ where } A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Now, $x = 1 \Rightarrow y = \frac{10 - 1}{2} = \frac{9}{2} \notin A.$

This shows that 1 is not related to any element in A. Similarly, we can observe that 3, 5, 7, 9 and 10 are not related to any element of A under the defined relation.

Further we find that:

For $x = 2$, $y = \frac{10 - 2}{2} = 4 \in A$. Therefore, $(2, 4) \in R$

For $x = 4$, $y = \frac{10 - 4}{2} = 3 \in A$. Therefore, $(4, 3) \in R$

For $x = 6$, $y = \frac{10 - 6}{2} = 2 \in A$. Therefore, $(6, 2) \in R$

For $x = 8$, $y = \frac{10 - 8}{2} = 1 \in A$. Therefore, $(8, 1) \in R$

Thus, $R = \{(2, 4), (4, 3), (6, 2), (8, 1)\} \Rightarrow R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, Dom$(R) = \{2, 4, 6, 8\} =$ Range (R^{-1}) and, Range $(R) = \{4, 3, 2, 1\} =$ Dom (R^{-1}).

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **ON EXAMINING WHETHER A SET OF ORDERED PAIRS REPRESENTS A RELATION OR NOT**

EXAMPLE 1 *If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, which of the following are relations from A to B? Give reasons in support of your answer:*

(i) $R_1 = \{(1, 4), (1, 5), (1, 6)\}$ (ii) $R_2 = \{(1, 5), (2, 4), (3, 6)\}$

(iii) $R_3 = \{(1, 4), (1, 5), (3, 6), (2, 6), (3, 4)\}$ (iv) $R_4 = \{(4, 2), (2, 6), (5, 1), (2, 4)\}$.

SOLUTION (i) Clearly, $R_1 \subseteq A \times B$. So, it is a relation from A to B.

(ii) Clearly, $R_2 \subseteq A \times B$. So, it is a relation from A to B.

(iii) Clearly, $R_3 \subseteq A \times B$. So, it is a relation from A to B.

(iv) Since $(4, 2) \in R_4$ but $(4, 2) \notin A \times B$. So, R_4 is not a relation from A to B.

Type II **ON DESCRIBING A RELATION AND ITS INVERSE AS A SET OF ORDERED PAIRS AND FINDING THEIR DOMAINS AND RANGES**

EXAMPLE 2 *A relation R is defined from a set $A = \{2, 3, 4, 5\}$ to a set $B = \{3, 6, 7, 10\}$ as follows: $(x, y) \in R \Leftrightarrow x$ divides y. Express R as a set of ordered pairs and determine the domain and range of R. Also, find R^{-1}.*

SOLUTION Recall that $a \mid b$ stands for 'a divides b'. For the elements of the given sets A and B, we find that $2 \mid 6$, $2 \mid 10$, $3 \mid 3$, $3 \mid 6$, and $5 \mid 10$.

$\therefore \quad (2, 6) \in R, (2, 10) \in R, (3, 3) \in R, (3, 6) \in R$ and $(5, 10) \in R$.

Thus, $\quad R = \{(2, 6), (2, 10), (3, 3), (3, 6), (5, 10)\}$.

Clearly, Dom $(R) = \{2, 3, 5\}$ and, Range $(R) = \{3, 6, 10\}$.

Also, $\quad R^{-1} = \{(6, 2), (10, 2), (3, 3), (6, 3), (10, 5)\}$.

EXAMPLE 3 *If R is the relation "less than" from $A = \{1, 2, 3, 4, 5\}$ to $B = \{1, 4, 5\}$, write down the set of ordered pairs corresponding to R. Find the inverse of R.*

SOLUTION It is given that $(x, y) \in R \Leftrightarrow x < y$, where $x \in A$ and $y \in B$.

For the elements of the given sets A and B, we find that

$$1 < 4,\ 1 < 5,\ 2 < 4,\ 2 < 5,\ 3 < 4,\ 3 < 5 \text{ and } 4 < 5$$

$\therefore \quad (1, 4) \in R, (1, 5) \in R, (2, 4) \in R, (2, 5) \in R, (3, 4) \in R, (3, 5) \in R$ and $(4, 5) \in R$.

Thus, $\quad R = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$.

$\therefore \quad R^{-1} = \{(4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4)\} = \{(x, y) : x \in B, y \in A \text{ and } x > y\}$.

EXAMPLE 4 *A relation R is defined on the set Z of integers as: $(x, y) \in R \Leftrightarrow x^2 + y^2 = 25$.*

Express R and R^{-1} as the sets of ordered pairs and hence find their respective domains.

SOLUTION We have,

$$(x, y) \in R \Leftrightarrow x^2 + y^2 = 25 \Leftrightarrow y = \pm\sqrt{25 - x^2}$$

We observe that from the above relation $x = 0$ gives $y = \pm 5$.

$\therefore \quad (0, 5) \in R$ and $(0, -5) \in R$

Similarly, $x = \pm 3 \Rightarrow y = \sqrt{25 - 9} = \pm 4$

$\therefore \quad (3, 4) \in R, (-3, 4) \in R, (3, -4) \in R$ and $(-3, -4) \in R$

$\quad x = \pm 4 \Rightarrow y = \sqrt{25 - 16} = \pm 3$

$\therefore \quad (4, 3) \in R, (-4, 3) \in R, (4, -3) \in R$ and $(-4, -3) \in R$

$\quad x = \pm 5 \Rightarrow y = \sqrt{25 - 25} = 0$

$\therefore \quad (5, 0) \in R$ and $(-5, 0) \in R$

We also notice that for any other integral value of x, the value of y given by $y = \pm\sqrt{25 - x^2}$ is not an integer.

$\therefore \quad R = \{(0, 5), (0, -5), (3, 4), (-3, 4), (3, -4), (-3, -4), (4, 3), (-4, 3), (4, -3), (-4, -3), (5, 0), (-5, 0)\}$

$\Rightarrow R^{-1} = \{(5, 0), (-5, 0), (4, 3), (4, -3), (-4, 3), (-4, -3), (3, 4), (3, -4), (-3, 4), (-3, -4), (0, 5), (0, -5)\}$

Clearly, Domain $(R) = \{0, 3, -3, 4, -4, 5, -5\}$ = Domain (R^{-1}).

EXAMPLE 5 *Let R be the relation on the set N of natural numbers defined by*

$$R = \{(a, b) : a + 3b = 12, a \in N, b \in N\}.$$

Find: (i) R (ii) Domain of R (iii) Range of R

SOLUTION (i) We have, $a + 3b = 12 \Rightarrow a = 12 - 3b$

Putting $b = 1, 2, 3$ respectively in the above relation, we get $a = 9, 6, 3$ respectively.
For $b = 4$, $a = 12 - 3b$ gives $a = 0$ which does not belong to N. Also, values of a given by $a = 12 - 3b$ do not belong to N for all $b > 4$.

$\therefore \quad R = \{(9, 1), (6, 2), (3, 3)\}$

(ii) Domain of $R = \{9, 6, 3\}$

(iii) Range of $R = \{1, 2, 3\}$

Type III ON REPRESENTING A RELATION BY USING AN ARROW DIAGRAM

EXAMPLE 6 *Let $A = \{1, 2, 3, 4, 5, 6\}$. Define a relation R on set A by $R = \{(x, y) : y = x + 1\}$*

(i) *Depict this relation using an arrow diagram* (ii) *Write down the domain, co-domain and range of R.*

SOLUTION (i) Putting $x = 1, 2, 3, 4, 5, 6$ respectively in $y = x + 1$, we get $y = 2, 3, 4, 5, 6, 7$ respectively.

$\therefore \quad (1, 2) \in R, (2, 3) \in R, (3, 4) \in R, (4, 5) \in R, (5, 6) \in R$ and $(6, 7) \notin R$.

For $x = 6$, we get $y = 7$ which does not belong to set A.

Hence, $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

The arrow diagram representing R is as follows.

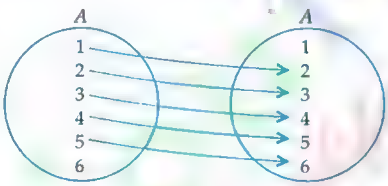


Fig. 2.13

(ii) Clearly, Domain $(R) = \{1, 2, 3, 4, 5\}$, Range $(R) = \{2, 3, 4, 5, 6\}$.

EXAMPLE 7 *Figure 2.14 shows a relation R between the sets P and Q. Write this relation R in (i) Roster form (ii) Set builder form. What is its domain and range?*

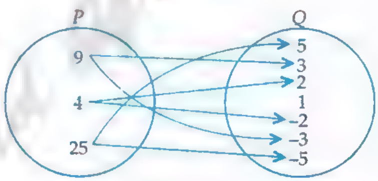


Fig. 2.14

SOLUTION (i) It is evident from the figure that

$R = \{(9, 3), (9, -3), (4, 2), (4 - 2), (25, 5), (25 - 5)\}$

(ii) It is evident from R, that it consists of elements (x, y), where x is the square of y i.e. $x = y^2$.

Therefore, relation R in set builder form is $R = \{(x, y) : x = y^2, x \in P, y \in Q\}$.

The domain and range of R are $\{9, 4, 25\}$ and $\{-5, -3, -2, 2, 3, 5\}$ respectively.

REMARK *In the above example, the range of relation R is not same as the set Q. The set Q is known as the co-domain.*

Type IV ON PROVING RESULTS BASED ON THE DEFINITION OF A RELATION

EXAMPLE 8 *Let R be a relation on Q defined by $R = \{(a, b) : a, b \in Q$ and $a - b \in Z\}$.*

Show that:

(i) $(a, a) \in R$ *for all* $a \in Q$ (ii) $(a, b) \in R \Rightarrow (b, a) \in R$

(iii) $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$. [NCERT]

SOLUTION (i) For any $a \in Q$, we have

$a - a = 0 \in Z$

$\Rightarrow \quad (a, a) \in R$

Hence, $(a, a) \in R$ for all $a \in Q$.

(ii) Let $(a, b) \in R$. Then,

$(a, b) \in R$

$\Rightarrow \quad a - b \in Z$, where $a, b \in Q$

$\Rightarrow \quad b - a \in Z \qquad [\because b - a = -(a - b)]$

$\Rightarrow \quad (b, a) \in R$

(iii) Let $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow \quad a - b \in Z$ and $b - c \in Z \Rightarrow (a - b) + (b - c) \in Z \Rightarrow a - c \in Z \Rightarrow (a, c) \in R$

EXAMPLE 9 *Let R be a relation on N defined by* $R = \{(a, b) : a, b \in N \text{ and } a = b^2\}$.

Are the following true :

(i) $(a, a) \in R$ *for all* $a \in N$ (ii) $(a, b) \in R \Rightarrow (b, a) \in R$

(iii) $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$ [NCERT]

Justify your answer in each case.

SOLUTION (i) We observe that $a = a^2$ is true for $a = 1 \in N$ only. Therefore, $(1, 1) \in R$. But, $(2, 2), (3, 3), (4, 4)$ etc do not belong to R. So, $(a, a) \in R$ for all $a \in N$ is not true.

(ii) We observe that $(4, 2) \in R$, because $4 = 2^2$. But, $(2, 4) \notin R$ as $2 \neq 4^2$.

So, $(a, b) \in R \Rightarrow (b, a) \in R$ is not true for all $a, b \in N$.

(iii) We observe that $(16, 4) \in R$ and $(4, 2) \in R$. However, $(16, 2) \notin R$.

So, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ is not true for all $a, b, c \in N$.

EXAMPLE 10 *Let a relation* R_1 *on the set R of all real numbers be defined as* $(a,b) \in R_1 \Leftrightarrow 1 + ab > 0$ *for all* $a, b \in R$. *Show that:* (i) $(a, a) \in R_1$ *for all* $a \in R$ (ii) $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$ *for all* $a, b \in R$

SOLUTION (i) For any $a \in R$, we find that $1 + a^2 > 0 \Rightarrow (a, a) \in R_1$. Thus, $(a, a) \in R_1$ for all $a \in R$.

(ii) Let $(a, b) \in R_1$. Then,

$(a, b) \in R_1 \Rightarrow 1 + ab > 0 \Rightarrow 1 + ba > 0 \Rightarrow (b, a) \in R_1$

Thus, $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$ for all $a, b \in R$.

EXAMPLE 11 *Let R be the relation on the set Z of all integers defined by* $(x, y) \in R \Rightarrow x - y$ *is divisible by n. Prove that:*

(i) $(x, x) \in R$ *for all* $x \in Z$ (ii) $(x, y) \in R \Rightarrow (y, x) \in R$ *for all* $x, y \in Z$

(iii) $(x, y) \in R$ *and* $(y, z) \in R \Rightarrow (x, z) \in R$ *for all* $x, y, z \in R$.

SOLUTION (i) For any $x \in Z$, we have

$x - x = 0 = 0 \times n \Rightarrow x - x$ is divisible by $n \Rightarrow (x, x) \in R$

Thus, $(x, x) \in R$ for all $x \in Z$.

(ii) Let $(x, y) \in R$. Then,

$(x, y) \in R$

$\Rightarrow \quad x - y$ is divisible by n

$\Rightarrow \quad x - y = \lambda n$ for same $\lambda \in Z$

$\Rightarrow \quad y - x = (-\lambda) n$

$\Rightarrow \quad y - x$ is divisible by n $\quad [\because \lambda \in Z \Rightarrow -\lambda \in Z]$

$\Rightarrow \quad (y, x) \in R$

Thus, $(x, y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in Z$.

(iii) Let $(x, y) \in R$ and $(y, z) \in R$. Then,

$(x, y) \in R \Rightarrow x - y$ is divisible by $n \Rightarrow x - y = \lambda n$ for some $\lambda \in Z$

$(y, z) \in R \Rightarrow y - z$ is divisible by $n \Rightarrow y - z = \mu n$ for some $\mu \in Z$

$\therefore \quad (x, y) \in R$ and $(y, z) \in R$

$\Rightarrow \quad x - y = \lambda n$ and $y - z = \mu n$

$\Rightarrow \quad (x - y) + (y - z) = \lambda n + \mu n$

$\Rightarrow \quad x - z = (\lambda + \mu) n$

$\Rightarrow \quad x - z$ is divisible by n $\quad [\because \lambda + \mu \in Z]$

$\Rightarrow \quad (x, z) \in R$

Thus, $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$.

EXERCISE 2.3

BASIC

1. If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, which of the following are relations from A to B? Give reasons in support of your answer.
   (i) $\{(1, 6), (3, 4), (5, 2)\}$ (ii) $\{(1, 5), (2, 6), (3, 4), (3, 6)\}$
   (iii) $\{(4, 2), (4, 3), (5, 1)\}$ (iv) $A \times B$.
2. A relation R is defined from a set $A = \{2, 3, 4, 5\}$ to a set $B = \{3, 6, 7, 10\}$ as follows:
   $(x, y) \in R \Leftrightarrow x$ is relatively prime to y
   Express R as a set of ordered pairs and determine its domain and range.
3. Let A be the set of first five natural numbers and let R be a relation on A defined as follows:
   $(x, y) \in R \Leftrightarrow x \le y$
   Express R and R^{-1} as sets of ordered pairs. Determine also
   (i) the domain of R^{-1} (ii) the range of R.
4. Find the inverse relation R^{-1} in each of the following cases:
   (i) $R = \{(1, 2), (1, 3), (2, 3), (3, 2), (5, 6)\}$
   (ii) $R = \{(x, y) : x, y \in N, x + 2y = 8\}$
   (iii) R is a relation from $\{11, 12, 13\}$ to $\{(8, 10, 12\}$ defined by $y = x - 3$.
5. Write the following relations as the sets of ordered pairs:
   (i) A relation R from the set $\{2, 3, 4, 5, 6\}$ to the set $\{1, 2, 3\}$ defined by $x = 2y$.
   (ii) A relation R on the set $\{1, 2, 3, 4, 5, 6, 7\}$ defined by $(x, y) \in R \Leftrightarrow x$ is relatively prime to y.
   (iii) A relation R on the set $\{0, 1, 2, ..., 10\}$ defined by $2x + 3y = 12$.
   (iv) A relation R from a set $A = \{5, 6, 7, 8\}$ to the set $B = \{10, 12, 15, 16, 18\}$ defined by $(x, y) \in R \Leftrightarrow x$ divides y.
6. Let R be a relation in N defined by $(x, y) \in R \Leftrightarrow x + 2y = 8$. Express R and R^{-1} as sets of ordered pairs.
7. Let $A = \{3, 5\}$ and $B = \{7, 11\}$. Let $R = \{(a, b) : a \in A, b \in B, a - b$ is odd $\}$. Show that R is an empty relation from A into B.
8. Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Find the total number of relations from A into B.
9. Determine the domain and range of the relation R defined by
   (i) $R = \{(x, x + 5) : x \in \{0, 1, 2, 3, 4, 5\}\}$ [NCERT]
   (ii) $R = \{(x, x^3) : x$ is a prime number less than 10 $\}$ [NCERT]

10. Determine the domain and range of the following relations:
   (i) $R = \{(a, b) : a \in N, a < 5, b = 4\}$
   (ii) $S = \{(a, b) : b = |a - 1|, a \in Z \text{ and } |a| \le 3\}$
11. Let $A = \{a, b\}$. List all relations on A and find their number.
12. Let $A = \{x, y, z\}$ and $B = \{a, b\}$. Find the total number of relations from A into B. [NCERT]
13. Let R be a relation from N to N defined by $R = \{(a, b) : a, b \in N \text{ and } a = b^2\}$.
   Are the following statements true?
   (i) $(a, a) \in R$ for all $a \in N$ (ii) $(a, b) \in R \Rightarrow (b, a) \in R$
   (iii) $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$
14. Let $A = \{1, 2, 3, \ldots, 14\}$. Define a relation on a set A by $R = \{(x, y) : 3x - y = 0, x, y \in A\}$. Depict this relationship using an arrow diagram. Write down its domain, co-domain and range.
15. Define a relation R on the set N of natural numbers by
   $R = \{(x, y) : y = x + 5, x \text{ is a natural number less than } 4, x, y \in N\}$.
   Depict this relationship using (i) roster form (ii) an arrow diagram. Write down the domain and range or R. [NCERT]
16. $A = \{1, 2, 3, 5\}$ and $B = \{4, 6, 9\}$. Define a relation R from A to B by
   $R = \{(x, y) :$ the difference between x and y is odd, $x \in A, y \in B\}$. Write R in Roster form. [NCERT]
17. Write the relation $R = \{(x, x^3) : x \text{ is a prime number less than } 10\}$ in roster form. [NCERT]
18. Let $A = \{1, 2, 3, 4, 5, 6\}$. Let R be a relation on A defined by
   $R = \{(a, b) : a, b \in A, b \text{ is exactly divisible by } a\}$
   (i) Write R in roster form (ii) Find the domain of R (iii) Find the range of R. [NCERT]
19. Figure 2.15 shows a relationship between the sets P and Q. Write this relation in
   (i) set builder form (ii) roster form. What is its domain and range? [NCERT]

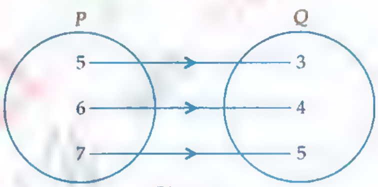


Fig. 2.15

20. Let R be the relation on Z defined by $R = \{(a, b) : a, b \in Z, a - b \text{ is an integer}\}$. Find the domain and range of R. [NCERT]
21. For the relation R_1 defined on R by the rule $(a, b) \in R_1 \Leftrightarrow 1 + ab > 0$.
   Prove that: $(a, b) \in R_1$ and $(b, c) \in R_1 \Rightarrow (a, c) \in R_1$ is not true for all $a, b, c \in R$.
22. Let R be a relation on $N \times N$ defined by $(a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c$ for all (a, b), $(c, d) \in N \times N$. Show that:
   (i) $(a, b)\ R\ (a, b)$ for all $(a, b) \in N \times N$
   (ii) $(a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in N \times N$
   (iii) $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$ for all $(a, b), (c, d), (e, f) \in N \times N$

ANSWERS

1. (i) It is not a relation from A to B.
   (ii) It is a subset of $A \times B$, so it is a relation from A to B.
   (iii) It is not a relation from A to B as it is not a subset of $A \times B$.
   (iv) It is a relation from A to B.

2. $R = \{(2, 3), (2, 7), (3, 7), (3, 10), (4, 3), (4, 7), (5, 3), (5, 6), (5, 7)\}$
3. $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 4), (4, 5), (5, 5)\}$

   $R^{-1} = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (2, 2), (3, 2), (4, 2), (5, 2), (3, 3), (4, 3), (5, 3), (4, 4), (5, 4), (5, 5)\}$

   Domain of $R^{-1} = \{1, 2, 3, 4, 5\}$ = Range of R.
4. (i) $R^{-1} = \{(2, 1), (3, 1), (3, 2), (2, 3), (6, 5)\}$ (ii) $R^{-1} = \{(3, 2), (2, 4), (1, 6)\}$
   (iii) $R^{-1} = \{(8, 11), (10, 13)\}$
5. (i) $\{(2, 1), (4, 2), (6, 3)\}$
   (ii) $\{(2, 3), (2, 5), (2, 7), (3, 2), (3, 4), (3, 5), (3, 7), (4, 3), (4, 5), (4, 7), (5, 2), (5, 3), (5, 4), (5, 6), (5, 7), (6, 5), (6, 7), (7, 2), (7, 3), (7, 4), (7, 5), (7, 6)\}$
   (iii) $\{(0, 4), (3, 2), (6, 0)\}$ (iv) $\{(5, 10), (5, 15), (6, 12), (6, 18), (8, 16)\}$
6. $R = \{(2, 3), (4, 2), (6, 1)\}$ $R^{-1} = \{(3, 2), (2, 4), (1, 6)\}$ **8.** 16
9. (i) Domain $R = \{0, 1, 2, 3, 4, 5\}$, Range $R = \{5, 6, 7, 8, 9, 10\}$
   (ii) Domain $R = \{2, 3, 5, 7\}$, Range $R = \{8, 27, 125, 343\}$
10. (i) Domain $R = \{1, 2, 3, 4\}$, Range $R = \{4\}$
    Domain $S = \{0, -1, -2, -3, 1, 2, 3\}$, Range $S = \{0, 1, 2, 3, 4\}$
    (ii) $S = \{(0, 1)), (-1, 2), (-2, 3), (-3, 4), (1, 0), (2, 1), (3, 2)\}$
11. 16 **12.** 64 **13.** (i) No (ii) No (iii) No
14. Domain (R) = $\{1, 2, 3, 4\}$, Co-domain (R) = A, Range (R) = $\{3, 6, 9, 12\}$

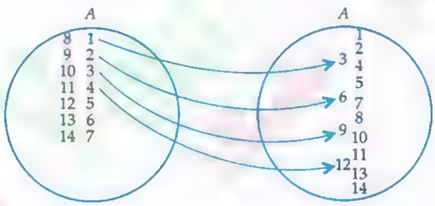


Fig. 2.16

15. (i) $R = \{(1, 6), (2, 7), (3, 8)\}$ (ii) Domain (R) = [1, 2, 3], Range (R) = { 6, 7, 8 }

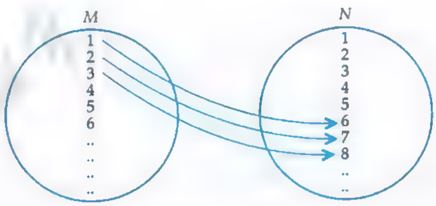


Fig. 2.17

16. $R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$
17. $R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$
18. (i) $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5.5), (6, 6)\}$
    (ii) Domain $(R) = \{1, 2, 3, 4, 5, 6\}$ (iii) Range $(R) = \{1, 2, 3, 4, 5, 6\}$
19. (i) $R = \{(x, y) : y = x - 2, x \in P, y \in Q\}$ (ii) $R = \{(5, 3), (6, 4), (7, 5)\}$
    Domain $(R) = \{5, 6, 7\}$, Range $(R) = 3, 4, 5$
20. (i) Domain $(R) = Z$, Range $(R) = Z$

HINTS TO SELECTED PROBLEMS

8. We have, $n(A) = 2, n(B) = 2$. Therefore, $n(A \times B) = 2 \times 2 = 4$. So, there are $2^4 = 16$ relations from A to B.

9. (i) $R = \{(x, x+5): x \in \{0, 1, 2, 3, 4, 5\}\} = \{(0,5), (1, 6), (2,7), (3, 8), (4, 9), (5, 10)\}$

$\therefore$ Domain $(R) = \{0, 1, 2, 3, 4, 5\}$ and, Range $(R) = \{5, 6, 7, 8, 9, 10\}$

(ii) $R = \{(x, x^3): x$ is a prime number less than $10\} = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

$\therefore$ Domain $(R) = \{2, 3, 5, 7\}$, and Range $(R) = \{8, 27, 125, 343\}$

10. (i) $R = \{(1, 4), (2, 4), (3, 4), (4, 4)\}$. Therefore, Domain $(R) = \{1, 2, 3, 4\}$, Range $(R) = \{4\}$

(ii) $S = \{(-3, 4), (-2, 3), (-1, 2), (0, 1), (1, 0), (2, 1), (3, 2)\}$

$\therefore$ Domain $(S) = \{-3, -2, -1, 0, 1, 2, 3\}$, and Range $(S) = \{0, 1, 2, 3, 4\}$

12. Here A has 3 elements and B has 2 elements. Therefore, total number of relations from A to B is $2^{3 \times 2} = 64$.

13. (i) No, because $(2, 2) \notin R$. (ii) No, because $(4, 2) \in R$ but $(2, 4) \notin R$.

(iii) No, because $(16, 4) \in R$ and $(4, 2) \in R$ but $(16, 2) \notin R$.

14. $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$. Domain $(R) = \{1, 2, 3, 4\}$, and Range $(R) = \{3, 6, 9, 12\}$.

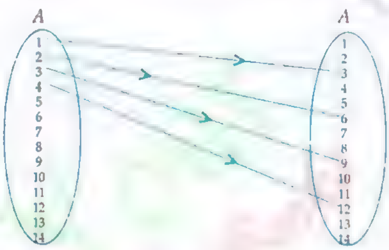


Fig. 2.18

15. (i) $R = \{(x, y): y = x + 5, x$ is a natural number less than $4, x, y \in N\} = \{(1, 6), (2, 7), (3, 8)\}$

(ii) Domain $(R) = \{1, 2, 3\}$, and Range $(R) = \{6, 7, 8\}$

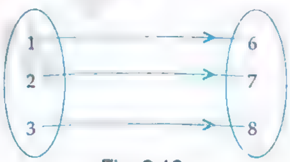


Fig. 2.19

16. We have, $A = \{1, 2, 3, 5\}, B = \{4, 6, 9\}$

$R = \{(x, y)$: the difference between x and y is odd, $x \in A, y \in B\}$

$\Rightarrow R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

17. We have, $R = \{(x, x^3): x$ is a prime number less than $10\}$

$\therefore R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

18. (i) We have, $R = \{(a, b): a, b \in A, b$ is exactly divisible by $a\}$, where $A = \{1, 2, 3, 4, 5, 6\}$.

$\therefore R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

(i) Domain $R = \{1, 2, 3, 4, 5, 6\}$ (ii) Range $R = \{1, 2, 3, 4, 5, 6\}$

19. (i) $\{(x, y): y = x - 2, x \in \{5, 6, 7\}, y \in \{3, 4, 5\}\}$ (ii) $\{(5, 3), (6, 4), (7, 5)\}$

Domain $R = \{5, 6, 7\}$, and Range $R = \{3, 4, 5\}$

20. The relation R on Z is defined by $R = \{(a, b) : a, b \in Z, a - b \text{ is an integer}\}$
Since $a - b$ is an integer for all $a, b \in Z$. So, domain $(R) = Z =$ Range (R).

21. We find that: $\left(1, -\frac{1}{2}\right) \in R_1$ and $\left(-\frac{1}{2}, -4\right) \in R_1$ as $1 + \left(\frac{-1}{2}\right) > 0$ and $1 + \left(\frac{-1}{2}\right)(-4) > 0$.
But, $1 + 1 \times -4 \not> 0$. So, $(1, -4) \notin R_1$.

22. (i) We know that $a + b = b + a$ for all $a, b \in N$. Therefore, $(a, b)\ R\ (a, b)$ for all $a, b \in N$
(ii) $(a, b)\ R\ (c, d) \Rightarrow a + d = b + c \Rightarrow c + b = d + a \Rightarrow (c, d)\ R\ (a, b)$
(iii) $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$
$\Rightarrow a + d = b + c$ and $c + f = d + e$
$\Rightarrow a + d + c + f = b + c + d + e \Rightarrow a + f = b + e \Rightarrow (a, b)\ R\ (e, f)$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. Let $n(A) = m$ and $n(B) = n$. Then, the total number of non-empty relations that can be defined from A to B is ....
2. The smallest reflexive relation on a set A is the ....
3. If A and B are two sets such that $n(A) = 5$ and $n(B) = 7$, then the total number of relations on $A \times B$ is ....
4. A relation R on a set A is a symmetric relation iff.............................. .
5. If R and S are two equivalence relations on a set A, then $R \cap S$ is ....
6. If $(1, 3), (2, 5)$ and $(3, 3)$ are three elements of $A \times B$ and $n(A \times B) = 6$, then the remaining three elements of $A \times B$ are ....
7. The total number of reflexive relations on a finite set having n elements is ....
8. If $R = \{(x, y) : x, y \in W, x^2 + y^2 = 25\}$, then Domain $(R) =$ .... and Range $(R) =$.......... .
9. The relation $R = \{(x, y) : x, y \in Z, x^2 + y^2 = 64\}$ is ....
10. If $A = \{x : x \in W, x < 2\}$, $B = \{x : x \in N, 1 < x < 5\}$ and $C = \{3, 5\}$, then $A \times (B \cap C) =$.................... .
11. If $R = \{(x, y) :$ where $x \in R$ and $-5 \le x \le 5\}$ is a relation, then range (R) = ...
12. If $n(A \cap B') = 9$, $n(A' \cap B) = 10$ and $n(A \cup B) = 24$, then $n(A \times B) =$ ....
13. If $A = \{3, 5, 6, 9\}$ and R is a relation in A defined as $R = \{(x, y) : x + y < 18\}$, then R in roster form is ....
14. If $n(A \times B) = 200$ and $n(A) = 50$, then the number of elements in $P(B)$ is ....
15. If $A = \{1, 2, 3, 4, 4, 5, 6\}$, then the number of subsets of A containing elements 2, 3 and 5 is ....

ANSWERS

1. $2^{mn} - 1$
2. Identity relation
3. 2^{35}
4. $R = R^{-1}$
5. an equivalence relation
6. $(1, 5), (2, 3), (3, 5)$
7. $2^{n^2 - n}$
8. $\{0, 3, 4, 5\}, \{0, 3, 4, 5\}$
9. symmetric
10. $\{(0, 3), (1, 3)\}$
11. $[-3, 17]$
12. 210

13. R = {(3, 3), (3, 5), (3, 6), (3, 9), (5, 3), (5, 6), (5, 9(, (6, 3), (6, 5), (6, 6), (6, 9), (9, 3), (9, 5), (9,6)}

14. 2^4 15. 8

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $A = \{1, 2, 4\}$, $B = \{2, 4, 5\}$ and $C = \{2, 5\}$, write $(A - C) \times (B - C)$.
2. If $n(A) = 3$, $n(B) = 4$, then write $n(A \times A \times B)$.
3. If R is a relation defined on the set Z of integers by the rule $(x, y) \in R \Leftrightarrow x^2 + y^2 = 9$, then write domain of R.
4. If $R = \{(x, y) : x, y \in Z, x^2 + y^2 \le 4\}$ is a relation defined on the set Z of integers, then write domain of R.
5. If R is a relation from set $A = \{11, 12, 13\}$ to set $B = \{8, 10, 12\}$ defined by $y = x - 3$, then write R^{-1}.
6. Let $A = \{1, 2, 3\}$ and $R = \{(a, b) : |a^2 - b^2| \le 5, a, b \in A\}$. Then write R as set of ordered pairs.
7. Let $R = \{(x, y) : x, y \in Z, y = 2x - 4\}$. If $(a, -2)$ and $(4, b^2) \in R$, then write the values of a and b.
8. If $R = \{(2, 1), (4, 7), (1, -2), \ldots\}$, then write the linear relation between the components of the ordered pairs of the relation R.
9. If $A = \{1, 3, 5\}$ and $B = \{2, 4\}$, list the elements of R, if $R = \{(x, y) : x, y \in A \times B \text{ and } x > y\}$.
10. If $R = \{(x, y) : x, y \in W, 2x + y = 8\}$, then write the domain and range of R.
11. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1), (y, 2), (z, 1)$ are in $A \times B$, write A and B.
12. Let $A = \{1, 2, 3, 5\}$, $B = \{4, 6, 9\}$ and R be a relation from A to B defined by $R = \{(x, y) : x - y \text{ is odd }\}$. Write R in roster form.

ANSWERS

1. $\{(1, 4), (4, 4)\}$ 2. 36 3. Domain $(R) = \{-3, 0, 3\}$
4. Domain $(R) = \{-2, -1, 0, 1, 2\}$ 5. $\{(8, 11), (10, 13)\}$
6. $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$ 7. $a = 1, b = \pm 2$
8. $y = 3x - 5$ 9. $\{(3, 2), (5, 2), (5, 4)\}$
10. Domain $(R) = \{0, 1, 2, 3, 4\}$, Range $(R) = \{0, 2, 4, 6, 8\}$
11. $A = \{x, y, z\}$, $B = \{1, 2\}$ 12. $R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $A = \{1, 2, 4\}$, $B = \{2, 4, 5\}$, $C = \{2, 5\}$, then $(A - B) \times (B - C)$ is
   (a) $\{(1, 2), (1, 5), (2, 5)\}$ (b) $\{(1, 4)\}$
   (c) $(1, 4)$ (d) none of these.

2. If R is a relation on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ given by $x\ R\ y \Leftrightarrow y = 3x$, then $R=$
   (a) $\{(3, 1), (6, 2), (8, 2), (9, 3)\}$ (b) $\{(3, 1), (6, 2), (9, 3)\}$
   (c) $\{(3, 1), (2, 6), (3, 9)\}$ (d) none of these.
3. Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$. If relation R from A to B is given by $R = \{(1, 3), (2, 5), (3, 3)\}$. Then, R^{-1} is
   (a) $\{(3, 3), (3, 1), (5, 2)\}$ (b) $\{(1, 3), (2, 5), (3, 3)\}$
   (c) $\{(1, 3), (5, 2)\}$ (d) none of these.
4. If $A = \{1, 2, 3\}$, $B = \{1, 4, 6, 9\}$ and R is a relation from A to B defined by `x is greater than y. The range of R is
   (a) $\{1, 4, 6, 9\}$ (b) $\{4, 6, 9\}$ (c) $\{1\}$ (d) none of these.
5. If $R = \{(x, y) : x, y \in Z, x^2 + y^2 \le 4\}$ is a relation on Z, then domain of R is
   (a) $\{0, 1, 2\}$ (b) $\{0, -1, -2\}$ (c) $\{-2, -1, 0, 1, 2\}$ (d) none of these.
6. A relation R is defined from $\{2, 3, 4, 5\}$ to $\{3, 6, 7, 10\}$ by : $x\ R\ y \Leftrightarrow x$ is relatively prime to y. Then, domain of R is
   (a) $\{2, 3, 5\}$ (b) $\{3, 5\}$ (c) $\{2, 3, 4\}$ (d) $\{2, 3, 4, 5\}$.
7. A relation ϕ from C to R is defined by $x\ \phi\ y \Leftrightarrow |x| = y$. Which one is correct?
   (a) $(2 + 3i)\ \phi\ 13$ (b) $3\ \phi(-3)$ (c) $(1 + i)\ \phi\ 2$ (d) $i\ \phi\ 1$.
8. Let R be a relation on N defined by $x + 2y = 8$. The domain of R is
   (a) $\{2, 4, 8\}$ (b) $\{2, 4, 6, 8\}$ (c) $\{2, 4, 6\}$ (d) $\{1, 2, 3, 4\}$.
9. R is a relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by $y = x - 3$. Then, R^{-1} is
   (a) $\{(8, 11), (10, 13)\}$ (b) $\{(11, 8), (13, 10)\}$
   (c) $\{(10, 13), (8, 11), (12, 10)\}$ (d) none of these.
10. If the set A has p elements, B has q elements, then the number of elements in $A \times B$ is
   (a) $p + q$ (b) $p + q + 1$ (c) pq (d) p^2
11. Let R be a relation from a set A to a set B, then
   (a) $R = A \cup B$ (b) $R = A \cap B$ (c) $R \subseteq A \times B$ (d) $R \subseteq B \times A$.
12. If R is a relation from a finite set A having m elements to a finite set B having n elements, then the number of relations from A to B is
   (a) 2^{mn} (b) $2^{mn} - 1$ (c) $2mn$ (d) m^n
13. If R is a relation on a finite set having n elements, then the number of relations on A is
   (a) 2^n (b) 2^{n^2} (c) n^2 (d) n^n.
14. Let $n(A) = m$ and $n(B) = n$. Then the total number of non-empty relations that can be defined from A to B is
   (a) m^n (b) $n^m - 1$ (c) $mn - 1$ (d) $2^{mn} - 1$

[NCERT EXEMPLAR]

ANSWERS

1. (b) 2. (d) 3. (a) 4. (c) 5. (c) 6. (d) 7. (d) 8. (c) 9. (a)
10. (c) 11. (c) 12. (a) 13. (b) 14. (d)

ACTIVITY

OBJECTIVE To explain the meaning of Cartesian product of two sets.

MATERIALS REQUIRED Cardboard, chart paper, pencil, scale, nails etc.

STEPS OF CONSTRUCTION

Step I Take a cardboard and a chart paper. Fix the chart paper on the cardboard.

Step II Draw two mutually perpendicular lines on the chart paper, one horizontal and one vertical.

Step III Take two sets A and B. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3, b_4\}$.

Step IV Mark points A_1, A_2, A_3, A_4, A_5 on horizontal line to represent points a_1, a_2, a_3, a_4 and a_5 respectively. Similarly, mark points B_1, B_2, B_3 and B_4 on the vertical line to represent b_1, b_2, b_3 and b_4 respectively.

Step V Draw vertical lines through A_1, A_2, A_3, A_4, A_5 and horizontal lines through B_1, B_2, B_3 and B_4. At the points of intersection of the horizontal and vertical lines fix nails. Points represented by these nails represent $A \times B$.

STEPS OF DEMONSTRATION

Step I Let the points of intersection of horizontal and vertical lines be $C_{11}, C_{12}, C_{13}, C_{14}, C_{15}, C_{21}, C_{22}, C_{23}, C_{24}, C_{25}, C_{31}, C_{32}, C_{33}, C_{34}, C_{35}, C_{41}, C_{42}, C_{43}, C_{44}, C_{45}$.

Step II Clearly, point C_{ij} in step II represents the order pair $(a_i, b_j); i = 1, 2, 3, 4, 5 = 1, 2, 3, 4$.

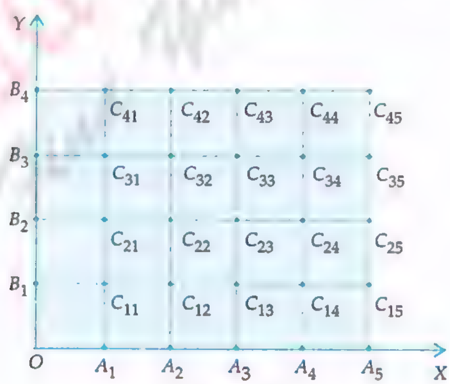


Fig. 2.20

SUMMARY

1. An ordered pair consists of two objects or elements in a given fixed order.
2. $(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$
3. If A and B are two non-empty sets, then $A \times B = \{(a, b) : a \in A, b \in B\}$ is called the cartesian product of A and B. If A and B are finite sets having m and n elements respectively, then $A \times B$ has mn elements.

4. $R \times R = \{(x, y) : x, y \in R\}$ is the set of all points in xy-plane.
5. $R \times R \times R = \{(x, y, z) : x, y, z \in R\}$ set of all points in three dimensional space.
6. For any three sets A, B, C, we have
   (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   (iii) $A \times (B - C) = A \times B - A \times C$ (iv) $A \times B = B \times A \Leftrightarrow A = B$
   (v) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$ (vi) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
   (vii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$ (viii) $A \times B = A \times C \Rightarrow B = C$
7. Let A and B be two sets. A relation from A to B is a subset of $A \times B$.
8. If A and B are finite sets having m and n elements respectively. Then, 2^{mn} relations can be defined from A to B.
9. If R is a relation from set A to set B, then
   Domain $(R) = \{x : (x, y) \in R\}$, Range $(R) = \{y : (x, y) \in R\}$
10. A relation from a set A to itself is called a relation on A.
11. Let A, B be two sets and let R be a relation from set A to set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.

    Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

    Domain (R) = Range (R^{-1}), and Range (R) = Domain (R^{-1}).

CHAPTER 3

FUNCTIONS

3.1 INTRODUCTION

In this chapter, we shall study about one of the most important concepts in mathematics known as a function. Functions form one of the most important building blocks of Mathematics. The word "Function" is derived from a Latin word meaning operation and the words mapping and map are synonimus to it. Functions play a very important role in differential and integral calculus. In this chapter, we shall introduce the concept of a function as a correspondence between two sets. We shall also study function as a relation from one set to the other set.

3.2 FUNCTION AS A SPECIAL KIND OF RELATION

DEFINITION *Let A and B be two non-empty sets. A relation f from A to B, i.e., a sub-set of $A \times B$, is called a function (or a mapping or a map) from A to B, if*

(i) *for each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$*

(ii) *$(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$.*

Thus, a non-void subset f of $A \times B$ is a function from A to B if each element of A appears in some ordered pair in f and no two ordered pairs in f have the same first element.

If $(a, b) \in f$, then b is called the image of a under f.

ILLUSTRATION 1 *Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and f_1, f_2 and f_3 be three subsets of $A \times B$ as given below:*

$$f_1 = \{(1, 2), (2, 3), (3, 4)\},\ f_2 = \{(1, 2), (1, 3), (2, 3), (3, 4)\},\ f_3 = \{(1, 3), (2, 4)\}.$$

Then, f_1 is a function from A to B but f_2 and f_3 are not functions from A to B. f_2 is not a function from A to B, because $1 \in A$ has two images 2 and 3 in B and f_3 is not a function from A to B because $3 \in A$ has no image in B.

If a function f is expressed as the set of ordered pairs, the domain of f is the set of all first components of members of f and the range of f is the set of second components of members of f i.e. Domain of $f = \{a : (a, b) \in f\}$, and Range of $f = \{b : (a, b) \in f\}$

ILLUSTRATION 2 If $x, y \in \{1, 2, 3, 4\}$, then which of the following are functions in the given set ?

(a) $f_1 = \{(x, y) : y = x + 1\}$ (b) $f_2 = \{(x, y) : x + y > 4\}$

(c) $f_3 = \{(x, y) : y < x\}$ (d) $f_4 = \{(x, y) : x + y = 5\}$

Also, in case of a function give its range.

SOLUTION If we express f_1, f_2, f_3 and f_4 as sets of ordered pairs, then we have

$$f_1 = \{(1, 2), (2, 3), (3, 4)\},$$
$$f_2 = \{(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)\},$$
$$f_3 = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\} \text{ and } f_4 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}.$$

(a) We have, $f_1 = \{(1, 2), (2, 3), (3, 4)\}$. We observe that an element 4 of the given set has not appeared in first place of any ordered pair of f_1. So, f_1 is not a function from the given set to itself.

(b) We have, $f_2 = \{(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)\}$. We observe that 2, 3, 4

have appeared more than once as first components of the ordered pairs in f_2. So, f_2 is not a function.

(c) We have, $f_3 = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$. We observe that 3 and 4 have appeared more than once as first components of the ordered pairs in f_3. So, f_3 is not a function.

(d) We have, $f_4 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$. We observe that each element of the given set has appeared as first components in one and only one ordered pair of f_4. So, f_4 is a function in the given set. In this case, Range of $f = \{1, 2, 3, 4\}$.

ILLUSTRATION 3 *Let f be a relation on the set N of natural numbers defined by $f = \{(n, 3n) : n \in N\}$. Is f a function from N to N. If so, find the range of f.*

SOLUTION We find that for each $n \in N$, there exists a unique $3n \in N$ such that $(n, 3n) \in f$. Therefore, f is a function from N to N. Clearly, Range of $f = \{f(n) : n \in N\} = \{3n : n \in N\}$.

ILLUSTRATION 4 *Let f be a subset of $Z \times Z$ defined by $f = \{(ab, (a + b)) : a, b \in Z\}$. Is f a function from Z into Z. Justify your answers.* [NCERT]

SOLUTION We observe that: $1 \times 6 = 6$ and $2 \times 3 = 6$

$$(1 \times 6, 1 + 6) \in f \text{ and } (2 \times 3, 2 + 3) \in f \Rightarrow (6, 7) \in f \text{ and } (6, 5) \in f$$

Thus, $(6, 7) \in f$ and $(6, 5) \in f$ but $5 \neq 7$. Hence, f is not a function from Z to Z.

3.3 FUNCTION AS A CORRESPONDENCE

DEFINITION *Let A and B be two non-empty sets. Then a function 'f' from set A to set B is a rule or method or correspondence which associates elements of set A to elements of set B such that:*

(i) *all elements of set A are associated to elements in set B.*

(ii) *an element of set A is associated to a unique element in set B.*

In other words, a function 'f' from a set A to a set B associates each element of set A to a unique element of set B.

Terms such as "map" (or "mapping"), "correspondence" are used as synonyms for "function". If f is a function from a set A to a set B, then we write $f : A \to B$ or $A \to B$, which is read as *f is a function from A to B* or *f maps A to B*.

If an element $a \in A$ is associated to an element $b \in B$, then b is called *'the f-image of a'* or *'image of a under f'* or *'the value of the function f at a'*. Also, a is called the pre-image of b under the function f. We write it as: $b = f(a)$

ILLUSTRATION *Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$ be two sets and let f_1, f_2, f_3 and f_4 be rules associating elements of set A to elements of set B as shown in figures 3.1 – 3.4.*

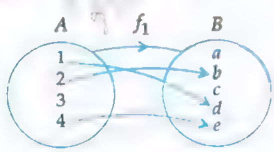


Fig. 3.1

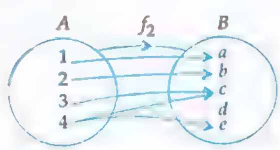


Fig. 3.2

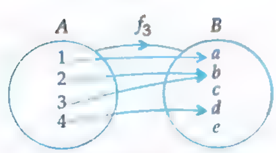


Fig. 3.3

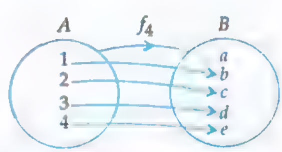


Fig. 3.4

We observe that f_1 is not a function from set A to set B, because there is an element $3 \in A$ which is not associated to any element of B.

Also, f_2 is not a function from A to B because an element $4 \in A$ is associated to two elements c and e in B. But, f_3 and f_4 are functions from A to B, because under f_3 and f_4 each element in A is associated to a unique element in B.

3.3.1 DESCRIPTION OF A FUNCTION

Let $f : A \to B$ be a function such that the set A consists of a finite number of elements. Then, $f(x)$ can be described by listing the values which it attains at different points of its domain. For example, if $A = \{-1, 1, 2, 3\}$ and B is the set of real numbers, then a function $f : A \to B$ can be described as $f(-1) = 3, f(1) = 0, f(2) = 3/2$ and $f(3) = 0$. In case, A is an infinite set, then f cannot be described by listing the images at points in its domain. In such cases functions are generally described by a formula. For example, $f : Z \to Z$ given by $f(x) = x^2 + 1$ or $f : R \to R$ given by $f(x) = e^x$ etc.

3.3.2 DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION

Let $f : A \to B$. Then, the set A is known as the domain of f and the set B is known as the co-domain of f. The set of all f-images of elements of A is known as the range of f or image set of A under f and is denoted by $f(A)$.

Thus, $f(A) = \{f(x) : x \in A\} =$ Range of f. Clearly, $f(A) \subseteq B$.

ILLUSTRATION 1 *Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 2, 3, 4, 5, 6\}$. Consider a rule $f(x) = x^2$. Under this rule, we obtain $f(-2) = (-2)^2 = 4$, $f(-1) = (-1)^2 = 1$, $f(0) = 0^2 = 0$, $f(1) = 1^2 = 1$ and $f(2) = 2^2 = 4$. We observe that each element of A is associated to a unique element of B. So, $f : A \to B$ given by $f(x) = x^2$ is a function. Clearly, domain $(f) = A = \{-2, -1, 0, 1, 2\}$ and range $(f) = \{0, 1, 4\}$.*

ILLUSTRATION 2 *Consider a rule $f(x) = 2x - 3$ associating elements of N (set of natural numbers) to elements of N. This rule does not define a function from N to itself, because $f(1) = 2 \times 1 - 3 = -1 \notin N$ i.e. $1 \in N$ (domain) is not associated to any element of N (co-domain). Thus, every rule relating elements of one set to elements of another set need not be a function.*

ILLUSTRATION 3 *Let $A = \{-2, -1, 0, 1, 2\}$ and $f : A \to Z$ be given by $f(x) = x^2 - 2x - 3$. Find:*
(i) the range of f (ii) pre-images of 6, −3 and 5.

SOLUTION (i) We have, $f(x) = x^2 - 2x - 3$.

$\therefore$ $f(-2) = (-2)^2 - 2(-2) - 3 = 5$, $f(-1) = (-1)^2 - 2(-1) - 3 = 0$, $f(0) = -3$,

$f(1) = 1^2 - 2 \times 1 - 3 = -4$ and $f(2) = 2^2 - 2 \times 2 - 3 = -3$.

So, range $(f) = \{f(-2), f(-1), f(0), f(1), f(2)\} = \{0, 5, -3, -4\}$

(ii) Let x be a pre-image of 6. Then,

$$f(x) = 6 \Rightarrow x^2 - 2x - 3 = 6 \Rightarrow x^2 - 2x - 9 = 0 \Rightarrow x = 1 \pm \sqrt{10}$$

Since $x = 1 \pm \sqrt{10} \notin A$. So, there is no pre-image of 6.

Let x be a pre-image of -3. Then,

$$f(x) = -3 \Rightarrow x^2 - 2x - 3 = -3 \Rightarrow x^2 - 2x = 0 \Rightarrow x = 0, 2.$$

Clearly, $0, 2 \in A$. So, 0 and 2 are pre-images of -3.

Let x be a pre-image of 5. Then,

$$f(x) = 5 \Rightarrow x^2 - 2x - 3 = 5 \Rightarrow x^2 - 2x - 8 = 0 \Rightarrow (x-4)(x+2) = 0 \Rightarrow x = 4, -2.$$

Since, $-2 \in A$ but $4 \notin A$. So, -2 is a pre-image of 5.

3.4 EQUAL FUNCTIONS

DEFINITION *Two functions f and g are said to be equal iff*

(i) *domain of* f = *domain of* g (ii) *co-domain of* f = *co-domain of* g,

and (iii) $f(x) = g(x)$ *for every x belonging to their common domain.*

If two functions f and g are equal, then we write $f = g$.

ILLUSTRATION 1 *Let* $A = \{1, 2\}$, $B = \{3, 6\}$ *and* $f : A \to B$ *given by* $f(x) = x^2 + 2$ *and* $g : A \to B$ *given by* $g(x) = 3x$. *Then, we observe that f and g have the same domain and co-domain. Also we have,* $f(1) = 3 = g(1)$ *and* $f(2) = 6 = g(2)$. Hence, $f = g$.

ILLUSTRATION 2 *Let* $f : R - \{2\} \to R$ *be defined by* $f(x) = \frac{x^2 - 4}{x - 2}$ *and* $g : R \to R$ *be defined by* $g(x) = x + 2$. *Find whether* $f = g$ *or not.*

SOLUTION We have, $f(x) = \frac{x^2 - 4}{x - 2}$, $x \neq 2$.

$\Rightarrow \quad f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2$ for all $x \neq 2$. $\Rightarrow$ $f(x) = g(x)$ for all $x \neq 2$.

Thus, $f(x) = g(x)$ for all $x \in R - \{2\}$. But, $f(x)$ and $g(x)$ have different domains. Infact, domain of $f = R - \{2\}$ and domain of $g = R$. Therefore, $f \neq g$.

ILLUSTRATION 3 *Let* $f : Z \to Z$ *and* $g : Z \to Z$ *be functions defined by* $f = \{(n, n^2) : n \in Z\}$ *and,* $g = \{(n, |n|^2) : n \in Z\}$. *Show that:* $f = g$.

SOLUTION Clearly, Domain of f = Domain of $g = Z$ and, Co-domain of f = Co-domain of $g = Z$.

We have, $f(n) = n^2$ and $g(n) = |n|^2 = n^2$ $\quad [\because |n|^2 = n^2]$

$\therefore \quad f(n) = g(n)$ for all $n \in Z$.

Hence, $f = g$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Express the following functions as sets of ordered pairs and determine their ranges*

(i) $f : A \to R, f(x) = x^2 + 1$, *where* $A = \{-1, 0, 2, 4\}$.

(ii) $g : A \to N, g(x) = 2x$, *where* $A = \{x : x \in N, x \leq 10\}$.

SOLUTION (i) We have, $f(x) = x^2 + 1$.

$\therefore \quad f(-1) = (-1)^2 + 1 = 2, f(0) = 0^2 + 1 = 1, f(2) = 2^2 + 1 = 5$ and $f(4) = 4^2 + 1 = 17$

So, $\quad f = \{(x, f(x)) : x \in A\} = \{(-1, 2), (0, 1), (2, 5), (4, 17)\}$. Hence, Range of $(f) = \{2, 1, 5, 17\}$

(ii) We have, $g(x) = 2x$ and $A = \{1, 2, 3, \ldots\ldots 10\}$. Therefore,

$g(1) = 2 \times 1 = 2, \quad g(2) = 2 \times 2 = 4, \ g(3) = 2 \times 3 = 6, \ g(4) = 2 \times 4 = 8, g(5) = 2 \times 5 = 10,$

$g(6) = 2 \times 6 = 12, \quad g(7) = 2 \times 7 = 14, g(8) = 2 \times 8 = 16, \ g(9) = 2 \times 9 = 18$ and, $g(10) = 2 \times 10 = 20$.

$\therefore g = \{(x, g(x) : x \in A\} = \{(1, 2), (2, 4), (3, 6), \ldots, (10, 20)\}$.

Hence, Range of $g = g(A) = \{g(x) : x \in A\} = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$.

EXAMPLE 2 *Find the domain for which the functions* $f(x) = 2x^2 - 1$ *and* $g(x) = 1 - 3x$ *are equal.*

[NCERT EXEMPLAR]

SOLUTION The values of x for which $f(x)$ and $g(x)$ are equal are given by

$$f(x)=g(x) \Rightarrow 2x^2-1=1-3x \Rightarrow 2x^2+3x-2=0 \Rightarrow (x+2)(2x-1)=0 \Rightarrow x=-2, 1/2.$$

Thus, $f(x)$ and $g(x)$ are equal on the set $\{-2, 1/2\}$.

EXAMPLE 3 *Is $g=\{(1,1),(2,3),(3,5),(4,7)\}$ a function? If this is described by the formula, $g(x)=\alpha x+\beta$, then what values should be assigned to α and β?* [NCERT EXEMPLAR]

SOLUTION Since no two ordered pairs in g have the same first component. So, g is a function such that $g(1)=1$, $g(2)=3$, $g(3)=5$ and $g(4)=7$.

It is given that $g(x)=\alpha x+\beta$.

$\therefore$ $g(1)=1$ and $g(2)=3 \Rightarrow \alpha+\beta=1$ and $2\alpha+\beta=3 \Rightarrow \alpha=2, \beta=-1$.

EXAMPLE 4 *Given $A=\{-1, 0, 2, 5, 6, 11\}$, $B=\{-2, -1, 0, 18, 28, 108\}$ and $f(x)=x^2-x-2$. Is $f(A)=B$? Find $f(A)$.*

SOLUTION We have, $f(x)=x^2-x-2$.

$\therefore$ $f(-1)=(-1)^2-(-1)-2=0$, $f(0)=0^2-0-2=-2$, $f(2)=2^2-2-2=0$,

$f(5)=5^2-5-2=18$, $f(6)=6^2-6-2=28$ and $f(11)=11^2-11-2=108$.

Hence, $f(A)=\{f(x): x \in A\}=\{f(-1), f(0), f(2), f(5), f(6), f(11)\}=\{0, -2, 18, 28, 108\}$

We observe that $-1 \in B$, but $-1 \notin f(A)$. So, $f(A) \neq B$.

EXAMPLE 5 *Let $f: R \rightarrow R$ be given by $f(x)=x^2+3$. Find : (i) $\{x : f(x)=28\}$ (ii) the pre-images of 39 and 2 under f.*

SOLUTION (i) We have, $f(x)=x^2+3$

$\therefore$ $f(x)=28 \Rightarrow x^2+3=28 \Rightarrow x^2=25 \Rightarrow x=\pm 5$

Hence, $\{x : f(x)=28\}=\{-5, 5\}$.

(ii) Let x be a pre-image of 39. Then, the image of x under f is 39.

i.e. $f(x)=39 \Rightarrow x^2+3=39 \Rightarrow x^2=36 \Rightarrow x=\pm 6$

So, pre-images of 39 are -6 and 6.

Let x be a pre-image of 2. Then, the image of x under f is 2.

i.e. $f(x)=2 \Rightarrow x^2+3=2 \Rightarrow x^2=-1$

We find that no real value of x satisfies the equation $x^2=-1$. Therefore, 2 does not have any pre-image under f.

EXAMPLE 6 *Let $f: R \rightarrow R$ be a function given by $f(x)=x^2+1$. Find:*

(i) $f^{-1}\{-5\}$ (ii) $f^{-1}\{26\}$ (iii) $f^{-1}\{10, 37\}$

SOLUTION Recall that if $f: A \rightarrow B$ is a function and $y \in B$. Then, $f^{-1}(y)=\{x \in A : f(x)=y\}$. In other words, $f^{-1}(y)$ is the set of pre-images of y.

(i) Let $f^{-1}(-5)=x$. Then, $f(x)=-5 \Rightarrow x^2+1=-5 \Rightarrow x^2=-6$. Clearly, this equation is not solvable in R. Therefore, there is no pre-image of -5. So, $f^{-1}\{-5\}=\phi$.

(ii) Let $f^{-1}(26)=x$. Then, $f(x)=26 \Rightarrow x^2+1=26 \Rightarrow x=\pm 5$. So, pre-images of 26 are -5 and 5. i.e. $f^{-1}\{26\}=\{-5, 5\}$.

(iii) Let $f^{-1}(10)=x$. Then, $f(x)=10 \Rightarrow x^2+1=10 \Rightarrow x^2=9 \Rightarrow x=\pm 3$. So, pre-images of 10 are -3 and 3. i.e. $f^{-1}(10)=\{-3, 3\}$.

Let $f^{-1}(37) = x$. Then, $f(x) = 37 \Rightarrow x^2 + 1 = 37 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$. So, pre-images of 37 are -6 and 6. Hence, $f^{-1}\{10, 37\} = \{3, -3, 6, -6\}$.

EXAMPLE 7 *Let $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a function described by the formula $f(x) = ax + b$ for some integers a, b. Determine a, b.* [NCERT]

SOLUTION Clearly, $f(1) = 1$, $f(2) = 3$, $f(0) = -1$ and $f(-1) = -3$. It is given that $f(x) = ax + b$.

$\therefore$ $f(1) = 1$ and $f(2) = 3 \Rightarrow a + b = 1$ and $2a + b = 3 \Rightarrow a = 2, b = -1$.

Substituting the values of a and b in $f(x) = ax + b$, we get: $f(x) = 2x - 1$.

$\therefore$ Clearly, $f(0) = -1$ and $f(-1) = -3$ are true.

Hence, $a = 2$ and $b = -1$.

EXAMPLE 8 *If $f : R \to R$ be defined as* $f(x) = \begin{cases} 1, & \text{if } x \in Q \\ -1, & \text{if } x \notin Q. \end{cases}$

Find (i) $f(1/2), f(\pi), f(\sqrt{2})$ (ii) Range of f (iii) pre-images of 1 and -1.

SOLUTION (i) It is evident from the definition of f that at every rational point the function attains value 1 and at every irrational point attains value -1.

$$\therefore \quad \frac{1}{2} \in Q \Rightarrow f\left(\frac{1}{2}\right) = 1, \pi \notin Q \Rightarrow f(\pi) = -1 \text{ and } \sqrt{2} \notin Q \Rightarrow f(\sqrt{2}) = -1.$$

(ii) We find that $f(x)$ attains values 1 or -1 according as x is rational or irrational and a real number is either rational or irrational. Thus, all rational numbers have image 1 and all irrational numbers have image -1. Hence, Range of $f = \{1, -1\}$.

(iii) Since $f(x) = 1$ for all $x \in Q$. Therefore, pre-images of 1 are rational numbers i.e $f^{-1}(1) = Q$. Also, -1 is the image of every real number which is not rational. Therefore,

$f^{-1}(-1) = R - Q =$ Set of irrational numbers.

EXAMPLE 9 *Let $f : R \to R$ be such that $f(x) = 2^x$. Determine:*

(i) *Range of f* (ii) $\{x : f(x) = 1\}$ (iii) *whether $f(x + y) = f(x) f(y)$ holds.*

SOLUTION (i) Since 2^x is positive for every $x \in R$. So, $f(x) = 2^x$ is a positive real number for every $x \in R$. Moreover, for every positive real number x, there exist $\log_2 x \in R$ such that

$$f(\log_2 x) = 2^{\log_2 x} = x \qquad [\because a^{\log_a x} = x]$$

Hence, the range of f is the set of all positive real numbers.

(ii) $f(x) = 1 \Rightarrow 2^x = 1 \Rightarrow 2^x = 2^0 \Rightarrow x = 0$. Therefore, $\{x : f(x) = 1\} = \{0\}$.

(iii) We have, $f(x) = 2^x$.

$$\therefore \quad f(x + y) = 2^{x+y} = 2^x \times 2^y = f(x) f(y)$$

Hence, $f(x + y) = f(x) f(y)$ holds for all $x, y \in R$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 10 *Let A be the set of two positive integers. Let $f : A \to Z^+$ (set of positive integers) be defined by $f(n) = p$, where p is the highest prime factor of n. If range of $f = \{3\}$. Find set A. Is A uniquely determined?*

SOLUTION It is given that the set A consists of two positive integers. So, let $A = \{n, m\}$. Since range of $f = \{3\}$.

$\therefore$ $f(n) = 3$ and $f(m) = 3$

$\Rightarrow$ Highest prime factors of n and m both are equal to 3.

$\Rightarrow$ $(n = 3$ and $m = 6)$ or $(n = 3$ and $m = 9)$ or $(n = 3$ and $m = 12)$ or $(n = 6$ and $m = 12)$ etc.

$\Rightarrow$ Either $A = \{3, 6\}$, or $A = \{3, 9\}$, or $A = \{3, 12\}$, or $A = \{6, 12\}$ etc.

Clearly, A is not uniquely determined.

EXAMPLE 11 *Let $A \subseteq N$ and $f : A \to A$ be defined by : $f(n)$ = the highest prime factor of n. If range of f is A, determine A. Is A uniquely determined?*

SOLUTION For any $n \in A$, we have

$f(n)$ = Highest prime factor of n

$\Rightarrow$ $f(n)$ takes prime values only $\Rightarrow$ Range of f consists of prime numbers only

But, it is given that range of f is A. Therefore, set A consists of prime numbers only.

Hence, A = Set of some prime numbers. Clearly, A is not uniquely determined.

EXERCISE 3.1

BASIC

1. Define a function as a set of ordered pairs.
2. Define a function as a correspondence between two sets.
3. What is the fundamental difference between a relation and a function ? Is every relation a function ?
4. Let $A = \{-2, -1, 0, 1, 2\}$ and $f : A \to Z$ be a function defined by $f(x) = x^2 - 2x - 3$. Find:
   (i) range of f i.e. $f(A)$ (ii) pre-images of 6, –3 and 5.
5. If a function $f : R \to R$ be defined by $f(x) = \begin{cases} 3x - 2, & x < 0 \\ 1, & x = 0 \\ 4x + 1, & x > 0 \end{cases}$. Find: $f(1), f(-1), f(0), f(2)$.
6. A function $f : R \to R$ is defined by $f(x) = x^2$. Determine
   (i) range of f (ii) $\{x : f(x) = 4\}$ (iii) $\{y : f(y) = -1\}$.
7. Let $f : R^+ \to R$, where R^+ is the set of all positive real numbers be such that $f(x) = \log_e x$. Determine: (i) the image set of the domain of f (ii) $\{x : f(x) = -2\}$ (iii) whether $f(xy) = f(x) + f(y)$ holds.
8. Write the following relations as sets of ordered pairs and find which of them are functions:
   (i) $\{(x, y) : y = 3x, x \in \{1, 2, 3\}, y \in \{3, 6, 9, 12\}\}$.
   (ii) $\{(x, y) : y > x + 1, x = 1, 2$ and $y = 2, 4, 6\}$ (iii) $\{(x, y) : x + y = 3, x, y \in \{0, 1, 2, 3\}\}$
9. Let $f : R \to R$ and $g : C \to C$ be two functions defined as $f(x) = x^2$ and $g(x) = x^2$. Are they equal functions?
10. If f, g, h are three functions defined from R to R as follows:
    (i) $f(x) = x^2$ (ii) $g(x) = \sin x$ (iii) $h(x) = x^2 + 1$
    Find the range of each function.
11. Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 5, 9, 11, 15, 16\}$. Determine which of the following sets are functions from X to Y
    (i) $f_1 = \{(1, 1), (2, 11), (3, 1), (4, 15)\}$ (ii) $f_2 = \{(1, 1), (2, 7), (3, 5)\}$
    (iii) $f_3 = \{(1, 5), (2, 9)\ (3, 1), (4, 5), (2, 11)\}$. [NCERT]
12. Let $A = \{12, 13, 14, 15, 16, 17\}$ and $f : A \to Z$ be a function given by $f(x)$ = highest prime factor of x. Find range of f.
13. If $f : R \to R$ be defined by $f(x) = x^2 + 1$, then find $f^{-1}\{17\}$ and $f^{-1}\{-3\}$.
14. Let $A = \{p, q, r, s\}$ and $B = \{1, 2, 3\}$. Which of the following relations from A to B is not a function?

(i) $R_1 = \{(p, 1), (q, 2), (r, 1), (s, 2)\}$ (ii) $R_2 = \{(p, 1), (q, 1), (r, 1), (s, 1)\}$
(iii) $R_3 = \{(p, 1), (q, 2), (p, 2), (s, 3)\}$ (iv) $R_4 = \{(p, 2), (q, 3), (r, 2), (s, 2)\}$.

15. Let $A = \{9, 10, 11, 12, 13\}$ and let $f : A \to N$ be defined by $f(n)$ = the highest prime factor of n. Find the range of f. [NCERT]

16. The function f is defined by $f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 3x, & 3 \le x \le 10 \end{cases}$. The relation g is defined by

$g(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ 3x, & 2 \le x \le 10 \end{cases}$ Show that f is a function and g is not a function. [NCERT]

17. If $f(x) = x^2$, find $\dfrac{f(1.1) - f(1)}{(1.1) - 1}$. [NCERT]

18. Express the function $f : X \to R$ given by $f(x) = x^3 + 1$ as set of ordered pairs, where $X = \{-1, 0, 3, 9, 7\}$. [NCERT EXEMPLAR]

ANSWERS

4. (i) $f(A) = \{-4, -3, 0, 5\}$ (ii) ϕ, $\{0, 2\}$, -2. 5. $f(1) = 5$, $f(-1) = -5$, $f(0) = 1$, $f(2) = 9$
6. (i) R^+ (set of all real numbers greater than or equal to zero) (ii) $\{-2, 2\}$ (iii) ϕ
7. (i) R (ii) $\{e^{-2}\}$ (iii) Yes
8. (i) $\{(1, 3), (2, 6), (3, 9)\}$; Function (ii) $\{(1, 4), (1, 6), (2, 4), (2, 6)\}$; Not a function
(iii) $\{(0, 3), (1, 2), (2, 1), (3, 0)\}$; Function.
9. No, Since domain of $f \ne$ domain of g.
10. (i) $R^+ = \{x \in R \mid x \ge 0\}$ (ii) $\{x \in R : -1 \le x \le 1\}$ (iii) $\{x \in R : x \ge 1\}$.
11. (i) 12. $\{3, 13, 7, 5, 2, 17\}$ 13. $f^{-1}(17) = \{-4, 4\}$, $f^{-1}(-3) = \phi$
14. (iii) 15. Range $(f) = \{3, 5, 11, 13\}$. 17. 2.1
18. $f = \{(-1, 0), (0, 1), (3, 28), (9, 730), (7, 344)\}$

HINTS TO SELECTED PROBLEMS

11. (iii) $f_3 = \{(1,5), (2, 9), (3, 1), (4, 5), (2, 11)\}$ is not a function because $(2, 9)$ and $(2, 11) \in f_3$, which means that 2 is related to two elements in y.

12. Clearly, $f(12)$ = highest prime factor of 12 = 3. Similarly, $f(13) = 13$, $f(14) = 7$, $f(15) = 5$, $f(16) = 2$ and $f(17) = 17$. Hence, range $(f) = \{3, 13, 7, 5, 2, 17\}$.

15. $A = \{9, 10, 11, 12, 13\}$ and $f : A \to N$ is defined by $f(n)$ = the highest prime factor of n.

$\therefore$ $f(9) = 3$, $f(10) = 5$, $f(11) = 11$, $f(12) = 3$ and $f(13) = 13$

Hence, range $(f) = \{f(9), f(10), f(11), f(12), f(13)\} = \{3, 5, 11, 13\}$

16. We observe that $f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 3x, & 3 \le x \le 10 \end{cases}$ associates all numbers in $[0, 10]$ to numbers in R and no number in $[0, 10]$ is associated to two or more numbers. Hence, f is a function. But, g is not a function because 2 is associated to two distinct elements viz. 4 and 6.

17. We have, $f(x) = x^2$

$$\therefore \quad \frac{f(1.1) - f(1)}{(1.1) - 1} = \frac{(1.1)^2 - 1^2}{(1.1) - 1} = \frac{(1.1 + 1)(1.1 - 1)}{(1.1 - 1)} = 2.1$$

3.5 REAL FUNCTIONS

In this section, we will discuss functions having domain and co-domain both as subsets of the set R of all real numbers. Such functions are called real functions or real valued functions of the real variable as defined below.

REAL VALUED FUNCTION *A function $f: A \to B$ is called a real valued function, if B is a subset of R (set of all real numbers).*

If A and B both are subsets of R, then f is called a real function.

In section 3.3.1, we have discussed the description of a function. Generally, domain and co-domain both are infinite subsets of R in case of real functions of real variable. Therefore, a real function is generally described by some general formula. In other words, images of various elements in the domain of a real function are provided by some general formula. For example, $f: R \to R$ given by $f(x) = x^2 + x + 1$ or, $f: A \to B$ given by $f(x) = \frac{x-1}{x^2-4}$ etc. In practice, real functions are described by giving the general expressions or formulae describing them without mentioning their domains and co-domains. Following are some examples of real functions.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *If $f(x) = 3x^4 - 5x^2 + 9$, find $f(x-1)$.*

SOLUTION We have, $f(x) = 3x^4 - 5x^2 + 9$. Replacing x by $(x-1)$, we obtain

$$f(x-1) = 3(x-1)^4 - 5(x-1)^2 + 9 = 3x^4 - 12x^3 + 13x^2 - 2x + 7$$

EXAMPLE 2 *If $f(x) = x + \frac{1}{x}$, prove that $\{f(x)\}^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$.*

SOLUTION We have,

$$f(x) = x + \frac{1}{x} \Rightarrow f(x^3) = x^3 + \frac{1}{x^3}, \{f(x)\}^3 = \left(x + \frac{1}{x}\right)^3 \text{ and } f\left(\frac{1}{x}\right) = \frac{1}{x} + x.$$

$$\therefore \quad \left(x + \frac{1}{x}\right)^3 = \left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right) \Rightarrow \{f(x)\}^3 = f(x^3) + 3f(x)$$

EXAMPLE 3 *If $f(x) = \frac{1}{2x+1}$, $x \neq -\frac{1}{2}$, then show that $f(f(x)) = \frac{2x+1}{2x+3}$, provided that $x \neq -\frac{3}{2}$.*

SOLUTION We have, $f(x) = \dfrac{1}{2x+1}$

$$\Rightarrow \quad f(f(x)) = f\left(\frac{1}{2x+1}\right) = \frac{1}{2\left(\frac{1}{2x+1}\right)+1} = \frac{1}{\frac{2}{2x+1}+1} = \frac{2x+1}{2+2x+1} = \frac{2x+1}{2x+3}$$

Clearly, $f(f(x)) = \frac{2x+1}{2x+3}$ is real for $2x + 3 \neq 0$ i.e. $f(f(x))$ is defined for $2x + 3 \neq 0$ i.e. $x \neq -\frac{3}{2}$.

Hence, $f(f(x)) = \frac{2x+1}{2x+3}$, provided that $x \neq -\frac{3}{2}$.

EXAMPLE 4 *If $f(x) = \frac{x-1}{x+1}$, $x \neq -1$, then show that $f(f(x)) = -\frac{1}{x}$, provided that $x \neq 0$.*

SOLUTION We have, $f(x) = \dfrac{x-1}{x+1}, x \neq -1.$

$\therefore \quad f(f(x)) = f\left(\dfrac{x-1}{x+1}\right) = \dfrac{\dfrac{x-1}{x+1} - 1}{\dfrac{x-1}{x+1} + 1}$ $\left[\text{Replacing } x \text{ by } \dfrac{x-1}{x+1} \text{ in the formula for } f(x)\right]$

$$= \frac{x-1-x-1}{x-1+x+1} = \frac{-2}{2x} = -\frac{1}{x}.$$

We find that $-\dfrac{1}{x}$ is meaningful for $x \neq 0$. Hence, $f(f(x)) = -\dfrac{1}{x}$, provided that $x \neq 0$.

EXAMPLE 5 *Let f be defined by* $f(x) = x - 4$ *and g be defined by* $g(x) = \begin{cases} \dfrac{x^2-16}{x+4}, & x \neq -4 \\ \lambda, & x = -4 \end{cases}$

Find λ *such that* $f(x) = g(x)$ *for all* x.

SOLUTION We have,

$f(x) = g(x) \quad$ for all $x \in R$

$\Rightarrow \quad f(-4) = g(-4) \Rightarrow -4-4 = \lambda \Rightarrow \lambda = -8.$ $\quad [\because f(x) = x-4 \therefore f(-4) - 4 = -8]$

EXAMPLE 6 *If f is a real function defined by* $f(x) = \dfrac{x-1}{x+1}$, *then prove that:* $f(2x) = \dfrac{3f(x)+1}{f(x)+3}$.

SOLUTION We have, $f(x) = \dfrac{x-1}{x+1}$

$\Rightarrow \quad \dfrac{f(x)}{1} = \dfrac{x-1}{x+1} \Rightarrow \dfrac{f(x)+1}{f(x)-1} = \dfrac{x-1+x+1}{x-1-x-1}$ [Applying componendo and dividendo]

$\Rightarrow \quad \dfrac{f(x)+1}{f(x)-1} = -x \Rightarrow x = \dfrac{f(x)+1}{1-f(x)}$ ...(i)

Again, $f(x) = \dfrac{x-1}{x+1}$

$\Rightarrow \quad f(2x) = \dfrac{2x-1}{2x+1} = \dfrac{2\left\{\dfrac{f(x)+1}{1-f(x)}\right\} - 1}{2\left\{\dfrac{f(x)+1}{1-f(x)}\right\} + 1}$ [Using (i)]

$$= \frac{2f(x)+2-1+f(x)}{2f(x)+2+1-f(x)} = \frac{3f(x)+1}{f(x)+3}$$

EXERCISE 3.2

BASIC

1. If $f(x) = x^2 - 3x + 4$, then find the values of x satisfying the equation $f(x) = f(2x+1)$.
2. If $f(x) = (x-a)^2 (x-b)^2$, find $f(a+b)$.
3. If $y = f(x) = \dfrac{ax-b}{bx-a}$, show that $x = f(y)$. [NCERT EXEMPLAR]
4. If $f(x) = \dfrac{1}{1-x}$, show that $f[f\{f(x)\}] = x$.

5. If $f(x) = \dfrac{x+1}{x-1}$, show that $f[f(x)] = x$.

6. If $f(x) = \begin{cases} x^2, \text{ when } x < 0 \\ x, \text{ when } 0 \le x < 1 \\ \dfrac{1}{x}, \text{ when } x > 1 \end{cases}$ [NCERT]

Find: (i) $f(1/2)$ (ii) $f(-2)$ (iii) $f(1)$ (iv) $f(\sqrt{3})$ and (v) $f(\sqrt{-3})$.

7. If $f(x) = x^3 - \dfrac{1}{x^3}$, show that $f(x) + f\left(\dfrac{1}{x}\right) = 0$.

8. If $f(x) = \dfrac{2x}{1+x^2}$, show that $f(\tan\theta) = \sin 2\theta$.

9. If $f(x) = \dfrac{x-1}{x+1}$, then show that: (i) $f\left(\dfrac{1}{x}\right) = -f(x)$ (ii) $f\left(-\dfrac{1}{x}\right) = -\dfrac{1}{f(x)}$

BASED ON HOTS

10. If $f(x) = (a - x^n)^{1/n}$, $a > 0$ and $n \in N$, then prove that $f(f(x)) = x$ for all x.

11. If for non-zero x, $a\,f(x) + b\,f\left(\dfrac{1}{x}\right) = \dfrac{1}{x} - 5$, where $a \ne b$, then find $f(x)$.

ANSWERS

1. $x = -1, 2/3$ 2. $a^2 b^2$

6. (i) $\dfrac{1}{2}$ (ii) 4. (iii) 1 (iv) $\dfrac{1}{\sqrt{3}}$ (v) does not exist 11. $\dfrac{1}{a^2-b^2}\left\{\dfrac{a}{x} - bx\right\} - \dfrac{5}{a+b}$

HINTS TO SELECTED PROBLEMS

6. We have, $f(x) = \begin{cases} x^2, & \text{when } x < 0 \\ x, & \text{when } 0 \le x < 1 \\ \dfrac{1}{x}, & \text{when } x \ge 1 \end{cases}$. Therefore,

(i) $f\left(\dfrac{1}{2}\right) = \dfrac{1}{2}$ (ii) $f(-2) = (-2)^2 = 4$ (iii) $f(1) = \dfrac{1}{1} = 1$

(iv) $f(\sqrt{3}) = \dfrac{1}{\sqrt{3}}$ (iv) $f(-\sqrt{3}) = (-\sqrt{3})^2 = 3$

11. We have,

$$a\,f(x) + b\,f\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \quad \text{... (i)}$$

$$\Rightarrow a\,f\left(\frac{1}{x}\right) + b\,f(x) = x - 5 \quad \left[\text{Replacing } x \text{ by } \frac{1}{x}\right] \quad \text{... (ii)}$$

Adding (i) and (ii), we get

$$\left\{f(x) + f\left(\frac{1}{x}\right)\right\}(a+b) = x + \frac{1}{x} - 10$$

$$\Rightarrow \quad f(x)+f\left(\frac{1}{x}\right) = \frac{1}{a+b}\left\{x+\frac{1}{x}-10\right\} \quad \text{...(iii)}$$

Subtracting (ii) from (i), we get

$$f(x)-f\left(\frac{1}{x}\right)=\frac{1}{a-b}\left\{\frac{1}{x}-x\right\} \quad \text{... (iv)}$$

Add (iii) and (iv) to obtain $f(x)$.

3.6 DOMAIN AND RANGE OF A REAL A FUNCTION

Mathematically to define a function one has to provide its domain, co-domain and the images of elements in its domain either by giving a general formula or by listing them one by one. As the domain and co-domain of real functions are subsets of R. Therefore, conventionally, real functions are described by providing the general formula for finding the images of elements in its domain. In such cases, the domain of the real function $f(x)$ is the set of all those real numbers for which the expression for $f(x)$ or the formula for $f(x)$ assumes real values only. In otherwords, the domain of $f(x)$ is the set of all those real numbers for which $f(x)$ is meaningful.

For example, a real function $f(x)$ described by the general formula $f(x)=\dfrac{3x-2}{x^2-1}$ assumes real values for all $x \in R$ except for $x = \pm 1$, because denominator of $\dfrac{3x-2}{x^2-1}$ becomes zero for $x = \pm 1$.

So, domain of $f(x)$ is the set of all real numbers other than -1 and 1 i.e. domain$(f) = R - \{-1, 1\}$.

Following examples will illustrate the procedure for finding the domain of a real function of a real variable.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the domain of each of the following real valued functions:*

(i) $f(x) = \dfrac{1}{x+2}$ (ii) $f(x) = \dfrac{x-1}{x-3}$

(iii) $f(x) = \dfrac{2x-3}{x^2-3x+2}$ (iv) $f(x) = \dfrac{x^2+3x+5}{x^2-5x+4}$ [NCERT]

SOLUTION (i) We have, $f(x)=\dfrac{1}{x+2}$. Clearly, $f(x)$ assumes real values for all real values of x except for the values of x satisfying $x+2=0$ i.e. $x=-2$. Hence, Domain$(f) = R-\{-2\}$.

(ii) We have, $f(x) = \dfrac{x-1}{x-3}$. We observe that $f(x)$ is a rational function of x as $\dfrac{x-1}{x-3}$ is a rational expression. Clearly, $f(x)$ assumes real values for all x except for the values of x for which $x-3=0$ i.e. $x=3$. Hence, Domain$(f)=R-\{3\}$.

(iii) We have, $f(x)=\dfrac{2x-3}{x^2-3x+2}$. Clearly, $f(x)$ is a rational function of x as $\dfrac{2x-3}{x^2-3x+2}$ is a rational expression. We observe that $f(x)$ assumes real values for all x except for all those values of x for which $x^2-3x+2 = 0$ i.e. $x=1, 2$. Hence, Domain $(f)=R-\{1, 2\}$.

(iv) We have, $f(x) = \dfrac{x^2+3x+5}{x^2-5x+4}$. Clearly, $f(x)$ is a rational function of x as $\dfrac{x^2+3x+5}{x^2-5x+4}$ is a rational expression in x. We observe that $f(x)$ assumes real values for all x except for all those values of x for which $x^2-5x+4=0$ i.e. $x=1, 4$. Hence, Domain $(f) = R-\{1, 4\}$.

EXAMPLE 2 *Find the domain of each of the following functions:*

(i) $f(x) = \sqrt{x-2}$ (ii) $f(x) = \dfrac{1}{\sqrt{1-x}}$ (iii) $f(x) = \sqrt{4-x^2}$

SOLUTION (i) We have, $f(x) = \sqrt{x-2}$. Clearly, $f(x)$ assumes real values for all x satisfying $x-2 \geq 0 \Rightarrow x \geq 2 \Rightarrow x \in [2, \infty)$. Hence, Domain $(f) = [2, \infty)$.

(ii) We have, $f(x) = \dfrac{1}{\sqrt{1-x}}$. Clearly, $f(x)$ assumes real values for all x satisfying $1-x>0 \Rightarrow 1>x \Rightarrow x<1 \Rightarrow x \in (-\infty, 1)$. Hence, Domain $(f) = (-\infty, 1)$.

(iii) We have, $f(x) = \sqrt{4-x^2}$. Clearly, $f(x)$ assumes real values for all x satisfying

$$4-x^2 \geq 0 \Rightarrow -(x^2-4) \geq 0 \Rightarrow x^2-4 \leq 0 \Rightarrow (x-2)(x+2) \leq 0 \Rightarrow x \in [-2, 2].$$

Hence, Domain $(f) = [-2, 2]$.

EXAMPLE 3 *Find the domain of the function $f(x)$ defined by* $f(x) = \sqrt{4-x} + \dfrac{1}{\sqrt{x^2-1}}$.

SOLUTION Clearly, $f(x)$ is defined for all x satisfying

$4-x \geq 0$ and $x^2-1>0$

$\Rightarrow x-4 \leq 0$ and $(x-1)(x+1)>0 \Rightarrow x \leq 4$ and $(x<-1$ or $x>1) \Rightarrow x \in (-\infty, -1) \cup (1, 4]$.

Hence, Domain $(f) = (-\infty, -1) \cup (1, 4]$.

3.6.1 RANGE OF REAL FUNCTIONS

The range of a real function of a real variable is the set of all real values taken by $f(x)$ at points in its domain. In order to find the range of a real function $f(x)$, we may use the following algorithm.

ALGORITHM

Step I *Put $y = f(x)$.*

Step II *Solve the equation $y = f(x)$ for x in terms of y. Let $x = \phi(y)$.*

Step III *Find the values of y for which the values of x, obtained from $x = \phi(y)$, are real and in the domain of f.*

Step IV *The set of values of y obtained in step III is the range of f.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the domain and range of the function $f(x)$ given by* $f(x) = \dfrac{x-2}{3-x}$.

SOLUTION We have, $f(x) = \dfrac{x-2}{3-x}$.

Domain of f: Clearly, $f(x)$ is defined for all x satisfying $3-x \neq 0$ i.e. $x \neq 3$. Hence, Domain $(f) = R-\{3\}$.

Range of f: Let $y = f(x)$. Then,

$$y = \frac{x-2}{3-x} \Rightarrow 3y - xy = x - 2 \Rightarrow x(y+1) = 3y + 2 \Rightarrow x = \frac{3y+2}{y+1}$$

Clearly, x assumes real values for all y except $y + 1 = 0$ i.e. $y = -1$. Hence, Range $(f) = R - \{-1\}$.

EXAMPLE 2 *Find the range of each of the following functions:*

(i) $f(x) = \frac{1}{\sqrt{x-5}}$ (ii) $f(x) = \sqrt{16-x^2}$ (iii) $f(x) = \frac{x}{1+x^2}$ (iv) $f(x) = \frac{3}{2-x^2}$

SOLUTION (i) We have, $f(x) = \frac{1}{\sqrt{x-5}}$. Clearly, $f(x)$ takes real values for all x satisfying $x - 5 > 0 \Rightarrow x > 5 \Rightarrow x \in (5, \infty)$. Therefore, Domain $(f) = (5, \infty)$.

For any $x > 5$, we have

$$x - 5 > 0 \Rightarrow \sqrt{x-5} > 0 \Rightarrow \frac{1}{\sqrt{x-5}} > 0 \Rightarrow f(x) > 0.$$

Thus, $f(x)$ takes all real values greater than zero. Hence, Range $(f) = (0, \infty)$.

(ii) We have, $f(x) = \sqrt{16 - x^2}$. We observe that $f(x)$ is defined for all x satisfying

$$16 - x^2 \geq 0 \Rightarrow x^2 - 16 \leq 0 \Rightarrow (x-4)(x+4) \leq 0 \Rightarrow -4 \leq x \leq 4 \Rightarrow x \in [-4, 4]$$

$\therefore$ Domain $(f) = [-4, 4]$.

Let $y = f(x)$. Then,

$$y = \sqrt{16 - x^2} \Rightarrow y^2 = 16 - x^2 \Rightarrow x^2 = 16 - y^2 \Rightarrow x = \sqrt{16 - y^2}$$

Clearly, x will take real values, if

$$16 - y^2 \geq 0 \Rightarrow y^2 - 16 \leq 0 \Rightarrow (y-4)(y+4) \leq 0 \Rightarrow -4 \leq y \leq 4 \Rightarrow y \in [-4, 4]$$

Also, $y = \sqrt{16 - x^2} \geq 0$ for all $x \in [-4, 4]$. Therefore, $y \in [0, 4]$ for all $x \in [-4, 4]$.

Hence, Range $f) = [0, 4]$.

(iii) We have, $f(x) = \frac{x}{1+x^2}$. We observe that $f(x)$ takes real values for all $x \in R$. Hence, domain $(f) = R$. Let $y = f(x)$. Then,

$$y = f(x) \Rightarrow y = \frac{x}{1+x^2} \Rightarrow x^2 y - x + y = 0 \Rightarrow x = \frac{1 \pm \sqrt{1-4y^2}}{2y}$$

Clearly, x will assume real values, if

$$1 - 4y^2 \geq 0 \text{ and } y \neq 0 \Rightarrow 4y^2 - 1 \leq 0 \text{ and } y \neq 0 \Rightarrow y^2 - \frac{1}{4} \leq 0 \text{ and } y \neq 0$$

$$\Rightarrow \left(y - \frac{1}{2}\right)\left(y + \frac{1}{2}\right) \leq 0 \text{ and } y \neq 0 \Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2} \text{ and } y \neq 0 \Rightarrow y \in \left[-\frac{1}{2}, \frac{1}{2}\right] - \{0\}$$

Also, $y = 0$ for $x = 0$. Hence, Range $(f) = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

(iv) We have, $f(x) = \frac{3}{2-x^2}$. For $f(x)$ to be real, we must have $2 - x^2 \neq 0 \Rightarrow x \neq \pm\sqrt{2}$.

$\therefore$ Domain $(f) = R - \{-\sqrt{2}, \sqrt{2}\}$. Let $f(x) = y$. Then,

$$y = f(x) \Rightarrow y = \frac{3}{2-x^2} \Rightarrow 2y - x^2 y = 3 \Rightarrow x^2 y = 2y - 3 \Rightarrow x = \pm\sqrt{\frac{2y-3}{y}}$$

We observe that x will take real values other than $-\sqrt{2}$ and $\sqrt{2}$, if

$$\frac{2y-3}{y} > 0 \Rightarrow y \in (-\infty, 0) \cup [3/2, \infty)$$ [See Fig. 3.5]

Fig. 3.5 Signs of $\frac{2y-3}{y}$

Hence, range $(f) = (-\infty, 0) \cup [3/2, \infty)$.

EXAMPLE 3 *Find the domain and range of the function* $f(x) = \frac{x^2-9}{x-3}$.

SOLUTION We have, $f(x) = \frac{x^2-9}{x-3}$.

Domain of f: Clearly, $f(x)$ is not defined for $x - 3 = 0$ i.e. $x = 3$. Therefore, Domain $(f) = R - \{3\}$.

Range of f: Let $f(x) = y$. Then, $f(x) = y \Rightarrow \frac{x^2-9}{x-3} = y \Rightarrow x + 3 = y$ $[\because x \neq 3]$

It follows from the above relation that y takes all real values except 6 when x takes values in the set $R - \{3\}$. Therefore, Range $(f) = R - \{6\}$.

EXAMPLE 4 *Find the domain and range of the real valued function* $f(x)$ *given by* $f(x) = \frac{4-x}{x-4}$.

SOLUTION We have, $f(x) = \frac{4-x}{x-4}$.

Domain of f: We observe that $f(x)$ is defined for all x except at $x = 4$. At $x = 4$, $f(x)$ takes the indeterminate form $\frac{0}{0}$. Therefore, Domain $(f) = R - \{4\}$.

Range of f: For any $x \in$ Domain (f) i.e. for any $x \neq 4$, we have

$$f(x) = \frac{4-x}{x-4} = \frac{-(x-4)}{x-4} = -1.$$ Therefore, Range $(f) = \{-1\}$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Let* $f = \left\{\left(x, \frac{x^2}{1+x^2}\right) : x \in R\right\}$ *be a function from R into R. Determine the range of f.*

[NCERT]

SOLUTION We have, $f(x) = \frac{x^2}{x^2+1}$

Domain of f: Clearly, $f(x)$ is defined for all $x \in R$ as $x^2 + 1 \neq 0$ for any $x \in R$. So, Domain $(f) = R$.

Range of f: Let $f(x) = y$. Then,

$$f(x) = y$$

$$\Rightarrow \quad \frac{x^2}{x^2+1} = y \Rightarrow x^2 = x^2 y + y \Rightarrow x^2(1-y) = y \Rightarrow x^2 = \frac{y}{1-y} \Rightarrow x = \pm\sqrt{\frac{y}{1-y}}$$

Clearly, x will take real values, if

$$\frac{y}{1-y} \geq 0$$

Fig. 3.6 Signs of $\frac{y}{y-1}$

$\Rightarrow \quad \dfrac{y-0}{y-1} \leq 0 \Rightarrow 0 \leq y < 1 \Rightarrow y \in [0, 1)$ [See Fig. 3.6]

Hence, range $(f) = [0, 1)$.

EXAMPLE 6 *Find the domain and range of the function* $f = \left\{\left(x : \dfrac{1}{1-x^2}\right) : x \in R, x \neq \pm 1\right\}$.

SOLUTION We have, $f(x) = \dfrac{1}{1-x^2}$

Domain of f: Clearly, $f(x)$ is defined for all $x \in R$ except for which $x^2 - 1 \neq 0$ i.e. $x = \pm 1$. Therefore, Domain of $f = R - \{-1, 1\}$.

Range of f: Let $f(x) = y$. Then,

$$f(x) = y \Rightarrow \frac{1}{1-x^2} = y \Rightarrow 1 - x^2 = \frac{1}{y} \Rightarrow x^2 = 1 - \frac{1}{y} = \frac{y-1}{y} \Rightarrow x = \pm\sqrt{\frac{y-1}{y-0}}$$

Clearly, x will take real values, if

$$\frac{y-1}{y-0} \geq 0$$

Fig. 3.7 Signs of $\frac{y-1}{y}$

$\Rightarrow \quad y < 0$ or $y \geq 1 \Rightarrow y \in (-\infty, 0) \cup [1, \infty)$ [See Fig. 3.7]

Hence, range $(f) = (-\infty, 0) \cup [1, \infty)$.

EXAMPLE 7 *Find the domain and range of the function* $f(x) = \dfrac{1}{2 - \sin 3x}$.

SOLUTION We have, $f(x) = \dfrac{1}{2 - \sin 3x}$

Domain of f: We know that

$-1 \leq \sin 3x \leq 1$ for all $x \in R$

$\Rightarrow \quad -1 \leq -\sin 3x \leq 1$ for all $x \in R$

$\Rightarrow \quad 1 \leq 2 - \sin 3x \leq 3$ for all $x \in R$ [Adding 2 throughout]

$\Rightarrow \quad 2 - \sin 3x \neq 0$ for any $x \in R \Rightarrow f(x) = \dfrac{1}{2 - \sin 3x}$ is defined for all $x \in R$

Hence, domain $(f) = R$.

Range of f: As discussed above

$1 \leq 2 - \sin 3x \leq 3$ for all $x \in R$

$\Rightarrow \quad \dfrac{1}{3} \leq \dfrac{1}{2 - \sin 3x} \leq 1$ for all $x \in R \Rightarrow \dfrac{1}{3} \leq f(x) \leq 1$ for all $x \in R \Rightarrow f(x) \in \left[\dfrac{1}{3}, 1\right]$

Hence, range $(f) = \left[\dfrac{1}{3}, 1\right]$

EXERCISE 3.3

BASIC

1. Find the domain of each of the following real valued functions of real variable:

(i) $f(x) = \dfrac{1}{x}$ (ii) $f(x) = \dfrac{1}{x-7}$ (iii) $f(x) = \dfrac{3x-2}{x+1}$ (iv) $f(x) = \dfrac{2x+1}{x^2-9}$

(v) $f(x) = \dfrac{x^2+2x+1}{x^2-8x+12}$ [NCERT]

2. Find the domain of each of the following real valued functions of real variable:

(i) $f(x) = \sqrt{x-2}$ (ii) $f(x) = \dfrac{1}{\sqrt{x^2-1}}$ (iii) $f(x) = \sqrt{9-x^2}$ (iv) $f(x) = \sqrt{\dfrac{x-2}{3-x}}$

BASED ON LOTS

3. Find the domain and range of each of the following real valued functions:

(i) $f(x) = \dfrac{ax+b}{bx-a}$ (ii) $f(x) = \dfrac{ax-b}{cx-d}$

(iii) $f(x) = \sqrt{x-1}$ [NCERT] (iv) $f(x) = \sqrt{x-3}$

(v) $f(x) = \dfrac{x-2}{2-x}$ (vi) $f(x) = |x-1|$ [NCERT]

(vii) $f(x) = -|x|$ [NCERT] (viii) $f(x) = \sqrt{9-x^2}$ [NCERT]

(ix) $f(x) = \dfrac{1}{\sqrt{16-x^2}}$ (x) $f(x) = \sqrt{x^2-16}$

ANSWERS

1. Domain

(i) $R-\{0\}$
(ii) $R-\{7\}$
(iii) $R-\{-1\}$
(iv) $R-\{-3, 3\}$
(v) $R-\{2, 6\}$

2.

	Domain	Range
(i)	$[2, \infty)$	$[0, \infty)$
(ii)	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, -1] \cup (0, \infty)$
(iii)	$[-3, 3]$	$[0, 3]$
(iv)	$[2, 3)$	$[0, \infty)$

3.

	Domain	Range		Domain	Range
(i)	$R-\left\{\dfrac{a}{b}\right\}$	$R-\left\{\dfrac{a}{b}\right\}$	(ii)	$R-\left\{\dfrac{d}{c}\right\}$	$R-\left\{\dfrac{a}{c}\right\}$
(iii)	$[1, \infty)$	$[0, \infty)$	(iv)	$[3, \infty)$	$[0, \infty)$
(v)	$R-\{2\}$	$\{-1\}$	(vi)	R	$[0, \infty)$
(vii)	R	$(-\infty, 0]$	(viii)	$[-3, 3]$	$[0, 3]$
(ix)	$(-4, 4)$	$\left[\dfrac{1}{4}, \infty\right)$	(x)	$(-\infty, -4] \cup [4, \infty)$	$[0, \infty)$

HINTS TO SELECTED PROBLEMS

1. (v) $f(x) = \dfrac{x^2+2x+1}{x^2-8x+12} = \dfrac{(x+1)^2}{(x-6)(x-2)}$ is defined for all x satisfying

$(x-6)(x-2) \neq 0$ i.e. $x \neq 2, 6$. Therefore, Domain $(f) = R - \{2, 6\}$

3. (iii) $f(x) = \sqrt{x-1}$ is defined for all x satisfying $x-1 \geq 0$ i.e. $x \geq 1$. So, domain $(f) = [1, \infty)$.

Let $y = \sqrt{x-1}$. Clearly, $y \geq 0$ for all $x \in [1, \infty)$. So, range $(f) = [0, \infty)$.

(vi) $f(x) = |x-1|$. Clearly, $f(x)$ is defined for all $x \in R$. So, domain $(f) = R$.

Also, $f(x) = |x-1| \geq 0$ for all $x \in R$. So, range $(f) = [0, \infty)$.

(vii) $f(x) = -|x|$. We observe that $f(x)$ is defined for all $x \in R$. So, domain $(f) = R$.

Also, $|x| \geq 0$ for all $x \in R \Rightarrow -|x| \leq 0$ for all $x \in R \Rightarrow f(x) \leq 0$ for all $x \in R$.

So, range $(f) = (-\infty, 0]$.

(viii) We have, $f(x) = \sqrt{9-x^2}$. Clearly, $f(x)$ takes real values if

$9 - x^2 \geq 0 \Rightarrow x^2 - 9 \leq 0 \Rightarrow (x-3)(x+3) \leq 0 \Rightarrow x \in [-3, 3]$

$\therefore$ Domain $(f) = [-3, 3]$

Also, $f(x) = \sqrt{9-x^2} \geq 0$ for all $x \in [-3, 3]$.

Let $y = \sqrt{9-x^2}$. Then, $y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9 \Rightarrow x = \sqrt{9-y^2}$

Clearly, $x \in R$, if $y \in [-3, 3]$. But, $y \geq 0$. Therefore, $y \in [0, 3]$. Hence, range $(f) = [0, 3]$

3.7 SOME STANDARD REAL FUNCTIONS AND THEIR GRAPHS

In this section, we shall discuss some standard real functions which frequently occur in the study of calculus.

CONSTANT FUNCTION *If k is a fixed real number, then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function.*

Sometimes we also call it the constant function k.

We observe that the domain of the constant function $f(x) = k$ is the set R of all real numbers and range of f is the singleton set $\{k\}$.

The graph of a constant function $f(x) = k$ is a straight line parallel to x-axis (See Fig. 3.8) which is above or below x-axis according as k is positive or negative. If $k = 0$, then the straight line is coincident to x-axis.

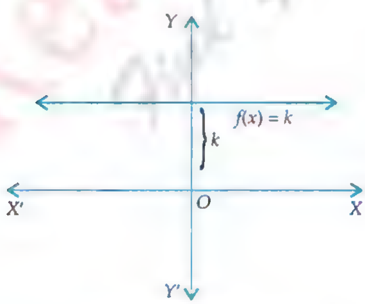


Fig. 3.8 Constant function

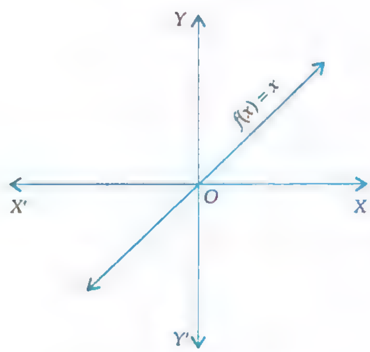


Fig. 3.9 Identity function

IDENTITY FUNCTION *The function that associates each real number to itself is called the identity function and is usually denoted by I.*

Thus, the function $I : R \rightarrow R$ defined by $I(x) = x$ for all $x \in R$ is called the identity function.

Clearly, the domain and range of the identity function are both equal to R.

The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with X-axis.

MODULUS FUNCTION *The function $f(x)$ defined by $f(x)=|x|=\begin{cases} x \ , & when\ x \ge 0 \\ -x \ , & when\ x<0 \end{cases}$ is called the modulus function.*

It is also called the absolute value function.

We observe that the domain of the modulus function is the set R of all real numbers and the range is the set of all non-negative real numbers i.e. $R^+ = \{x \in R : x \ge 0\}$.

The graph of the modulus function is as shown in Fig. 3.10. for $x \ge 0$, the graph coincides with the graph of the identity function i.e. the line $y=x$ and for $x<0$, it is coincident to the line $y=-x$.

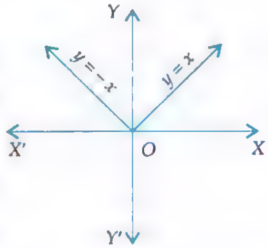


Fig. 3.10 Modulus function

PROPERTIES OF MODULUS FUNCTION The modulus function has the following properties:

(a) For any real number x, $\sqrt{x^2}=|x|$.

For example, $\sqrt{\cos^2 x} = |\cos x| = \begin{cases} \cos x \ , & 0 \le x \le \frac{\pi}{2} \\ -\cos x \ , & \frac{\pi}{2} < x \le \pi \end{cases}$

(b) If a, b are positive real numbers, then

(i) $x^2 \le a^2 \Leftrightarrow |x| \le a \Leftrightarrow -a \le x \le a$ (ii) $x^2 \ge a^2 \Leftrightarrow |x| \ge a \Leftrightarrow x \le -a$ or, $x \ge a$

(iii) $x^2 < a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$ (iv) $x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow x < -a$ or, $x > a$

(v) $a^2 \le x^2 \le b^2 \Leftrightarrow a \le |x| \le b \Leftrightarrow x \in [-b, -a] \cup [a, b]$

(vi) $a^2 < x^2 < b^2 \Leftrightarrow a < |x| < b \Leftrightarrow x \in (-b, -a) \cup (a, b)$

(c) For real numbers x and y, we have

(i) $|x+y| = |x|+|y| \Leftrightarrow (x \ge 0$ and $y \ge 0)$ or, $(x<0$ and $y<0)$

(ii) $|x-y| = |x|-|y| \Leftrightarrow (x \ge 0, y \ge 0$ and $|x| \ge |y|)$ or, $(x \le 0, y \le 0$ and $|x| \ge |y|)$

(iii) $|x \pm y| \le |x|+|y|$

(iv) $|x \pm y| \ge \big||x| - |y|\big|$

GREATEST INTEGER FUNCTION (FLOOR FUNCTION) For any real number x, we use the symbol $[x]$ or, $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x.

For example, $[2.75] = 2, [3] = 3, [0.74] = 0, [-7.45] = -8$ etc.

The function $f: R \to R$ defined by $f(x) = [x]$ for all $x \in R$ is called the greatest integer function or the floor function.

It is also called a step function.

Clearly, domain of the greatest integer function is the set R of all real numbers and the range is the set Z of all integers as it attains only integer values.

The graph of the greatest integer function is shown in Fig 3.11.

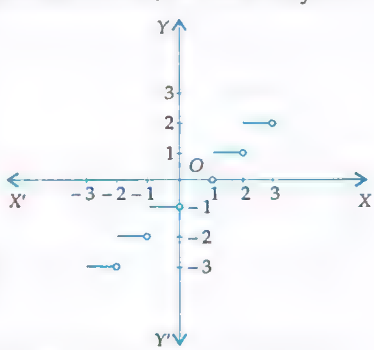


Fig. 3.11 Greatest integer function

PROPERTIES OF GREATEST INTEGER FUNCTION *If n is an integer and x is a real number between n and $n+1$, then*

(i) $[-n] = -[n]$ (ii) $[x+k] = [x]+k$ for any integer k.

(iii) $[-x] = -[x]-1$

(iv) $[x]+[-x] = \begin{cases} -1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$

(v) $[x]-[-x] = \begin{cases} 2[x]+1, & \text{if } x \notin Z \\ 2[x], & \text{if } x \in Z \end{cases}$

(vi) $[x] \geq k \Rightarrow x \geq k$, where $k \in Z$

(vii) $[x] \leq k \Rightarrow x < k+1$, where $k \in Z$

(viii) $[x] > k \Rightarrow x > k+1$, were $k \in Z$

(ix) $[x] < k \Rightarrow x < k$, where $k \in Z$

(x) $[x+y] = [x]+[y+x-[x]]$ for all $x, y \in R$

(xi) $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right] = [nx], n \in N.$

SMALLEST INTEGER FUNCTION (CEILING FUNCTION) For any real number x, we use the symbol $\lceil x \rceil$ to denote the smallest integer greater than or equal to x.

For example, $\lceil 4.7 \rceil = 5, \lceil -7.2 \rceil = -7, \lceil 5 \rceil = 5, \lceil 0.75 \rceil = 1$ etc.

The function $f: R \to R$ defined by $f(x) = \lceil x \rceil$ for all $x \in R$ is called the smallest integer function or the ceiling function.

It is also a step function.

We observe that the domain of the smallest integer function is the set R of all real numbers and its range is the set Z of all integers. The graph of the smallest integer function is as shown in Fig. 3.12.

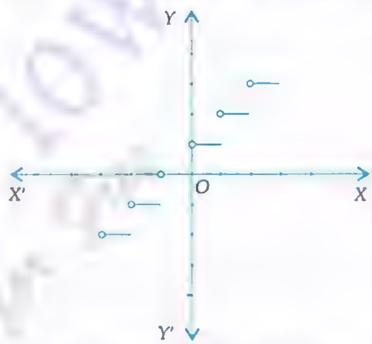


Fig. 3.12 Smallest integer function

PROPERTIES OF SMALLEST INTEGER FUNCTION Following are some properties of smallest integer function:

(i) $\lceil -n \rceil = -\lceil n \rceil$, where $n \in Z$

(ii) $\lceil -x \rceil = -\lceil x \rceil + 1$, where $x \in R - Z$

(iii) $\lceil x+n \rceil = \lceil x \rceil + n$, where $x \in R - Z$ and $n \in Z$

(iv) $\lceil x \rceil + \lceil -x \rceil = \begin{cases} 1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$

(v) $\lceil x \rceil - \lceil -x \rceil = \begin{cases} 2\lceil x \rceil - 1, & \text{if } x \notin Z \\ 2\lceil x \rceil, & \text{if } x \in Z \end{cases}$

FRACTIONAL PART FUNCTION For any real number x, we use the symbol $\{x\}$ to denote the fractional part or decimal part of x.

For example, $\{3.45\} = 0.45, \{-2.75\} = 0.25, \{-0.55\} = 0.45$, $\{3\} = 0, \{-7\} = 0$ etc.

The function $f: R \to R$ defined by $f(x) = \{x\}$ for all $x \in R$ is called the fractional part function.

We observe that the domain of the fractional part function is the set R of all real numbers and the range is the set $[0, 1)$.

It is evident from the definition that $f(x) = \{x\} = x - [x]$ for all $x \in R$

The graph of the fractional part function is as shown in Fig. 3.13.

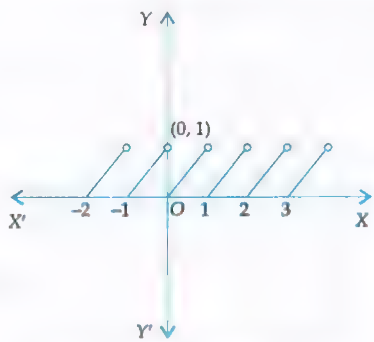


Fig. 3.13 Fractional part function

SIGNUM FUNCTION *The function f defined by* $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ or, $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

is called the signum function.

The domain of the signum function is the set R of all real numbers and the range is the set $\{-1, 0, 1\}$. The graph of the signum function is as shown in Fig. 3.14.

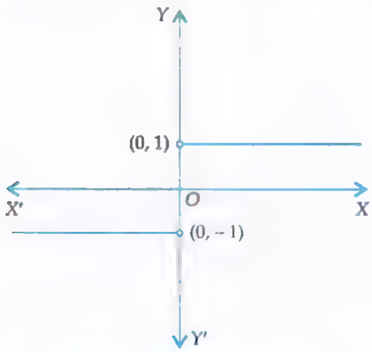


Fig. 3.14 Signum function

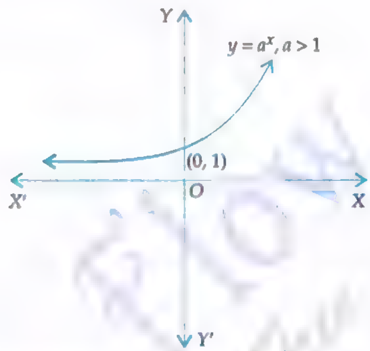


Fig. 3.15 Exponential function $f(x) = a^x$ for $a > 1$

EXPONENTIAL FUNCTION *If a is a positive real number other than unity, then a function that associates each $x \in R$ to a^x is called the exponential function.*

In other words, a function $f : R \to R$ defined by $f(x) = a^x$, where $a > 0$ and $a \neq 1$ is called the exponential function.

We observe that the domain of an exponential function is R the set of all real numbers and the range is the set $(0, \infty)$ as it attains only positive values.

As $a > 0$ and $a \neq 1$. So, we have the following cases:

CASE I *When* $a > 1$: We observe that the values of $y = f(x) = a^x$ increase as the values of x increase.

Also, $$f(x) = a^x \begin{cases} < 1 & \text{for } x < 0 \\ = 1 & \text{for } x = 0 \\ > 1 & \text{for } x > 0. \end{cases}$$

Thus, the graph of $f(x) = a^x$ for $a > 1$ as shown in Fig. 3.15.

We also observe that:

$$2^x < 3^x < 4^x < \ldots \text{ for all } x > 1$$

$$2^x = 3^x = 4^x = \ldots = 1 \text{ for } x = 0$$

$$2^x > 3^x > 4^x > \ldots \text{ for all } x < 1$$

So, the graphs of $f(x) = 2^x$, $f(x) = 3^x$, $f(x) = 4^x$ etc. are as shown in Fig. 3.16.

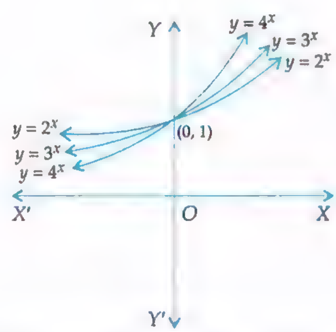


Fig. 3.16 Graphs of exponential functions

CASE II *When* $0 < a < 1$: In this case, the values of $y = f(x) = a^x$ decrease with the increase in x and $y > 0$ for all $x \in R$.

Also,

$$y = f(x) = a^x \begin{cases} > 1 & \text{for } x < 0 \\ = 1 & \text{for } x = 0 \\ < 1 & \text{for } x > 0 \end{cases}$$

Thus, the graph of $f(x) = a^x$ for $0 < a < 1$ is as shown in Fig. 3.17.

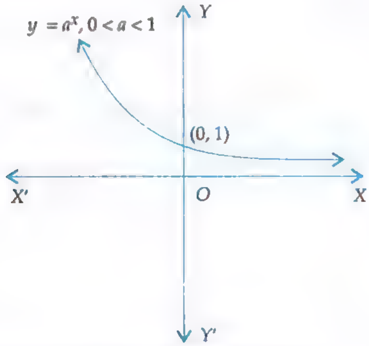


Fig. 3.17 Graph of exponential function $f(x) = a^x$ for $0 < a < 1$

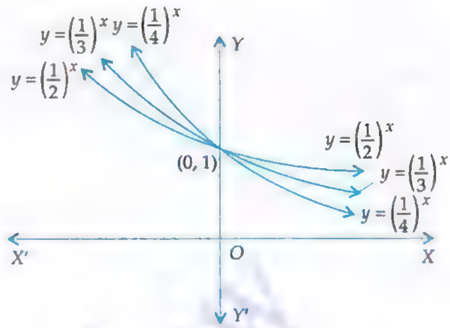


Fig. 3.18 Exponential functions

The graphs of $f(x) = a^x$, $0 < a < 1$ for different values of a are shown in Fig. 3.18.

REMARK 1 *We have, $2 < e < 3$. Therefore, graph of $f(x) = e^x$ is identical to that of $f(x) = a^x$ for $a > 1$ and the graph of $f(x) = e^{-x}$ is identical to that of $f(x) = a^x$ for $0 < a < 1$.*

LOGARITHMIC FUNCTION *If $a > 0$ and $a \neq 1$, then the function defined by $f(x) = \log_a x$, $x > 0$ is called the logarithmic function.*

We have learnt that the logarithmic function and the exponential function are inverse functions i.e. $\log_a x = y \Leftrightarrow x = a^y$.

We observe that the domain of the logarithmic function is the set of all non-negative real numbers i.e. $(0, \infty)$ and the range is the set R of all real numbers.

As $a > 0$ and $a \neq 1$. So, we have the following cases.

Case I *When $a > 1$*: In this case, we have $y = \log_a x \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1 \end{cases}$

Also, the values of y increase with the increase in x. So, the graph of $y = \log_a x$ is as shown in Fig.3.19.

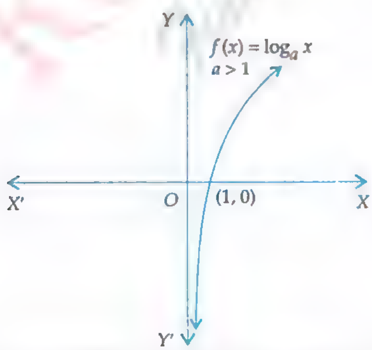


Fig. 3.19 Logarithmic function $f(x) = \log_a x$ for $a > 1$

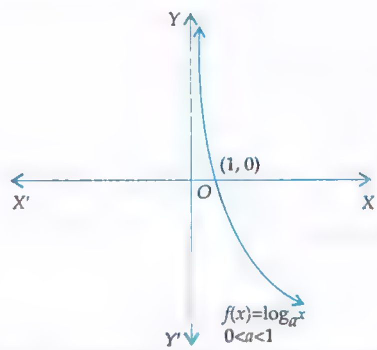


Fig. 3.20 Logarithmic function $f(x) = \log_a x$ for $0<a<1$

Case II *When $0 < a < 1$*: In this case, we have $y = \log_a x \begin{cases} > 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ < 0 & \text{for } x > 1 \end{cases}$

Also, the values of y decrease with the increase in x. So, the graph of $y = \log_a x$ is as shown in Fig. 3.20.

PROPERTIES OF LOGARITHMIC FUNCTION Following are some useful properties of logarithmic function:

(i) $\log_a 1 = 0$, where $a > 0, a \neq 1$

(ii) $\log_a a = 1$, where $a > 0, a \neq 1$

(iii) $\log_a (xy) = \log_a |x| + \log_a |y|$, where $a > 0, a \neq 1$ and $xy > 0$

(iv) $\log_a \left(\frac{x}{y}\right) = \log_a |x| - \log_a |y|$, where $a > 0, a \neq 1$ and $\frac{x}{y} > 0$

(v) $\log_a (x^n) = n \log_a |x|$, where $a > 0, a \neq 1$ and $x^n > 0$

(vi) $\log_{a^n} x^m = \frac{m}{n} \log_a x$, where $a > 0, a \neq 1$ and $x > 0$

(vii) $x^{\log_a y} = y^{\log_a x}$, where $x > 0, y > 0, a > 0, a \neq 1$

(viii) If $a > 1$, then the values of $f(x) = \log_a x$ increase with the increase in x.

i.e $x < y \Leftrightarrow \log_a x < \log_a y$. Also, $\log_a x \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1. \end{cases}$

(ix) If $0 < a < 1$, then the values of $f(x) = \log_a x$ decrease with the increase in x.

i.e $x < y \Leftrightarrow \log_a x > \log_a y$. Also, $\log_a x \begin{cases} > 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ < 0 & \text{for } x > 1 \end{cases}$

(x) $\log_a x = \frac{1}{\log_x a}$ for $a > 0, a \neq 1$ and $x > 0, x \neq 1$.

REMARK 2 *Functions $f(x) = \log_a x$ and $g(x) = a^x$ are inverse of each other. So, their graphs are mirror images of each other in the line mirror $y = x$.*

RECIPROCAL FUNCTION The function that associates a real number x to its reciprocal $1/x$ is called the reciprocal function. Since $1/x$ is not defined for $x = 0$. So, we define the reciprocal function as follows:

DEFINITION *The function $f : R - \{0\} \to R$ defined by $f(x) = \frac{1}{x}$ is called the reciprocal function.*

Clearly, domain of the reciprocal function is $R - \{0\}$ and its range is also $R - \{0\}$.

We observe that the sign of $\frac{1}{x}$ is same as that of x and $\frac{1}{x}$ decreases with the increase in x. So, the graph of $f(x) = \frac{1}{x}$ is as shown in Fig. 3.21.

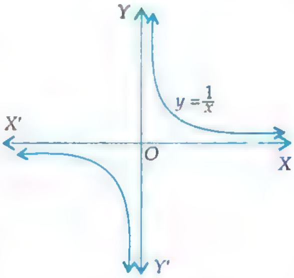


Fig. 3.21 Reciprocal function

SQUARE ROOT FUNCTION The function that associates a real number x to $+\sqrt{x}$ is called the square root function. Since $\sqrt{x}$ is real for $x > 0$. So, we defined the square root function as follows:

DEFINITION *The function $f : R^+ \to R$ defined by $f(x) = +\sqrt{x}$ is called the square root function.*

Clearly, domain of the square root function is R^+ *i.e.* $[0, \infty)$ and its range is also $[0, \infty)$.

We observe that the values of $f(x) = +\sqrt{x}$ increase with the increase in x. So, the graph of $f(x) = +\sqrt{x}$ is as shown in Fig. 3.22.

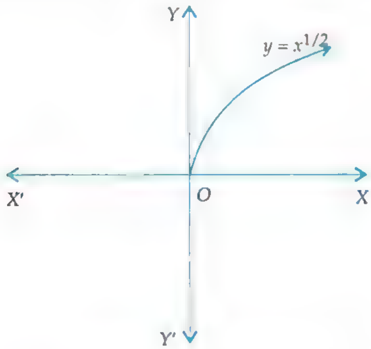


Fig. 3.22 Square root function

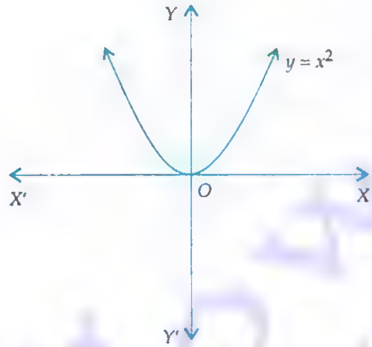


Fig. 3.23 Square function

SQUARE FUNCTION The function that associates a real number x to its square i.e. x^2 is called the square function. Since x^2 is defined for all $x \in R$. So, we define the square function as follows:

DEFINITION *The function $f: R \to R$ defined by $f(x) = x^2$ is called the square function.*

Clearly, domain of the square function is R and its range is the set of all non-negative real numbers i.e. $[0, \infty)$. The graph of $f(x) = x^2$ is parabola as shown in Fig. 3.23.

CUBE FUNCTION The function that associate a real number x to its cube is called the cube function. We observe that x^3 is meaningful for all $x \in R$. So, we define the cube function as follows:

DEFINITION *The function $f: R \to R$ defined by $f(x) = x^3$ is called the cube function.*

We observe that the sign of x^3 is same as that of x and the values of x^3 increase with the increase in x. So, the graph of $f(x) = x^3$ is as shown in Fig. 3.24. Clearly, the graph is symmetrical in opposite quadrants.

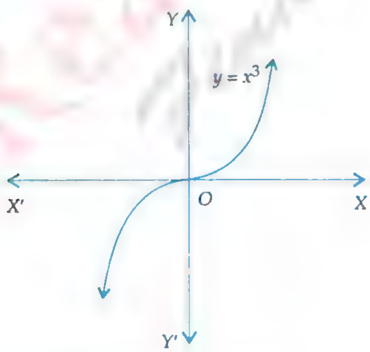


Fig. 3.24 Cube function

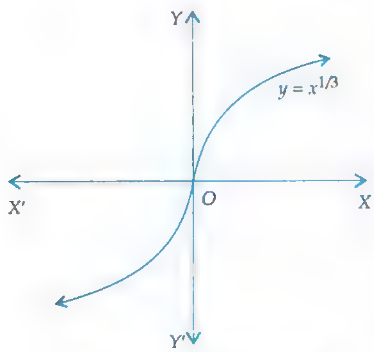


Fig. 3.25 Cube root function

CUBE ROOT FUNCTION The function that associates a real number x to its cube root $x^{1/3}$ is called the cube root function. Clearly, $x^{1/3}$ is defined for all $x \in R$. So, we define the cube root function as follows:

DEFINITION *The function $f: R \to R$ defined by $f(x) = x^{1/3}$ is called the cube root function.*

Clearly, domain and range of the cube root function are both equal to R.

Also, the sign of $x^{1/3}$ is same as that of x and $x^{1/3}$ increase with the increase in x. So, the graph of $f(x) = x^{1/3}$ is as shown in Fig. 3.25.

REMARK 3 *A function $f : R \to R$ is said to be a polynomial function if $f(x)$ is a polynomial in x. For example, $f(x) = x^2 - x + 4$, $g(x) = x^3 + 3x^2 + \sqrt{2}\,x - 1$ etc are polynomial functions.*

REMARK 4 *A function of the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function. The domain of a rational function $f(x) = \dfrac{p(x)}{q(x)}$ is the set of all real numbers, except points where $q(x) = 0$.*

3.8 OPERATIONS ON REAL FUNCTIONS

In this section, we shall introduce various operations, namely addition, subtraction, multiplication, division etc. on real functions.

ADDITION *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then, their sum $f + g$ is defined as that function from $D_1 \cap D_2$ to R which associates each $x \in D_1 \cap D_2$ to the number $f(x) + g(x)$.*

In other words, if $f : D_1 \to R$ and $g : D_2 \to R$ are two real functions, then their sum $f + g$ is a function from $D_1 \cap D_2$ to R such that

$$(f+g)(x) = f(x) + g(x) \quad \text{for all } x \in D_1 \cap D_2.$$

PRODUCT *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then, their product (or pointwise multiplication) $f\,g$ is a function from $D_1 \cap D_2$ to R and is defined as*

$$(f\,g)(x) = f(x)\,g(x) \quad \textit{for all } x \in D_1 \cap D_2$$

DIFFERENCE (SUBTRACTION) *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then the difference of g from f is denoted by $f - g$ and is defined as*

$$(f-g)(x) = f(x) - g(x) \quad \textit{for all } x \in D_1 \cap D_2$$

QUOTIENT *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then the quotient of f by g is denoted by $\dfrac{f}{g}$ and it is a function from $D_1 \cap D_2 - \{x : g(x) = 0\}$ to R defined by*

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \textit{for all } x \in D_1 \cap D_2 - \{x : g(x) = 0\}$$

MULTIPLICATION OF A FUNCTION BY A SCALAR *Let $f : D \to R$ ba a real function and α be a scalar (real number). Then the product $\alpha\,f$ is a function from D to R and is defined as*

$$(\alpha f)(x) = \alpha f(x) \quad \textit{for all } x \in D.$$

RECIPROCAL OF A FUNCTION *If $f : D \to R$ is a real function, then its reciprocal function $\dfrac{1}{f}$ is a function from $D - \{x : f(x) = 0\}$ to R and is defined as $\left(\dfrac{1}{f}\right)(x) = \dfrac{1}{f(x)}$.*

REMARK 1 *The sum, difference product and quotient are defined for real functions only on their common domain. These operations do not make any sense for general functions even if their domains are same, because the sum, difference, product and quotient may or may not be meaningful for the elements in their common domain.*

REMARK 2 *For any real function $f : D \cap R$ and $n \in N$, we define*

$$\underbrace{(f\,f\,f\ldots f)}_{n-\text{times}}(x) = \underbrace{f(x)\,f(x)\ldots f(x)}_{n-\text{times}} = \{f(x)\}^n \quad \text{for all } x \in D$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the sum and difference of the identity function and the modulus function.* [NCERT]

SOLUTION We know that $f : R \to R$ defined by $f(x) = x$ is the identity function and $g : R \to R$ defined by $g(x) = |x|$ is the modulus function. Clearly, f and g have the same domain. Therefore, $f + g : R \to R$ and $f - g : R \to R$ such that

$$(f+g)(x) = f(x) + g(x) = x + |x| = \begin{cases} x + x, & \text{if } x \geq 0 \\ x - x, & \text{if } x < 0 \end{cases} = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

and, $$(f-g)(x) = f(x) - g(x) = x - |x| = \begin{cases} x - x, & \text{if } x \geq 0 \\ x - (-x), & \text{if } x < 0 \end{cases} = \begin{cases} 0, & \text{if } x \geq 0 \\ 2x, & \text{if } x < 0 \end{cases}$$

Thus, $f + g : R \to R$ and $f - g : R \to R$ are defined as

$$(f+g)(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \quad \text{and, } (f-g)(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ 2x, & \text{if } x < 0 \end{cases}$$

EXAMPLE 2 *What are the sum and difference of the identity function and the reciprocal function?* [NCERT]

SOLUTION Let f and g denote respectively the identity function and the reciprocal function. Then, $f : R \to R$ and $g : R - \{0\} \to R$ such that $f(x) = x$ for all $x \in R$ and, $g(x) = \frac{1}{x}$ for all $x \in R - \{0\}$. The domains of f and g are R and $R - \{0\}$ respectively. Also, we have $R \cap R - \{0\} = R - \{0\}$. Therefore, $f + g : R - \{0\} \to R$ and $f - g : R - \{0\} \to R$ are given by

$$(f+g)(x) = f(x) + g(x) = x + \frac{1}{x} \text{ and, } (f-g)(x) = f(x) - g(x) = x - \frac{1}{x}$$

EXAMPLE 3 *Let $f : [2, \infty) \to R$ and $g : [-2, \infty) \to R$ be two real functions defined by $f(x) = \sqrt{x-2}$ and $g(x) = \sqrt{x+2}$. Find $f + g$ and $f - g$.*

SOLUTION Let $D_1 = [2, \infty)$ and $D_2 = [-2, \infty)$. Then, $D_1 \cap D_2 = [2, \infty)$. Thus, $f + g : [2, \infty) \to R$ and $f - g : [2, \infty) \to R$ are given by

$$(f+g)(x) = f(x) + g(x) = \sqrt{x-2} + \sqrt{x+2} \quad \text{for all } x \in [2, \infty)$$

and, $$(f-g)(x) = f(x) - g(x) = \sqrt{x-2} - \sqrt{x+2} \quad \text{for all } x \in [2, \infty).$$

EXAMPLE 4 *Find the product of the identity function and the modulus function.*

SOLUTION Let f and g denote respectively the identity function and modulus function. Then, $f : R \to R$ such that $f(x) = x$ for all x and, $g : R \to R$ such that $g(x) = |x|$ for all x. Clearly, f and g have the same domain. Therefore, the product $f\,g$ is a function from R to itself and is given by

$$(f\,g)(x) = f(x)\,g(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

EXAMPLE 5 *Find the quotient of the identity function by the modulus function.*

SOLUTION Let f and g denote respectively the identity function and the modulus function. Then, $f : R \to R$ is defined as $f(x) = x$ and, $g : R \to R$ is defined as $g(x) = |x|$. Clearly, f and g have the same domain.

And, $\quad g(x)=0 \Rightarrow |x|=0 \Rightarrow x=0.$

Therefore, the quotient of f by g i.e. $\frac{f}{g}$ is a function from $R-\{0\} \to R$ and is defined as

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{x}{|x|}=\begin{cases}\frac{x}{x}=1, & x>0\\ \frac{x}{-x}=-1, & x<0\end{cases}$$

EXAMPLE 6 *Find the product of the identity function and the reciprocal function.*

SOLUTION Let f and g denote respectively the identity function and the reciprocal function. Then, $f: R \to R$ is defined as $f(x)=x$ for all $x \in R$ and, $g: R-\{0\} \to R$ is defined as $g(x)=\frac{1}{x}$ for all $x \in R-\{0\}$. We find that Domain $(f) \cap$ Domain $(g) = R \cap R-\{0\}=R-\{0\}$. Therefore, the product fg is a function from $R-\{0\}$ to R and is defined as

$$(fg)(x)=f(x)\,g(x)=x \times \frac{1}{x}=1 \text{ for all } x \in R-\{0\}$$

Thus, $fg: R-\{0\} \to R$ is given by $(fg)(x)=1$ for all $x \in R-\{0\}$.

EXAMPLE 7 *Find the quotient of the identity function by the reciprocal function.*

SOLUTION Let f and g denote respectively the identity function and the reciprocal function. Then, $f: R \to R$ is defined as $f(x)=x$ for all $x \in R$ and, $g: R-\{0\} - R$ is defined as $g(x)=\frac{1}{x}$ for all $x \in R-\{0\}$. We find that Domain $(f) \cap$ Domain $(g)=R \cap R-\{0\}=R-\{0\}$. And, $g(x) \neq 0$ for any $x \in R-\{0\}$.

$\therefore \quad \frac{f}{g}$ is a function from $R-\{0\} \to R$ and is given by $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{x}{1/x}=x^2$

Hence, $\frac{f}{g}: R-\{0\} \to R$ is given by $\left(\frac{f}{g}\right)(x)=x^2$ for all $x \in R-\{0\}$.

EXAMPLE 8 *Let c be a non-zero real number and $f: R \to R$ be a function defined by $f(x)=\frac{x}{c}$ for all $x \in R$. Find* (i) cf (ii) $c^2 f$ (iii) $\left(\frac{1}{c}\right) f$.

SOLUTION Clearly, cf, $c^2 f$ and $\left(\frac{1}{c}\right) f$ are functions from R to itself such that

(i) $\quad (cf)(x)=c\,f(x)=c \times \frac{x}{c}=x$ for all $x \in R$

(ii) $\quad (c^2 f)(x)=c^2 f(x)=c^2 \times \frac{x}{c}=cx$ for all $x \in R$

(iii) $\quad \left(\left(\frac{1}{c}\right) f\right)(x)=\left(\frac{1}{c}\right) f(x)=\frac{1}{c} \times \frac{x}{c}=\frac{x}{c^2}$ for all $x \in R$.

EXAMPLE 9 *Let f and g be two real functions defined by $f(x)=\frac{1}{x+4}$ and $g(x)=(x+4)^3$.*

Find the following: (i) $f+g$ (ii) $f-g$ (iii) fg (iv) $\frac{f}{g}$ (v) $2f$ (vi) $\frac{1}{f}$ (vii) $\frac{1}{g}$

SOLUTION We observe that $f(x)=\frac{1}{x+4}$ is defined for all $x \neq -4$. So, domain $(f)=R-\{-4\}$.

Clearly, $g(x)=(x+4)^3$ is defined for all $x \in R$. So, domain $(g)=R$. We find that

Domain $(f) \cap$ Domain $(g) = R - \{-4\}$. Therefore,

(i) $f+g : R-\{-4\} \to R$ is given by $(f+g)(x) = f(x) + g(x) = \dfrac{1}{x+4} + (x+4)^3 = \dfrac{(x+4)^4+1}{x+4}$

(ii) $f-g : R-\{-4\} \to R$ is defined as $(f-g)(x) = f(x) - g(x) = \dfrac{1}{x+4} - (x+4)^3 = \dfrac{1-(x+4)^4}{x+4}$

(iii) $fg : R-\{-4\} \to R$ is given by $(fg)(x) = f(x)\, g(x) = \dfrac{1}{x+4} \times (x+4)^3 = (x+4)^2$

(iv) $g(x) = 0 \Rightarrow (x+4)^3 = 0 \Rightarrow x = -4.$

$\therefore$ Domain $\left(\dfrac{f}{g}\right)$ = Domain $(f) \cap$ Domain $(g) - \{x : g(x) = 0\} = R - \{-4\}$. Therefore,

$\dfrac{f}{g} : R-\{-4\} \to R$ is given by $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{1}{(x+4)^4}$

(v) $2f : R-\{-4\} \to R$ is given by $(2f)(x) = 2(f(x)) = \dfrac{2}{x+4}$ for all $x \in R - \{-4\}$.

(vi) We observe that $f(x) \neq 0$ for any $x \in R - \{-4\}$. Therefore, $\dfrac{1}{f} : R-\{-4\} \to R$ is given by

$$\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)} = \frac{1}{1/(x+4)} = (x+4)$$

(vii) We observe that $g(x) = (x+4)^3 = 0$ for $x = -4$. Therefore, $\dfrac{1}{g} : R-\{-4\} \to R$ is given by

$$\left(\frac{1}{g}\right)(x) = \frac{1}{g(x)} = \frac{1}{(x+4)^3}$$

EXAMPLE 10 *Let f and g be real functions defined by $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{4-x^2}$. Then, find each of the following functions:*

(i) $f+g$ (ii) $f-g$ (iii) fg (iv) $\dfrac{f}{g}$ (v) ff (vi) gg

SOLUTION We have, $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{4-x^2}$. Clearly, $f(x)$ is defined for all x satisfying

$x+2 \geq 0 \Rightarrow x \geq -2 \Rightarrow x \in [-2, \infty)$. Therefore, Domain $(f) = [-2, \infty)$

We observe that $g(x)$ is defined for all x satisfying

$4 - x^2 \geq 0 \Rightarrow x^2 - 4 \leq 0 \Rightarrow (x-2)(x+2) \leq 0 \Rightarrow x \in [-2, 2]$. Therefore, Domain $(g) = [-2, 2]$.

We find that: Domain $(f) \cap$ Domain $(g) = [-2, \infty) \cap [-2, 2] = [-2, 2]$. Therefore,

(i) $f+g : [-2, 2] \to R$ is given by $(f+g)(x) = f(x) + g(x) = \sqrt{x+2} + \sqrt{4-x^2}$

(ii) $f-g : [-2, 2] \to R$ is given by $(f-g)(x) = f(x) - g(x) = \sqrt{x+2} - \sqrt{4-x^2}$

(iii) $fg : [-2, 2] \to R$ is given by

$(fg)(x) = f(x)\, g(x) = \sqrt{x+2} \times \sqrt{4-x^2} = \sqrt{(x+2)^2 (2-x)} = (x+2)\sqrt{2-x}$

(iv) We have, $g(x) = \sqrt{4-x^2}$. Therefore, $g(x) = 0 \Rightarrow 4 - x^2 = 0 \Rightarrow x = \pm 2$. Therefore,

Domain $\left(\dfrac{f}{g}\right) = [-2, 2] - \{-2, 2\} = (-2, 2)$

Thus, $\frac{f}{g}:(-2, 2) \to R$ is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{(x+2)}}{\sqrt{4-x^2}} = \frac{1}{\sqrt{2-x}}$

(v) Since domain $(f) = [-2, \infty)$. Therefore,

$$(ff)(x) = f(x)\, f(x) = [f(x)]^2 = (\sqrt{x+2})^2 = x + 2 \text{ for all } x \in [-2, \infty)$$

(vi) Since domain $(g) = [-2, 2)$. Therefore,

$$(gg)(x) = g(x)\, g(x) = [g(x)]^2 = \left(\sqrt{4-x^2}\right)^2 = 4 - x^2 \text{ for all } x \in [-2, 2]$$

EXAMPLE 11 *Let f be the exponential function and g be the logarithmic function. Find:*

(i) $(f+g)(1)$ (ii) $(fg)(1)$ (iii) $(3f)(1)$ (iv) $(5g)(1)$

SOLUTION We have, $f: R \to R$ given by $f(x) = e^x$ and, $g: R^+ \to R$ given by $g(x) = \log_e x$.

(i) We find that: Domain $(f) \cap$ Domain $(g) = R \cap R^+ = R^+$. Therefore, $f + g : R^+ \to R$ is given by

$$(f+g)(x) = f(x) + g(x) = e^x + \log_e x \text{ for all } x \in R^+$$

$$\Rightarrow \quad (f+g)(1) = e^1 + \log_e 1 = e + 0 = e.$$

(ii) Domain $(f) \cap$ Domain $(g) = R \cap R^+ = R^+$. Therefore, $fg : R^+ \to R$ is given by

$$(fg)(x) = f(x)\, g(x) = e^x . \log_e x \Rightarrow (fg)(1) = e^1 \times \log_e 1 = e \times 0 = 0$$

(iii) Clearly, $(3f)(x) = 3(f(x)) = 3e^x$. Therefore, $(3f)(1) = 3e^1 = 3e$

(iv) Clearly, $(5g)(x) = 5(g(x)) = 5 \log_e x$. Therefore, $(5g)(1) = 5 \log_e 1 = 5 \times 0 = 0$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 12 *Find the domain of each of the following functions given by*

(i) $f(x) = \frac{1}{\sqrt{x - |x|}}$ [NCERT EXEMPLAR] (ii) $f(x) = \frac{1}{\sqrt{x + |x|}}$ [NCERT EXEMPLAR]

(iii) $f(x) = \frac{1}{\sqrt{x - [x]}}$ (iv) $f(x) = \frac{1}{\sqrt{x + [x]}}$

SOLUTION (i) We have, $f(x) = \frac{1}{\sqrt{x-|x|}}$. We know that $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$$\therefore \quad x - |x| = \begin{cases} x - x = 0, & \text{if } x \geq 0 \\ x + x = 2x, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \quad x - |x| \leq 0 \quad \text{for all } x$$

$\Rightarrow \quad \frac{1}{\sqrt{x-|x|}}$ does not take real values for any $x \in R \Rightarrow f(x)$ is not defined for any $x \in R$.

Hence, Domain $(f) = \phi$.

(ii) We have, $f(x) = \frac{1}{\sqrt{x+|x|}}$. We know that $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$$\therefore \quad x + |x| = \begin{cases} x + x, & \text{if } x \geq 0 \\ x - x, & \text{if } x < 0 \end{cases} \Rightarrow x + |x| = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \quad ...(i)$$

Thus, $f(x) = \frac{1}{\sqrt{x+|x|}}$ assumes real values, if $x + |x| > 0 \Rightarrow x > 0 \Rightarrow x \in (0, \infty)$. [Using (i)]

Hence, Domain $(f) = (0, \infty)$.

(iii) We have, $f(x) = \frac{1}{\sqrt{x-[x]}}$. We know that $0 \le x-[x] < 1$ for all $x \in R$. Also, $x-[x]=0$ for $x \in Z$. Thus, $f(x) = \frac{1}{\sqrt{x-[x]}}$ is defined, if

$x-[x] > 0 \Rightarrow x \in R - Z$ $\quad [\because x-[x]=0$ for $x \in Z$ and $0 < x-[x] < 1$ for $x \in R-Z]$

Hence, Domain $(f) = R - Z$.

(iv) We have, $f(x) = \frac{1}{\sqrt{x+[x]}}$. We know that

$$\left.\begin{array}{lll} x+[x] > 0 & \text{for all } x > 0 \\ x+[x] = 0 & \text{for } x = 0 \\ x+[x] < 0 & \text{for all } x < 0 \end{array}\right\} \quad ...(i)$$

Clearly, $f(x) = \frac{1}{\sqrt{x+[x]}}$ is defined for all x satisfying $x+[x] > 0$. Therefore, from (i), we find that, Domain $(f) = (0, \infty)$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 13 *Find the domain of definition of the function $f(x)$ given by*

$$f(x) = \log_4 \left\{ \log_5 \left(\log_3 (18x - x^2 - 77 \right) \right\}$$

SOLUTION We have, $(x) = \log_4 \left\{ \log_5 \left(\log_3 (18x - x^2 - 77) \right) \right\}$. Since $\log_a x$ is defined for all $x > 0$. Therefore, $f(x)$ is defined if

$\log_5 \{\log_3 (18x - x^2 - 77)\} > 0$ and $18x - x^2 - 77 > 0$

$\Rightarrow \log_3 (18x - x^2 - 77) > 5^0$ and $x^2 - 18x + 77 < 0$

$\Rightarrow \log_3 (18x - x^2 - 77) > 1$ and $(x-11)(x-7) < 0$

$\Rightarrow 18x - x^2 - 77 > 3^1$ and $7 < x < 11$

$\Rightarrow 18x - x^2 - 80 > 0$ and $7 < x < 11$

$\Rightarrow x^2 - 18x + 80 < 0$ and $7 < x < 11$

$\Rightarrow (x-10)(x-8) < 0$ and $7 < x < 11 \Rightarrow 8 < x < 10$ and $7 < x < 11 \Rightarrow 8 < x < 10 \Rightarrow x \in (8, 10)$.

Hence, the domain of $f(x)$ is $(8, 10)$.

EXAMPLE 14 *Find the domain of definition of the function $f(x)$ given by $f(x) = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$.*

SOLUTION We have, $f(x) = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$. Let $g(x) = \frac{1}{\log_{10}(1-x)}$ and $h(x) = \sqrt{x+2}$.

Then, $f(x) = g(x) + h(x)$. Therefore, Domain (f) = Domain $(g) \cap$ Domain (h).

Now, $g(x) = \frac{1}{\log_{10}(1-x)}$ is defined for all x for which $\log_{10}(1-x)$ is defined and

$\log_{10}(1-x) \ne 0 \Rightarrow 1-x > 0$ and $1-x \ne 1 \Rightarrow x < 1$ and $x \ne 0 \Rightarrow x \in (-\infty, 0) \cup (0, 1)$

$\therefore$ Domain $(g) = (-\infty, 0) \cup (0, 1)$.

And, $h(x) = \sqrt{x+2}$ is defined for all x satisfying $x+2 \geq 0 \Rightarrow x \geq -2 \Rightarrow x \in [-2, \infty)$.

$\therefore$ Domain $(h) = [-2, \infty)$.

Hence, Domain (f) = Domain $(g) \cap$ domain $(h) = (-\infty, 0) \cup (0, 1) \cap [-2, \infty) = [-2, 0) \cup (0, 1)$

EXAMPLE 15 *Find the range of each of the following functions:*

(i) $f(x) = |x-3|$ [NCERT EXEMPLAR] (ii) $f(x) = 1 - |x-2|$ [NCERT EXEMPLAR]

(iii) $f(x) = \dfrac{|x-4|}{x-4}$ [NCERT EXEMPLAR]

SOLUTION (i) We have, $(x) = |x-3|$. Clearly, $f(x)$ is defined for all $x \in R$. Therefore, Domain $(f) = R$. We find that

$|x-3| > 0$ for all $x \in R$

$\Rightarrow$ $0 \leq |x-3| < \infty$ for all $x \in R \Rightarrow 0 \leq f(x) < \infty$ for all $x \in R \Rightarrow f(x) \in [0, \infty)$ for all $x \in R$

Hence, Range $(f) = [0, \infty)$.

(ii) We have, $f(x) = 1 - |x-2|$. We observe that $f(x)$ is defined for all $x \in R$. Therefore, Domain $(f) = R$.

Now, $0 \leq |x-2| < \infty$ for all $x \in R$

$\Rightarrow$ $-\infty < -|x-2| \leq 0$ for all $x \in R$

$\Rightarrow$ $-\infty < 1 - |x-2| \leq 1$ for all $x \in R \Rightarrow -\infty < f(x) \leq 1$ for all $x \in R \Rightarrow f(x) \in (-\infty, 1]$

Hence, Range $(f) = (-\infty, 1]$

(iii) We have, $f(x) = \dfrac{|x-4|}{x-4}$. Clearly, $f(x)$ is defined for all $x \in R$ except at $x = 4$. Therefore,

Domain $(f) = R - \{4\}$. We find that

$$f(x) = \frac{|x-4|}{x-4} = \begin{cases} \dfrac{x-4}{x-4} = 1, & \text{if } x > 4 \\ \dfrac{-(x-4)}{x-4} = -1, & \text{if } x < 4 \end{cases}$$

Thus, $f(x)$ takes only two values -1 and 1. Hence, Range $(f) = \{-1, 1\}$.

EXAMPLE 16 *Find the domain and range of each of the following functions given by*

(i) $f(x) = \dfrac{1}{\sqrt{x-[x]}}$ (ii) $f(x) = 1 - |x-3|$

SOLUTION (i) We have, $f(x) = \dfrac{1}{\sqrt{x-[x]}}$.

Domain of f: We know that $0 \leq x - [x] < 1$ for all $x \in R$ and, $x - [x] = 0$ for $x \in Z$.

$\therefore$ $0 < x - [x] < 1$ for all $x \in R - Z \Rightarrow f(x) = \dfrac{1}{\sqrt{x-[x]}}$ exists for all $x \in R - Z$.

Hence, Domain $(f) = R - Z$.

Range of f: As discussed above

$0 < x - [x] < 1$ for all $x \in R - Z$

$\Rightarrow$ $0 < \sqrt{x-[x]} < 1$ for all $x \in R - Z$

$\Rightarrow$ $1 < \dfrac{1}{\sqrt{x-[x]}} < \infty$ for all $x \in R - Z \Rightarrow 1 < f(x) < \infty$ for all $x \in R - Z \Rightarrow$ Range $(f) = (1, \infty)$.

(ii) We have, $f(x) = 1 - |x-3|$. Clearly, $f(x)$ is defined for all $x \in R$. Therefore, Domain $(f) = R$.

Range of f: For any $x \in R$, we find that

$$|x-3| \geq 0 \Rightarrow -|x-3| \leq 0 \Rightarrow 1 - |x-3| \leq 1 \Rightarrow f(x) \leq 1 \Rightarrow f(x) \in (-\infty, 1]$$

Hence, Range$(f) = (-\infty, 1]$.

EXAMPLE 17 *Find the domain of the real function $f(x)$ defined by* $f(x) = \sqrt{\dfrac{1-|x|}{2-|x|}}$.

SOLUTION We have, $f(x) = \sqrt{\dfrac{1-|x|}{2-|x|}}$. We observe that $f(x)$ is defined for all x satisfying

$\dfrac{1-|x|}{2-|x|} \geq 0.$

Fig. 3.26 Signs of $\frac{1-|x|}{2-|x|}$ for different values of $|x|$

Now, $\dfrac{1-|x|}{2-|x|} \geq 0 \Rightarrow \dfrac{|x|-1}{|x|-2} \geq 0 \Rightarrow |x| \leq 1$ or, $|x| > 2$ [$\because$ See Fig. 3.26]

$\Rightarrow \quad x \in [-1, 1]$ or, $x \in (-\infty, -2) \cup (2, \infty) \Rightarrow x \in (-\infty, -2) \cup (2, \infty) \cup [-1, 1]$

Hence, domain $(f) = (-\infty, -2) \cup (2, \infty) \cup [-1, 1]$.

EXAMPLE 18 *Find the domain of the function f given by* $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$ [NCERT EXEMPALR]

SOLUTION We have, $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$

Clearly, $f(x)$ is defined for all x satisfying

$[x]^2 - [x] - 6 > 0$

$\Rightarrow \quad ([x]-3)([x]+2) > 0$

$\Rightarrow \quad [x] < -2$ or $[x] > 3$

$\Rightarrow \quad x \in (-\infty, -2)$ or $x \in [4, \infty) \Rightarrow x \in (-\infty, -2) \cup [4, \infty)$

Hence, domain $(f) = (-\infty, -2) \cup [4, \infty)$.

Fig. 3.27 Signs of $[x]^2 - [x] - 6$ for different values of x

EXERCISE 3.4

BASIC

1. Find $f+g$, $f-g$, $cf\ (c \in R, c \neq 0)$, fg, $\dfrac{1}{f}$ and $\dfrac{f}{g}$ in each of the following:

   (i) $f(x) = x^3 + 1$ and $g(x) = x + 1$ (ii) $f(x) = \sqrt{x-1}$ and $g(x) = \sqrt{x+1}$.

2. Let $f(x) = 2x + 5$ and $g(x) = x^2 + x$. Describe (i) $f+g$ (ii) $f-g$ (iii) fg (iv) f/g. Find the domain in each case.

3. If $f(x)$ be defined on $[-2, 2]$ and is given by $f(x) = \begin{cases} -1, & -2 \leq x \leq 0 \\ x-1, & 0 < x \leq 2 \end{cases}$

   and $g(x) = f(|x|) + |f(x)|$. Find $g(x)$.

4. Let f, g be two real functions defined by $f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{9-x^2}$. Then, describe each of the following functions:

   (i) $f+g$ (ii) $g-f$ (iii) fg (iv) f/g

   (v) $\dfrac{g}{f}$ (vi) $2f - \sqrt{5}\,g$ (vii) $f^2 + 7f$ (viii) $\dfrac{5}{g}$

5. If $f(x) = \log_e(1-x)$ and $g(x) = [x]$, then determine each of the following functions:

(i) $f+g$ (ii) fg (iii) $\dfrac{f}{g}$ (iv) $\dfrac{g}{f}$

Also, find $(f+g)(-1)$, $(fg)(0)$, $\left(\dfrac{f}{g}\right)\left(\dfrac{1}{2}\right)$, $\left(\dfrac{g}{f}\right)\left(\dfrac{1}{2}\right)$.

BASED ON LOTS

6. If f, g, h are real functions defined by $f(x) = \sqrt{x+1}$, $g(x) = \dfrac{1}{x}$ and $h(x) = 2x^2 - 3$, then find the values of $(2f + g - h)(1)$ and $(2f + g - h)(0)$.

7. The function f is defined by $f(x) = \begin{cases} 1-x, & x<0 \\ 1, & x=0 \\ x+1, & x>0 \end{cases}$. Draw the graph of $f(x)$. [NCERT]

8. Let $f, g : R \to R$ be defined, respectively by $f(x) = x+1$, $g(x) = 2x-3$. Find $f+g$, $f-g$ and $\dfrac{f}{g}$. [NCERT]

9. Let $f : [0, \infty) \to R$ and $g : R \to R$ be defined by $f(x) = \sqrt{x}$ and $g(x) = x$. Find $f+g$, $f-g$, fg and $\dfrac{f}{g}$. [NCERT]

10. Let $f(x) = x^2$ and $g(x) = 2x+1$ be two real functions. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ and $\left(\dfrac{f}{g}\right)(x)$. [NCERT]

ANSWERS

1. (i) $f+g : R \to R$ given by $(f+g)(x) = x^3 + x + 2$

$f-g : R \to R$ given by $(f-g)(x) = x^3 - x$

$cf : R \to R$ given by $(cf)(x) = c(x^3+1)$

$fg : R \to R$ given by $(fg)(x) = (x+1)^2(x^2-x+1)$

$\dfrac{1}{f} : R - \{-1\} \to R$ given by $\left(\dfrac{1}{f}\right)(x) = \dfrac{1}{x^3+1}$

$\dfrac{f}{g} : R - \{-1\} \to R$ given by $\left(\dfrac{f}{g}\right)(x) = x^2 + x + 1$

(ii) $f \pm g : [1, \infty) \to R$ defined by $(f+g)(x) = \sqrt{x-1} \pm \sqrt{x+1}$

$cf : [1, \infty) \to R$ defined by $(cf)(x) = c\sqrt{x-1}$

$fg : [1, \infty) \to R$ defined by $(fg)(x) = \sqrt{x^2-1}$

$\dfrac{1}{f} : (1, \infty) \to R$ defined by $\left(\dfrac{1}{f}\right)(x) = \dfrac{1}{\sqrt{x-1}}$

$\dfrac{f}{g} : [1, \infty) \to R$ defined by $\left(\dfrac{f}{g}\right)(x) = \sqrt{\dfrac{x-1}{x+1}}$

2. (i) $(f+g)(x) = x^2 + 3x + 5$; dom $(f+g) = R$ (ii) $(f-g)(x) = 5 + x - x^2$; dom $(f-g) = R$

(iii) $(fg)(x) = 2x^3 + 7x^2 + 5x$; dom $(fg) = R$ (iv) $\left(\dfrac{f}{g}\right)(x) = \dfrac{2x+5}{x^2+x}$, dom $\left(\dfrac{f}{g}\right) = R - \{0, 1\}$

3. $g(x) = \begin{cases} -x, & -2 \le x \le 0 \\ 0, & 0 < x < 1 \\ 2(x-1), & 1 \le x \le 2 \end{cases}$

4. (i) $f + g : [-1, 3] \to R$ defined by $(f + g)(x) = \sqrt{x+1} + \sqrt{9-x^2}$

(ii) $g - f : [-1, 3] \to R$ defined by $(g - f)(x) = \sqrt{9-x^2} - \sqrt{x+1}$

(iii) $fg : [-1, 3] \to R$ defined by $(fg)(x) = \sqrt{9 + 9x - x^2 - x^3}$

(iv) $\dfrac{f}{g} : [-1, 3] \to R$ defined by $\left(\dfrac{f}{g}\right)(x) = \sqrt{\dfrac{x+1}{9-x^2}}$

(v) $\dfrac{g}{f} : (-1, 3] \to R$ defined by $\left(\dfrac{g}{f}\right)(x) = \sqrt{\dfrac{9-x^2}{x+1}}$

(vi) $2f - \sqrt{5}\,g : [-1, 3] \to R$ defined by $(2f - \sqrt{5}\,g)(x) = 2\sqrt{x+1} - \sqrt{45 - 5x^2}$

(vii) $f^2 + 7f : [-1, \infty) \to R$ defined by $(f^2 + 7f)(x) = x + 1 + 7\sqrt{x+1}$

(viii) $\dfrac{5}{g} : (-3, 3) \to R$ defined by $\left(\dfrac{5}{g}\right)(x) = \dfrac{5}{\sqrt{9-x^2}}$

5. (i) $f + g : (-\infty, 1) \to R$ defined by $(f + g)(x) = \log_e(1 - x) + [x]$

(ii) $fg : (-\infty, 1) \to R$ defined by $(f\,g)(x) = [x]\log_e(1 - x)$

(iii) $\dfrac{f}{g} : (-\infty, 0) \to R$ defined by $\left(\dfrac{f}{g}\right)(x) = \dfrac{\log_e(1-x)}{[x]}$

(iv) $\dfrac{g}{f} : (-\infty, 0) \cup (0, 1) \to R$ defined by $\left(\dfrac{g}{f}\right)(x) = \dfrac{[x]}{\log_e(1-x)}$

$(f + g)(-1) = \log_e 2 - 1$ and, $(fg)(0) = 0$, $\left(\dfrac{f}{g}\right)\left(\dfrac{1}{2}\right)$ does not exist $\left(\dfrac{g}{f}\right)\left(\dfrac{1}{2}\right) = 0$

6. $2(\sqrt{2} + 1)$, 0, does not exist.

8. $f + g : R \to R$ defined by $(f + g)(x) = 3x - 2$; $f - g : R \to R$ defined by $(f - g)(x) = -x + 4$

$\dfrac{f}{g} : R - \left\{\dfrac{3}{2}\right\} \to R$ defined by $\left(\dfrac{f}{g}\right)(x) = \dfrac{x+1}{2x-3}$

9. $f + g : [0, \infty) \to R$ defined by $(f + g)(x) = \sqrt{x} + x$; $f - g : [0, \infty) \to$ defined by $(f - g)(x) = \sqrt{x} - x$

$fg : [0, \infty) \to R$ defined by $(fg)(x) = x^{3/2}$; $\dfrac{f}{g} : (0, \infty) \to R$ defined by $\left(\dfrac{f}{g}\right)(x) = \dfrac{1}{\sqrt{x}}$

10. $(f + g)(x) = (x + 1)^2$, $(f - g)(x) = x^2 - 2x - 1$, $(fg)(x) = 2x^3 + x^2$, $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^2}{2x+1}$.

7. We have, $f(x) = \begin{cases} 1-x, & x<0 \\ 1, & x=0 \\ x+1, & x>0 \end{cases}$.

Let $f(x) = y$. Then, $y = \begin{cases} 1-x, & x<0 \\ 1, & x=0 \\ x+1, & x>0 \end{cases}$

The graph of $f(x)$ is as shown in Fig. 3.28.

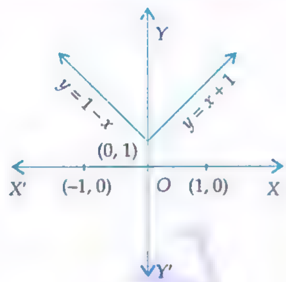


Fig. 3.28

8. $f: R \to R$ and $g: R \to R$ are given by $f(x) = x+1$ and $g(x) = 2x-3$. Clearly, $D(f) = R$ and $D(g) = R$. Therefore,

(i) $D(f+g) = D(f) \cap D(g) = R$ and, $(f+g)(x) = f(x) + g(x) = x+1+2x-3 = 3x-2$

(ii) $D(f-g) = D(f) \cap D(g) = R$ and, $(f-g)(x) = f(x) - g(x) = x+1-2x+3 = -x+4$

(iii) $D(fg) = D(f) \cap D(g) = R$ and, $(fg)(x) = f(x)\,g(x) = (x+1)(2x-3) = 2x^2 - x - 3$

(iv) $D\left(\frac{f}{g}\right) = D(f) \cap D(g) - \{x : g(x) = 0\} = R - \left\{\frac{3}{2}\right\}$ and, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x+1}{2x-3}$

9. It is given that $f:[0, \infty) \to R$ and $g: R \to R$ are such that $f(x) = \sqrt{x}$ and $g(x) = x$. We find that

$D(f+g) = [0, \infty) \cap R = [0, \infty)$. Therefore,

$f+g:[0, \infty) \to R$ is given by $(f+g)(x) = f(x) + g(x) = \sqrt{x} + x$

$D(f-g) = D(f) \cap D(g) = [0, \infty) \cap R = [0, \infty)$. Therefore,

$f-g:[0, \infty) \to R$ is given by $(f-g)(x) = f(x) - g(x) = \sqrt{x} - x$

$D(fg) = D(f) \cap D(g) = [0, \infty) \cap R = [0, \infty)$. Therefore,

$fg:[0, \infty) \to R$ is given by $(fg)(x) = f(x)\,g(x) = \sqrt{x}\;x = x^{3/2}$

$$D\left(\frac{f}{g}\right) = D(f) \cap D(g) - \{x : g(x) = 0\} = (0, \infty). \text{ Therefore,}$$

$\frac{f}{g}:(0, \infty) \to R$ is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$

10. We have, $f(x) = x^2$ and $g(x) = 2x+1$. Clearly, $D(f) = R$ and $D(g) = R$.

$\therefore$ $D(f \pm g) = D(f) \cap D(g) = R \cap R = R$

$D(fg) = D(f) \cap D(g) = R \cap R = R$

$$D\left(\frac{f}{g}\right) = D(f) \cap D(g) - \{x : g(x) = 0\} = R \cap R - \left\{-\frac{1}{2}\right\} = R - \left\{-\frac{1}{2}\right\}$$

Thus, $f+g: R \to R$ is given by $(f+g)(x) = f(x) + g(x) = x^2 + 2x + 1$

$f-g: R \to R$ is given by $(f-g)(x) = f(x) - g(x) = x^2 - 2x - 1$

$(fg) : R \to R$ is given by $(fg)(x) = f(x)\, g(x) = x(2x+1)$

$\left(\frac{f}{g}\right) : R - \left\{\frac{1}{2}\right\} \to R$ is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{2x+1}$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. Let A and B be any two sets such that $n(A) = p$ and $n(B) = q$, then the total number of functions from A to B is equal to ....
2. If $f(x) = \frac{x}{x-1} = \frac{1}{y}$, then $f(y) =$ ....
3. If $y = f(x) = \frac{ax+b}{cx-d}$, then $f(y) =$ ....
4. The domain of the function $f(x) = \frac{1}{\sqrt{|x| - x}}$ is ....
5. The range of the function $f(x) = [x] - x$ is ....
6. The range of the function $f(x) = \frac{x+2}{|x+2|}$ is ....
7. The range of the function $f(x) = \log_a x$, $a > 0$ is ....
8. Let f and g be two functions given by $f = \{(2, 4), (5, 6), (8, -1), (10, -3)\}$ and $g = \{(2, 5), (7, 1)\ (8, 4), (10, 13), (11, -5)\}$. Then, domain of $f + g$ is ....
9. Let f and g be two real functions given by $f = \{(10, 1), (2, 0), (3, -4), (4, 2), (5, 1)\}$ and $g = \{(1, 0), (2, 2), (3, -1), (4, 4), (5, 3)\}$. Then the domain fg is given by ....
10. The domain for which the functions $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal is ....
11. The domain of the function $f(x) = \frac{x^2+1}{x^2 - 3x + 2}$ is ....
12. If $f(x) = \frac{x-1}{x+1}$, then $f\left(\frac{1}{x}\right) + f(x)$ is equal to ....
13. If $f(x) = \frac{x-1}{x+1}$, then $f(x)\, f\left(-\frac{1}{x}\right)$ is equal to ....
14. If $f(x) = [x]^2 - 5[x] + 6$, then the set of values of x satisfying $f(x) = 0$ is ....
15. The domain of the function $f(x) = \sqrt{9-x} + \frac{1}{\sqrt{x^2 - 16}}$ is equal to ....
16. The domain and range of the function $f(x) = \frac{2-x}{x-2}$ are ....and.................. respectively.
17. The domain of the function $f(x) = \frac{1}{\sqrt{[x]^2 - 3[x] + 2}}$ is ....
18. The range of the function $f(x) = \frac{|x-4|}{x-4}$ is ....
19. The domain of the function $f(x) = x + [x]$ is ....
20. The range of the function $f(x) = \sqrt{1 - x^2}$ is ....

21. The domain of the function $f(x) = \sum_{n=1}^{10} \frac{1}{|2x-n|}$ is ....

22. The domain of the function $f(x) = \frac{|x|-2}{|x|-3}$ is ....

23. If $f(2x+3) = 4x^2 + 12x + 15$, then the value of $f(3x+2)$ is ....

24. The number of elements of an identity function defined on a set containing four elements is ....

ANSWERS

1. q^p
2. $1-x$
3. x
4. $(-\infty, 0)$
5. $(-1, 0]$
6. $\{-1, 1\}$
7. R
8. $\{2, 8, 10\}$
9. $\{2, 3, 4, 5\}$
10. $\left\{-1, \frac{4}{3}\right\}$
11. $R-\{1, 2\}$
12. 0
13. -1
14. $[2, 4)$
15. $(-\infty, -4) \cup (4, 9]$
16. $R-\{2\}, \{-1\}$
17. $(-\infty, 1) \cup [3, \infty)$
18. $\{-1, 1\}$
19. R
20. $[0, 1]$
21. $R-\left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5\right\}$
22. $R-\{-3, 3\}$
23. $9x^2 - 12x + 24$
24. 4

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the range of the real function $f(x) = |x|$.
2. If f is a real function satisfying $f\left(x+\frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ for all $x \in R-\{0\}$, then write the expression for $f(x)$.
3. Write the range of the function $f(x) = \sin[x]$, where $\frac{-\pi}{4} \le x \le \frac{\pi}{4}$.
4. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, where $[x]$ denotes the greatest integer less than or equal to x, then write the value of $f(\pi)$.
5. Write the range of the function $f(x) = \cos[x]$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
6. Write the range of the function $f(x) = e^{x-[x]}, x \in R$.
7. Let $f(x) = \frac{\alpha x}{x+1}, x \ne -1$. Then write the value of α satisfying $f(f(x)) = x$ for all $x \ne -1$.
8. If $f(x) = 1 - \frac{1}{x}$, then write the value of $f\left(f\left(\frac{1}{x}\right)\right)$.
9. Write the domain and range of the function $f(x) = \frac{x-2}{2-x}$.
10. If $f(x) = 4x - x^2, x \in R$, then write the value of $f(a+1) - f(a-1)$.
11. If f, g, h are real functions given by $f(x) = x^2$, $g(x) = \tan x$ and, $h(x) = \log_e x$, then write the value of $(hogof)\left(\sqrt{\frac{\pi}{4}}\right)$.

12. Write the domain and range of function $f(x)$ given by $f(x) = \dfrac{1}{\sqrt{x - |x|}}$.
13. Write the domain and range of $f(x) = \sqrt{x - [x]}$.
14. Write the domain and range of function $f(x)$ given by $f(x) = \sqrt{[x] - x}$.
15. Let A and B be two sets such that $n(A) = p$ and $n(B) = q$, write the number of functions from A to B.
16. Let f and g be two functions given by
$f = \{(2, 4), (5, 6), (8, -1), (10, -3)\}$ and $g = \{(2, 5), (7, 1), (8, 4), (10, 13), (11, -5)\}$.
Find the domain of $f + g$.
17. Find the set of values of x for which the functions $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal.
18. Let f and g be two real functions given by
$f = \{(0, 1), (2, 0), (3, -4), (4, 2), (5, 1)\}$ and $g = \{(1, 0), (2, 2), (3, -1), (4, 4), (5, 3)\}$.
Find the domain of fg.

ANSWERS

1. $[0, \infty)$
2. $f(x) = x^2 - 2$, where $|x| > 2$
3. $\{-\sin 1, 0\}$
4. 0
5. $\{1, \cos 1, \cos 2\}$
6. $[1, e)$
7. $\alpha = -1$
8. $\dfrac{x}{x-1}$
9. $D(f) = R - \{2\}$, $R(f) = \{-1\}$
10. $4(2 - a)$
11. 0
12. $D(f) = \phi = R(f)$
13. $D(f) = R$, $R(f) = [0, 1)$
14. $D(f) = z$, $R(f) = \{0\}$
15. q^p
16. $\{2, 8, 10\}$
17. $\{-1, 4/3\}$
18. $\{2, 3, 4, 5\}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, then which of the following is a function from A to B?
(a) $\{(1, 2), (1, 3), (2, 3), (3, 3)\}$
(b) $\{(1, 3), (2, 4)\}$
(c) $\{(1, 3), (2, 2), (3, 3)\}$
(d) $\{(1, 2), (2, 3), (3, 2), (3, 4)\}$.
2. If $f : Q \to Q$ is defined as $f(x) = x^2$, then $f^{-1}(9)$ is equal to
(a) 3
(b) -3
(c) $\{-3, 3\}$
(d) ϕ
3. Which one of the following is not a function?
(a) $\{(x, y) : x, y \in R, x^2 = y\}$
(b) $\{(x, y) : x, y \in R, y^2 = x\}$
(c) $\{(x, y) : x, y \in R, x = y^3\}$
(d) $\{(x, y) : x, y \in R, y = x^3\}$
4. If $f(x) = \cos(\log x)$, then $f(x^2) f(y^2) - \dfrac{1}{2}\left\{ f\left(\dfrac{x^2}{y^2}\right) + f(x^2 y^2)\right\}$ has the value
(a) -2
(b) -1
(c) $1/2$
(d) none of these
5. If $f(x) = \cos(\log x)$, then $f(x) f(y) - \dfrac{1}{2}\left\{ f\left(\dfrac{x}{y}\right) + f(xy)\right\}$ has the value
(a) -1
(b) $1/2$
(c) -2
(d) none of these
6. Let $f(x) = |x - 1|$. Then,

(a) $f(x^2) = [f(x)]^2$ (b) $f(x+y) = f(x) f(y)$
(c) $f(|x|) = |f(x)|$ (d) none of these

7. The range of $f(x) = \cos[x]$, for $-\pi/2 < x < \pi/2$ is
(a) $\{-1, 1, 0\}$ (b) $\{\cos 1, \cos 2, 1\}$ (c) $\{\cos 1, -\cos 1, 1\}$ (d) $[-1, 1]$

8. Which of the following are functions?
(a) $\{(x, y) : y^2 = x, x, y \in R\}$ (b) $\{(x, y) : y = |x|, x, y \in R\}$
(c) $\{(x, y) : x^2 + y^2 = 1, x, y \in R\}$ (d) $\{(x, y) : x^2 - y^2 = 1, x, y \in R\}$

9. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and $g(x) = \frac{3x + x^3}{1 + 3x^2}$, then $f(g(x))$ is equal to
(a) $f(3x)$ (b) $\{f(x)\}^3$ (c) $3f(x)$ (d) $-f(x)$

10. If $A = \{1, 2, 3\}$, $B = \{x, y\}$, then the number of functions that can be defined from A into B is
(a) 12 (b) 8 (c) 6 (d) 3

11. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to
(a) $\{f(x)\}^2$ (b) $\{f(x)\}^3$ (c) $2f(x)$ (d) $3f(x)$

12. If $f(x) = \cos(\log x)$, then value of $f(x) f(4) - \frac{1}{2}\left\{f\left(\frac{x}{4}\right) + f(4x)\right\}$ is
(a) 1 (b) -1 (c) 0 (d) ± 1

13. If $f(x) = \frac{2^x + 2^{-x}}{2}$, then $f(x+y) f(x-y)$ is equals to
(a) $\frac{1}{2}\{f(2x) + f(2y)\}$ (b) $\frac{1}{2}\{f(2x) - f(2y)\}$
(c) $\frac{1}{4}\{f(2x) + f(2y)\}$ (d) $\frac{1}{4}\{f(2x) - f(2y)\}$

14. If $2f(x) - 3f\left(\frac{1}{x}\right) = x^2$ $(x \neq 0)$, then $f(2)$ is equal to
(a) $-\frac{7}{4}$ (b) $\frac{5}{2}$ (c) -1 (d) none of these

15. Let $f : R \to R$ be defined by $f(x) = 2x + |x|$. Then $f(2x) + f(-x) - f(x) =$
(a) $2x$ (b) $2|x|$ (c) $-2x$ (d) $-2|x|$

16. The range of the function $f(x) = \frac{x^2 - x}{x^2 + 2x}$ is
(a) R (b) $R - \{1\}$ (c) $R - \{-1/2, 1\}$ (d) none of these

17. If $x \neq 1$ and $f(x) = \frac{x+1}{x-1}$ is a real function, then $f(f(f(2)))$ is
(a) 1 (b) 2 (c) 3 (d) 4

18. If $f(x) = \cos(\log_e x)$, then $f\left(\frac{1}{x}\right) f\left(\frac{1}{y}\right) - \frac{1}{2}\left\{f(xy) + f\left(\frac{x}{y}\right)\right\}$ is equal to
(a) $\cos(x-y)$ (b) $\log(\cos(x-y))$ (c) 1 (d) $\cos(x+y)$

19. Let $f(x) = x$, $g(x) = \frac{1}{x}$ and $h(x) = f(x) g(x)$. Then, $h(x) = 1$ for
(a) $x \in R$ (b) $x \in Q$ (c) $x \in R - Q$ (d) $x \in R, x \neq 0$

20. If $f(x) = \dfrac{\sin^4 x + \cos^2 x}{\sin^2 x + \cos^4 x}$ for $x \in R$, then $f(2002) =$

(a) 1 (b) 2 (c) 3 (d) 4

21. The function $f : R \to R$ is defined by $f(x) = \cos^2 x + \sin^4 x$. Then, $f(R) =$

(a) $[3/4, 1)$ (b) $(3/4, 1]$ (c) $[3/4, 1]$ (d) $(3/4, 1)$

22. Let $A = \{x \in R : x \neq 0, -4 \leq x \leq 4\}$ and $f : A \to R$ be defined by $f(x) = \dfrac{|x|}{x}$ for $x \in A$. Then A is

(a) $\{1, -1\}$ (b) $\{x : 0 \leq x \leq 4\}$ (c) $\{1\}$ (d) $\{x : -4 \leq x \leq 0\}$

23. If $f : R \to R$ and $g : R \to R$ are defined by $f(x) = 2x + 3$ and $g(x) = x^2 + 7$, then the values of x such that $g(f(x)) = 8$ are

(a) 1, 2 (b) −1, 2 (c) −1, −2 (d) 1, −2

24. If $f : [-2, 2] \to R$ is defined by $f(x) = \begin{cases} -1, \text{ for } -2 \leq x \leq 0 \\ x-1, \text{ for } 0 \leq x \leq 2 \end{cases}$, then

$\{x \in [-2, 2] : x \leq 0 \text{ and } f(|x|) = x\} =$

(a) $\{-1\}$ (b) $\{0\}$ (c) $\{-1/2\}$ (d) ϕ

25. If $e^{f(x)} = \dfrac{10 + x}{10 - x}$, $x \in (-10, 10)$ and $f(x) = k\, f\left(\dfrac{200\,x}{100 + x^2}\right)$, then $k =$

(a) 0.5 (b) 0.6 (c) 0.7 (d) 0.8

26. If f is a real valued function given by $f(x) = 27x^3 + \dfrac{1}{x^3}$ and α, β are roots of $3x + \dfrac{1}{x} = 12$. Then,

(a) $f(\alpha) \neq f(\beta)$ (b) $f(\alpha) = 10$ (c) $f(\beta) = -10$ (d) none of these

27. If $f(x) = 64x^3 + \dfrac{1}{x^3}$ and α, β are the roots of $4x + \dfrac{1}{x} = 3$. Then,

(a) $f(\alpha) = f(\beta) = -9$ (b) $f(\alpha) = f(\beta) = 63$
(c) $f(\alpha) \neq f(\beta)$ (d) none of these

28. If $3f(x) + 5f\left(\dfrac{1}{x}\right) = \dfrac{1}{x} - 3$ for all non-zero x, then $f(x) =$

(a) $\dfrac{1}{14}\left(\dfrac{3}{x} + 5x - 6\right)$ (b) $\dfrac{1}{14}\left(-\dfrac{3}{x} + 5x - 6\right)$
(c) $\dfrac{1}{14}\left(-\dfrac{3}{x} + 5x + 6\right)$ (d) none of these

29. If $f : R \to R$ be given by $f(x) = \dfrac{4^x}{4^x + 2}$ for all $x \in R$. Then,

(a) $f(x) = f(1-x)$ (b) $f(x) + f(1-x) = 0$
(c) $f(x) + f(1-x) = 1$ (d) $f(x) + f(x-1) = 1$

30. If $f(x) = \sin[\pi^2]x + \sin[-\pi^2]x$, where $[x]$ denotes the greatest integer less than or equal to x, then

(a) $f(\pi/2) = 1$ (b) $f(\pi) = 2$ (c) $f(\pi/4) = -1$ (d) none of these

31. The domain of the function $f(x) = \sqrt{2 - 2x - x^2}$ is

(a) $[-\sqrt{3}, \sqrt{3}]$ (b) $[-1-\sqrt{3}, -1+\sqrt{3}]$
(c) $[-2, 2]$ (d) $[-2-\sqrt{3}, -2+\sqrt{3}]$

32. The domain of definition of $f(x) = \sqrt{\dfrac{x+3}{(2-x)(x-5)}}$ is
    (a) $(-\infty, -3] \cup (2, 5)$ (b) $(-\infty, -3) \cup (2, 5)$
    (c) $(-\infty, -3] \cup [2, 5]$ (d) none of these
33. The domain of the function $f(x) = \sqrt{\dfrac{(x+1)(x-3)}{x-2}}$ is
    (a) $[-1, 2) \cup [3, \infty)$ (b) $(-1, 2) \cup [3, \infty)$
    (c) $[-1, 2] \cup [3, \infty)$ (d) none of these
34. The domain of definition of the function $f(x) = \sqrt{x-1} + \sqrt{3-x}$ is
    (a) $[1, \infty)$ (b) $(-\infty, 3)$ (c) $(1, 3)$ (d) $[1, 3]$
35. The domain of definition of the function $f(x) = \sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{1-x}{1+x}}$ is
    (a) $(-\infty, -2] \cup [2, \infty)$ (b) $[-1, 1]$
    (c) ϕ (d) none of these
36. The domain of definition of the function $f(x) = \log|x|$ is
    (a) R (b) $(-\infty, 0)$ (c) $(0, \infty)$ (d) $R - \{0\}$
37. The domain of definition of $f(x) = \sqrt{4x - x^2}$ is
    (a) $R - [0, 4]$ (b) $R - (0, 4)$ (c) $(0, 4)$ (d) $[0, 4]$
38. The domain of definition of $f(x) = \sqrt{x - 3 - 2\sqrt{x-4}} - \sqrt{x - 3 + 2\sqrt{x-4}}$ is
    (a) $[4, \infty)$ (b) $(-\infty, 4]$ (c) $(4, \infty)$ (d) $(-\infty, 4)$
39. The domain of the function $f(x) = \sqrt{5|x| - x^2 - 6}$ is
    (a) $(-3, -2) \cup (2, 3)$ (b) $[-3, -2) \cup [2, 3)$
    (c) $[-3, -2] \cup [2, 3]$ (d) none of these
40. The range of the function $f(x) = \dfrac{x}{|x|}$ is
    (a) $R - \{0\}$ (b) $R - \{-1, 1\}$ (c) $\{-1, 1\}$ (d) none of these
41. The range of the function $f(x) = \dfrac{x+2}{|x+2|}, x \neq -2$ is
    (a) $\{-1, 1\}$ (b) $\{-1, 0, 1\}$ (c) $\{1\}$ (d) $(0, \infty)$
42. The range of the function $f(x) = |x - 1|$ is
    (a) $(-\infty, 0)$ (b) $[0, \infty)$ (c) $(0, \infty)$ (d) R
43. Let $f(x) = \sqrt{x^2 + 1}$. Then, which of the following is correct?
    (a) $f(xy) = f(x)f(y)$ (b) $f(xy) \geq f(x)f(y)$ (c) $f(xy) \leq f(x)f(y)$ (d) none of these
    [NCERT EXEMPLAR]
44. If $[x]^2 - 5[x] + 6 = 0$, where $[\cdot]$ denotes the greatest integer function, then
    (a) $x \in [3, 4]$ (b) $x \in (2, 3]$ (c) $x \in [2, 3]$ (d) $x \in [2, 4)$
    [NCERT EXEMPLAR]
45. The range of $f(x) = \dfrac{1}{1 - 2\cos x}$ is
    (a) $[1/3, 1]$ (b) $[-1, 1/3]$
    (c) $(-\infty, -1) \cup [1/3, \infty)$ (d) $[-1/3, 1]$ [NCERT EXEMPLAR]
46. The domain of the function $f(x) = \sqrt{4 - x} + \dfrac{1}{\sqrt{x^2 - 1}}$ is equal to

(a) $(-\infty, -1) \cup (1, 4)$ (b) $(-\infty, -1] \cup (1, 4]$
(c) $(-\infty, -1) \cup [1, 4]$ (d) $(-\infty, -1) \cup [1, 4)$ [NCERT EXEMPLAR]

47. Domain of $f(x) = \sqrt{a^2 - x^2}$, $a > 0$ is
(a) $(-a, a)$ (b) $[-a, a]$ (c) $[0, a]$ (d) $(-a, 0]$
[NCERT EXEMPLAR]

48. If $f(x) = ax + b$, where a and b are integers, $f(-1) = -5$ and $f(x) = 3$, then a and b are equal
(a) $a = -3, b = -1$ (b) $a = 2, b = -3$ (c) $a = 0, b = 2$ (d) $a = 2, b = 3$
[NCERT EXEMPLAR]

49. The domain and range of the real function of defined by $f(x) = \dfrac{4-x}{x-4}$ is given by
(a) Domain = R, Range = $\{-1, 1\}$ (b) Domain = $R - \{1\}$, Range = R
(c) Domain = $R - \{4\}$, Range = $\{-1\}$ (d) Domain = $R - \{-4\}$, Range = $\{-1, 1\}$
[NCERT EXEMPLAR]

50. The domain and range of real function f defined by $f(x) = \sqrt{x-1}$ is given by
(a) Domain = $(1, \infty)$, Range = $(0, \infty)$ (b) Domain = $[1, \infty)$, Range = $(0, \infty)$
(c) Domain = $[1, \infty)$, Range = $[0, \infty)$ (d) Domain = $[1, \infty)$, Range = $[0, \infty)$
[NCERT EXEMPLAR]

51. The domain of the function f given by $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - x - 6}$
(a) $R - \{-2, 3\}$ (b) $R - \{-3, 2\}$ (c) $R - [-2, 3]$ (d) $R - (-2, 3)$
[NCERT EXEMPLAR]

52. The domain and range of the function f given by $f(x) = 2 - |x - 5|$, is
(a) Domain $= R^+$, Range $= (-\infty, 1]$ (b) Domain $= R$, Range $= (-\infty, 2]$
(c) Domain $= R$, Range $= (-\infty, 2)$ (d) Domain $= R^+$, Range $= (-\infty, 2]$
[NCERT EXEMPLAR]

53. If $f(x) = x^3 - \dfrac{1}{x^3}$, then $f(x) + f\left(\dfrac{1}{x}\right)$ is equal to
(a) $2x^3$ (b) $\dfrac{2}{x^3}$ (c) 0 (d) 1 [NCERT EXEMPLAR]

54. The domain of the function f defined by $f(x) = \dfrac{1}{\sqrt{x - |x|}}$ is
(a) R_0 (b) R^+ (c) R^- (d) none of these
[NCERT EXEMPLAR]

ANSWERS

1. (c)	2. (c)	3. (b)	4. (d)	5. (d)	6. (d)	7. (b)	8. (b)
9. (c)	10. (b)	11. (c)	12. (c)	13. (a)	14. (a)	15. (b)	16. (c)
17. (c)	18. (d)	19. (d)	20. (a)	21. (c)	22. (a)	23. (c)	24. (c)
25. (a)	26. (d)	27. (a)	28. (b)	29. (c)	30. (a)	31. (b)	32. (a)
33. (a)	34. (d)	35. (c)	36. (d)	37. (d)	38. (a)	39. (c)	40. (c)

41. (a)	42. (b)	43. (c)	44. (d)	45. (c)	46. (a)	47. (b)	48. (b)
49. (c)	50. (c)	51. (a)	52. (b)	53. (c)	54. (d)		

SUMMARY

1. Let A and B be two non-empty sets. Then a relation f from A to B is a function, if
   (i) for each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$
   (ii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$.
   In other words, f is a function from A to B if each element of A appears in some ordered pair in f and no two ordered pairs in f have the same first element.
   If $(a, b) \in f$, then b is called the image of a under f.
2. A function f from a set A to a set B is a rule associating elements of set A to elements of set B such that every element in set A is associated to a unique elements in set B.
   The set A is called the domain of f and the set B is called its co-domain.
3. The range of a function f is the set of images of elements in the domain.
4. A real function has the domain and co-domain both as subsets of set R.
5. If $f : D_1 \to R$ and $g : D_2 \to R$ are two real functions and $c \in R$, then
   (i) $f \pm g : D_1 \cap D_2 \to R$ is defined as $(f \pm g)(x) = f(x) \pm g(x)$
   (ii) $fg : D_1 \cap D_2 \to R$ is defined as $(fg)(x) = f(x)\, g(x)$
   (iii) $\dfrac{f}{g} : D_1 \cap D_2 - \{x : g(x) = 0\} \to R$ is defined as $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$
   (iv) $cf : D_1 \cap D_2 \to R$ is defined as $(cf)(x) = c\, f(x)$.

CHAPTER 4

MEASUREMENT OF ANGLES

4.1 INTRODUCTION

The word '*Trigonometry*' is derived from two Greek words : (i) trigonon and, (ii) metron. The word trigonon means a triangle and the word metron means a measure. Hence, trigonometry means the science of measuring triangles. In broader sense it is that branch of Mathematics which deals with the measurement of the sides and the angles of a triangle and the problems *allied* with angles.

4.2 ANGLES

ANGLE *Consider a ray* $\overrightarrow{OA}$*. If this ray rotates about its initial point O and takes the position OB, then we say that the angle* $\angle AOB$ *has been generated.*

Thus, an angle is considered as the figure obtained by rotating a given ray about its initial point.

The revolving ray is called the generating line of the angle. The initial position *OA* is called the *initial side* and the final position *OB* is called *terminal side* of the angle. The initial point *O* on the initial side about which the ray rotates is called the vertex of the angle.

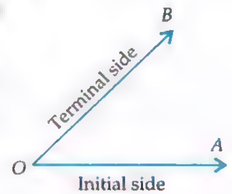


Fig. 4.1

MEASURE OF AN ANGLE *The measure of an angle is the amount of rotation performed to get the terminal side.*

SENSE OF AN ANGLE *The sense of an angle is determined by the direction of rotation of the initial side into the terminal side. The sense of an angle is said to be positive or negative according as the initial side rotates in anticlockwise or clockwise direction to get to the terminal side.*

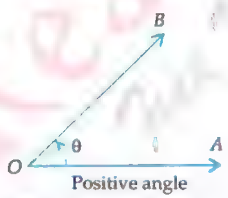


Fig. 4.2 Positive angle

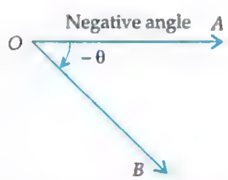


Fig. 4.3 Negative angle

RIGHT ANGLE *If the revolving ray starting from its initial position to final position describes one quarter of a circle, then we say that the measure of the angle formed is a right angle.*

4.3 SOME USEFUL TERMS

QUADRANTS *Let X'OX and YOY' be two lines at right angles in the plane of the paper. These lines divide the plane of the paper into four equal parts which are known as quadrants. The lines X'OX and YOY' are known as x-axis and y-axis respectively. These two lines taken together are known as the coordinate axes. The regions XOY, YOX', X' OY' and Y' OX are known as the first, the second, the third and the fourth quadrant respectively.*

ANGLE IN STANDARD POSITION *An angle is said to be in standard position if its vertex coincides with the origin O and the initial side coincides with OX i.e. the positive direction of x-axis.*

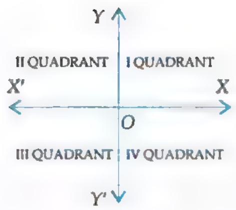


Fig. 4.4

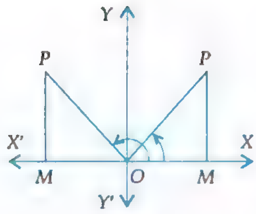


Fig. 4.5

ANGLE IN A QUADRANT *An angle in standard position is said to be in a particular quadrant, if the terminal side of the angle in standard position lies in that quadrant.*

QUADRANT ANGLE *An angle in standard position is said to be a quadrant angle, if the terminal side coincides with one of the axes.*

TRIANGLE OF REFERENCE *If from any point P on the terminal side of an angle in standard position a perpendicular PM is drawn on x-axis, then the right angled triangle OMP, thus formed, is called the triangle of reference of the $\angle XOP$. (See Fig. 4.5)*

CO-TERMINAL ANGLES *Two angles with different measures but having the same initial sides and the same terminal sides are known as co-terminal angles.*

Angles of measure 30°, 390° and – 330° are co-terminal angles.

4.4 SYSTEMS OF MEASUREMENT OF ANGLES

There are three systems for measuring angles, viz. (i) Sexagesimal or English system, (ii) Centesimal or French system, (iii) Circular system.

4.4.1 SEXAGESIMAL SYSTEM

In this system a right angle is divided into 90 equal parts, called degrees. The symbol $1°$ is used to denote one degree. Thus, one degree is one-ninetieth part of a right angle. Each degree is divided into 60 equal parts, called minutes. The symbol $1'$ is used to denote one minute. And each minute is divided into 60 equal parts, called seconds. The symbol $1''$ is used to denote one second.

Thus, 1 right angle = 90 degrees (90°)

$$1° = 60 \text{ minutes } (= 60')$$

$$1' = 60 \text{ seconds } (= 60'')$$

REMARK *Instead of defining degree as $\left(\frac{1}{90}\right)^{\text{th}}$ part of a right angle, we may define it in terms of one complete revolution as follows:*

ONE COMPLETE REVOLUTION *If the terminal side of an angle coincides with the initial side after rotation in anticlockwise direction, then we say that the terminal side has made one complete revolution*

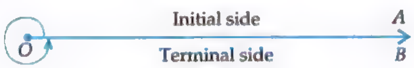
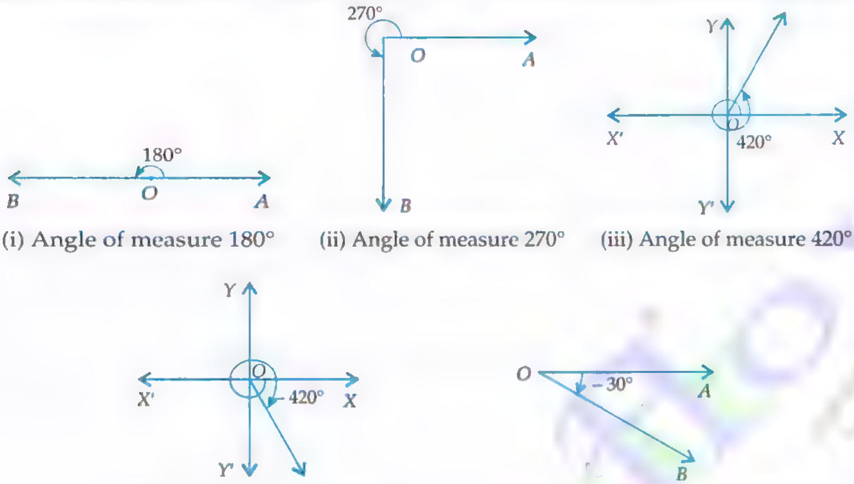


Fig. 4.6

DEGREE MEASURE *If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{\text{th}}$ of a revolution, the angle is said to have a measure of one degree.*

It is evident from this definition of degree measure that 1 complete revolution = 360°

The angles of measures 180°, 270°, 420°, – 30°, – 420° are shown in the following figures.



(i) Angle of measure 180° (ii) Angle of measure 270° (iii) Angle of measure 420°

(iv) Angle of measure – 420° (v) Angle of measure – 30°

Fig. 4.7 Angle of measure

4.4.2 CENTESIMAL SYSTEM

In this system a right angle is divided into 100 equal parts, called grades; each grade is subdivided into 100 minutes, and each minute into 100 seconds.

The symbols 1^g, $1'$ and $1''$ are used to denote a grade, a minute, and a second respectively.

Thus,
$$1 \text{ right angle} = 100 \text{ grades } (= 100^g)$$
$$1 \text{ grade} = 100 \text{ minutes } (= 100')$$
$$1 \text{ minute} = 100 \text{ seconds } (= 100'')$$

4.4.3 CIRCULAR SYSTEM

In this system the unit of measurement is radian as defined below.

RADIAN *One radian, written as* 1^c, *is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.*

Consider a circle of radius r having centre at O (see Fig. 4.8). Let A be a point on the circle. Now, cut off an arc AP whose length is equal to the radius r of the circle. Then by the definition the measure of $\angle AOP$ is 1 radian $(= 1^c)$.

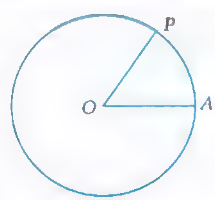


Fig. 4.8

Since a radian is chosen as the unit of measurement of an angle, therefore it should be a constant quantity. This we shall show in the following two theorems.

THEOREM 1 *Radian is a constant angle.*

PROOF Consider a circle with centre O and radius r. Take a point A on the circle and cut off an arc AP whose length is equal to the radius r. Join OA and OP. Then, by definition $\angle AOP = 1^c$.

Produce AO to meet the circle at B so that

$$\angle AOB = \text{a straight angle} = 2 \text{ right angles.}$$

Since the angles at the centre of a circle are proportional to the arcs subtending them.

$$\therefore \quad \frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } APB}$$

$$\Rightarrow \quad \frac{\angle AOP}{\angle AOB} = \frac{r}{\pi r} \qquad \left[\because \text{ arc } APB = \frac{1}{2}(2\pi r) = \pi r\right]$$

$$\Rightarrow \quad \frac{\angle AOP}{\angle AOB} = \frac{1}{\pi}$$

$$\Rightarrow \quad \angle AOP = \frac{1}{\pi}\angle AOB = \frac{\text{a straight angle}}{\pi}$$

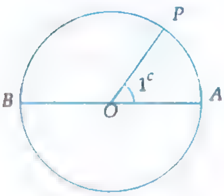


Fig. 4.9

$$\Rightarrow \quad 1^c = \frac{\text{a straight angle}}{\pi} \qquad [\because \angle AOP = 1^c]$$

$$\Rightarrow \quad 1^c = \text{Constant} \qquad [\because \text{ A straight angle and } \pi \text{ both are constants}]$$

Hence, radian is a constant angle **Q.E.D.**

THEOREM 2 *The number of radians in an angle subtended by an arc of a circle at the centre is equal to* $\frac{arc}{radius}$.

PROOF Consider a circle with centre O and radius r. Let $\angle AOQ = \theta^c$ and let arc $AQ = s$. Let P be a point on the arc AQ such that arc $AP = r$. Then, $\angle AOP = 1^c$. Since angles at the centre of a circle are proportional to the arcs subtending them.

$$\therefore \quad \frac{\angle AOQ}{\angle AOP} = \frac{\text{arc } AQ}{\text{arc } AP}$$

$$\Rightarrow \quad \angle AOQ = \left(\frac{\text{arc } AQ}{\text{arc } AP} \times 1\right)^c \qquad [\because \angle AOP = 1^c]$$

$$\Rightarrow \quad \theta = \frac{s}{r} \text{ radians}.$$

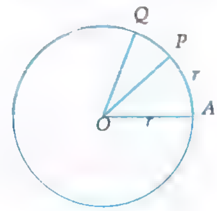


Fig. 4.10

Q.E.D

4.5 RELATION BETWEEN DEGREES AND RADIANS

Consider a circle with centre O and radius r. Let A be a point on the circle. Join OA and cut off an arc OP of length equal to the radius of the circle. Then, $\angle AOP = 1$ radian. Produce AO to meet the circle at B.

$$\therefore \quad \angle AOB = \text{a straight angle} = 2 \text{ right angles}$$

We know that the angles at the centre of a circle are proportional to the arcs subtending them.

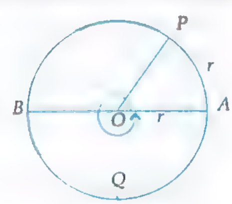


Fig. 4.11

$$\therefore \quad \frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } APB}$$

$$\Rightarrow \quad \frac{\angle AOP}{2 \text{ right angles}} = \frac{r}{\pi r} \qquad \left[\because \text{ arc } APB = \frac{1}{2}(\text{Circumference})\right]$$

$$\Rightarrow \quad \angle AOP = \frac{2 \text{ rightangles}}{\pi}$$

$$\Rightarrow \quad \angle AOP = \frac{180°}{\pi}$$

Hence, One radian $= \frac{180°}{\pi} \Rightarrow \pi$ radians $= 180°$.

REMARK 1 *In Fig. 4.11, arc (APBQA) = $2\pi r$. So, the measure of angle made by this arc at this centre O of the circle is $\frac{2\pi r}{r}$ radians i.e. 2π radians. Also, the measure of this angle is the measure of one revolution i.e. 360°.*

$\therefore \quad 2\pi$ *radians* $= 360° \Rightarrow \pi$ *radians* $= 180°$

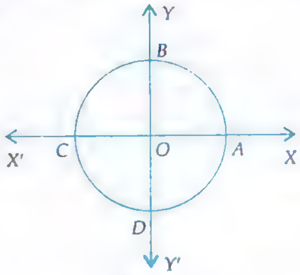


Fig. 4.12

Since one complete revolution subtends an angle of 2π radians at the centre of the unit circle shown in Fig. 4.12.

$\therefore \quad \angle AOB = \frac{1}{4} \times 2\pi = \frac{\pi}{2}$ *radians*, $\angle AOC = \frac{1}{2}(2\pi) = \pi$ *radians*

and, $\angle AOD = \frac{3}{4}(2\pi) = \frac{3\pi}{2}$ radians.

Thus, quadrant angles are integral multiples of $\frac{\pi}{2}$. All integral multiples of $\frac{\pi}{2}$ are called the quadrant angles.

REMARK 2 *When an angle is expressed in radians, the word radian is generally omitted.*

REMARK 3 *We know that 180° = π radians. Therefore, $1° = \frac{\pi}{180}$ radian.*

Hence, $30° = \frac{\pi}{180} \times 30 = \frac{\pi}{6}$ *radians,* $45° = \frac{\pi}{180} \times 45 = \frac{\pi}{4}$ *radians,* $60° = \frac{\pi}{180} \times 60 = \frac{\pi}{3}$ *radians,*

$$90° = \frac{\pi}{180} \times 90 = \frac{\pi}{2} \text{ radians etc.}$$

REMARK 4 *We have, π radians = 180°*

$$\therefore \quad 1 \text{ radian} = \frac{180°}{\pi} = \left(\frac{180}{22} \times 7\right)^{\circ} = 57° 16' 22'' \text{ (approx).}$$

Again, $\quad 180° = \pi$ radians

$$\therefore \quad 1° = \frac{\pi}{180} \text{ radian} = \left(\frac{22}{7 \times 180}\right) \text{radian} = 0.01746 \text{ radian.}$$

4.6 RELATION BETWEEN RADIANS AND REAL NUMBERS

In this section, we shall show that radian measures and real numbers can be considered as one and the same i.e. every radian measure is a real number and every real number can be considered as radian measure of some angle. For this, let us consider the unit circle with centre O and let A be any point on it as shown in Fig. 4.13.

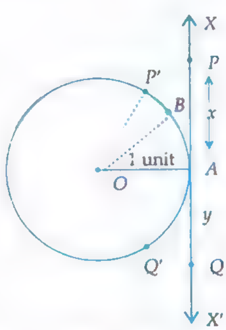


Fig. 4.13

Consider OA as initial side of an angle $\angle AOB$, where B is a point on the circle.

Now, $\quad \angle AOB = \frac{\text{arc } AB}{OA} \Rightarrow \angle AOB = \text{arc } AB \qquad [\because OA = 1 \text{ unit}]$

So, the length of arc AB is the radian measure of $\angle AOB$. Let XAX' be the tangent to the circle at A. Let the point A represent the real number zero, positive real number are represented by points on AX and negative real number by points on AX'. Let a positive real number x be represented by a point P on AX. Let us now rope the line AP in anticlockwise direction along the circle such that AP = arc AP'. Then,

arc $AP' =$ radian measure of $\angle AOP' \Rightarrow AP =$ radian measure of $\angle AOP'$

$\Rightarrow$ $x =$ radian measure of $\angle AOP'$

Thus, every positive real number is the radian measure of some positive angle and vice-versa.

Now, let y be a negative real number represented by a point Q on AX'. If we rope the line AQ in clockwise direction along the circle such that $AQ =$ arc AQ', then

arc $AQ' =$ radian measure of $\angle AOQ'$

$\Rightarrow$ $AQ =$ radian measure of $\angle AOQ'$ $\left[\because \angle AOQ' = \dfrac{\text{Arc } AQ'}{1} = \text{Arc } AQ'\right]$

$\Rightarrow$ $y =$ radian measure of $\angle AOQ'$

Thus, every negative real number is the radian measure of some negative angle and vice-versa.

Hence, radian measures of angles and real numbers can be considered as one and the same.

4.7 RELATION BETWEEN THREE SYSTEMS OF MEASUREMENT OF AN ANGLE

Let D be the number of degrees, R be the number of radians and G be the number of grades in an angle θ.

$\because$ $90° = 1$ right angle $\Rightarrow 1° = \dfrac{1}{90}$ right angle $\Rightarrow D° = \dfrac{D}{90}$ right angles

$\Rightarrow$ $\theta = \dfrac{D}{90}$ right angles ... (ii)

Also, π radians $= 2$ right angles $\Rightarrow 1$ radian $= \dfrac{2}{\pi}$ right angles $\Rightarrow R$ radians $= \dfrac{2R}{\pi}$ right angles

$\Rightarrow$ $\theta = \dfrac{2R}{\pi}$ right angles ... (ii)

And, 100 grades = 1 right angle

$\Rightarrow$ 1 grade $= \dfrac{1}{100}$ right angle $\Rightarrow$ G grades $= \dfrac{G}{100}$ right angles

$\Rightarrow$ $\theta = \dfrac{G}{100}$ right angles ... (iii)

From (i), (ii) and (iii), we obtain: $\dfrac{D}{90} = \dfrac{G}{100} = \dfrac{2R}{\pi}$

This is the required relation between the three systems of measurement of an angle.

SOME USEFUL POINTS

(i) The angle between two consecutive digits in a clock is $30° \left(= \dfrac{\pi}{6} \text{ radians}\right)$.

(ii) The hour hand rotates through an angle of $30°$ in one hour i.e. $(1/2)°$ in one minute.

(iii) The minute hand rotates through an angle of $6°$ in one minute.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the degree measure corresponding to the following radian measures:*

(i) $\left(\dfrac{2\pi}{15}\right)^c$ (ii) $\left(\dfrac{\pi}{8}\right)^c$ (iii) $\left(\dfrac{1}{4}\right)^c$ (iv) -2^c (v) 6^c (vi) $\left(\dfrac{11}{16}\right)^c$

SOLUTION We know that π radians $= 180°$ and so, $1^c = \left(\dfrac{180}{\pi}\right)^\circ$. Therefore,

(i) $\left(\dfrac{2\pi}{15}\right)^c = \left(\dfrac{2\pi}{15} \times \dfrac{180}{\pi}\right)^\circ = 24°$

(ii) $\left(\frac{\pi}{8}\right)^c = \left(\frac{\pi}{8} \times \frac{180}{\pi}\right)^\circ = \left(\frac{45}{2}\right)^\circ = \left(22\frac{1}{2}\right)^\circ = 22^\circ\left(\frac{1}{2} \times 60\right)' = 22^\circ\ 30'$

(iii) $\left(\frac{1}{4}\right)^c = \left(\frac{1}{4} \times \frac{180}{\pi}\right)^\circ = \left(\frac{1}{4} \times \frac{180}{22} \times 7\right)^\circ = \left(\frac{315}{22}\right)^\circ = \left(14\frac{7}{22}\right)^\circ$

$= 14^\circ\left(\frac{7}{22} \times 60\right)' = 14^\circ\left(19\frac{1}{11}\right)' = 14^\circ\ 19'\left(\frac{1}{11} \times 60\right)'' = 14^\circ 19'\ 5''$

(iv) $(-2)^c = \left(\frac{180}{\pi} \times -2\right)^\circ = \left(\frac{180}{22} \times 7 \times (-2)\right)^\circ = \left(-114\frac{6}{11}\right)^\circ = \left\{-114^\circ\left(\frac{6}{11} \times 60\right)'\right\}$

$= -\left[114^\circ\left(32\frac{8}{11}\right)'\right] = -\left[114^\circ\ 32'\left(\frac{8}{11} \times 60\right)''\right] = -(114^\circ\ 32'\ 44'')$

(v) $6^c = \left(\frac{180}{\pi} \times 6\right)^\circ = \left(\frac{180}{22} \times 7 \times 6\right)^\circ = \left(\frac{90 \times 7 \times 6}{11}\right)^\circ = \left(\frac{3780}{11}\right)^\circ = \left(343\frac{7}{11}\right)^\circ$

$= 343^\circ\left(\frac{7}{11} \times 60\right)' = 343^\circ\left(\frac{420}{11}\right)' = 343'\ 38'\left(\frac{2}{11} \times 60\right)'' = 343^\circ\ 38'\ 11''$

(vi) $\left(\frac{11}{16}\right)^c = \left(\frac{180}{\pi} \times \frac{11}{16}\right)^\circ = \left(\frac{180}{22} \times 7 \times \frac{11}{16}\right)^\circ = \left(\frac{315}{8}\right)^\circ = \left(39\frac{3}{8}\right)^\circ$

$= 39^\circ\left(\frac{3}{8} \times 60\right)' = 39^\circ\ 22'\left(\frac{1}{2} \times 60\right)'' = 39^\circ\ 22'\ 30''$

EXAMPLE 2 *Find the radian measures corresponding to the following degree measures :*
(i) 340° (ii) 75° (iii) $-37^\circ\ 30'$ (iv) $5^\circ\ 37'\ 30''$ (v) $40^\circ\ 20'$ (vi) 520°

SOLUTION We know that $180^\circ = \pi^c$ and, $1^\circ = \left(\frac{\pi}{180}\right)^c$. Therefore,

(i) $340^\circ = \left(340 \times \frac{\pi}{180}\right)^c = \left(\frac{17\pi}{9}\right)^c$

(ii) $75^\circ = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$

(iii) Clearly, $30' = \left(\frac{30}{60}\right)^\circ = \frac{1}{2}^\circ$.

$\therefore \quad -37^\circ\ 30' = -\left(37\frac{1}{2}\right)^\circ = -\left(\frac{75}{2}\right)^\circ = -\left(\frac{75}{2} \times \frac{\pi}{180}\right)^c = -\left(\frac{5\pi}{24}\right)^c$

(iv) Clearly, $30'' = \left(\frac{30}{60}\right)' = \left(\frac{1}{2}\right)'$. Therefore, $37'\ 30'' = \left(37\frac{1}{2}\right)' = \left(\frac{75}{2}\right)' = \left(\frac{75}{2} \times \frac{1}{60}\right)^\circ = \left(\frac{5}{8}\right)^\circ$

So, $5^\circ\ 37'\ 30'' = \left(5\frac{5}{8}\right)^\circ = \left(\frac{45}{8}\right)^\circ = \left(\frac{45}{8} \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{32}\right)^c$

(v) Clearly, $20' = \left(\frac{20}{60}\right)^\circ = \frac{1}{3}^\circ$. Therefore, $40^\circ\ 20' = \left(40\frac{1}{3}\right)^\circ = \left(\frac{121}{3}\right)^\circ = \left(\frac{121}{3} \times \frac{\pi}{180}\right)^c = \left(\frac{121\pi}{540}\right)^c$

(vi) Clearly, $520^\circ = \left(520 \times \frac{\pi}{180}\right)^c = \left(\frac{26\pi}{9}\right)^c$

EXAMPLE 3 *Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring* 15°.

SOLUTION Let s be the length of the arc subtending an angle θ^c at the centre of a circle of radius r.

Then, $\theta = \frac{s}{r}$. Here, $r = 5$ cm and $\theta = 15° = \left(15 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{12}\right)^c$

$$\therefore \quad \theta = \frac{s}{r} \Rightarrow \frac{\pi}{12} = \frac{s}{5} \Rightarrow s = \frac{5\pi}{12} \text{ cm.}$$

EXAMPLE 4 *Find in degrees the angle subtended at the centre of a circle of diameter 50 cm by an arc of length 11 cm.*

SOLUTION Here, $r = 25$ cm and $s = 11$ cm

$$\therefore \quad \theta = \left(\frac{s}{r}\right)^c$$

$$\Rightarrow \quad \theta = \left(\frac{11}{25}\right)^c = \left(\frac{11}{25} \times \frac{180}{\pi}\right)^\circ = \left(\frac{11}{25} \times \frac{180}{22} \times 7\right)^\circ = \left(\frac{126}{5}\right)^\circ = \left(25\frac{1}{5}\right)^\circ = 25°\left(\frac{1}{5} \times 60\right)' = 25°\,12'$$

EXAMPLE 5 *Find in degrees the angle through which a pendulum swings if its length is 50 cm and the tip describes an arc of length 10 cm.*

SOLUTION Here, $r = 50$ cm and $s = 10$ cm.

$$\therefore \quad \theta = \left(\frac{s}{r}\right)^c$$

$$\Rightarrow \theta = \left(\frac{10}{50}\right)^c = \left(\frac{1}{5}\right)^c = \left(\frac{1}{5} \times \frac{180}{\pi}\right)^\circ = \left(\frac{36}{22} \times 7\right)^\circ = \left(\frac{126}{11}\right)^\circ = \left(11\frac{5}{11}\right)^\circ = 11°\left(\frac{5}{11} \times 60\right)' = 11°\,27'\,16''$$

EXAMPLE 6 *A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 metres when it has traced out 72° at the centre, find the length of the rope.*

SOLUTION Let the post be at point P and let PA be the length of the rope in tight position. Suppose the horse moves along the arc AB so that $\angle APB = 72°$ and arc $AB = 88$ m. Let r be the length of the rope i.e. $PA = r$ metres.

Here, $\theta = 72° = \left(72 \times \frac{\pi}{180}\right)^c = \left(\frac{2\pi}{5}\right)^c$ and $s = 88$ m

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{2\pi}{5} = \frac{88}{r} \Rightarrow r = 88 \times \frac{5}{2\pi} = 70 \text{ metres.}$$

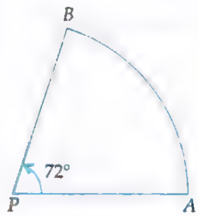


Fig. 4.14

BASED ON LOTS

EXAMPLE 7 *In a circle of diameter 40 cm the length of a chord is 20 cm. Find the length of minor arc corresponding to the chord.*

SOLUTION Let arc $AB = s$. It is given that $OA = 20$ cm and chord $AB = 20$ cm. Therefore, ΔOAB is an equilateral triangle.

Hence, $\angle AOB = 60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{\pi}{3} = \frac{s}{20} \Rightarrow s = \frac{20\pi}{3} \text{ cm.}$$

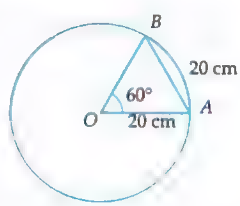


Fig. 4.15

EXAMPLE 8 *The angles of a triangle are in A.P. The number of degrees in the least is to the number of radians in the greatest as 60 : π. Find the angles in degrees.*

SOLUTION Let the measures of angles of the triangle be $(a-d)°$, $a°$ and $(a+d)°$ Then,

$$(a-d)+a+(a+d) = 180° \Rightarrow 3a = 180° \Rightarrow a = 60°$$

So, the angles are $(60-d)°$, $60°$, $(60+d)°$. Clearly, $(60-d)°$ is the least angle and $(60+d)°$ is the greatest angle.

$$\text{Greatest angle} = (60+d)° = \left\{(60+d)\frac{\pi}{180}\right\}^c$$

It is given that:

$$\frac{\text{Number of degrees in the least angle}}{\text{Number of radians in the greatest angle}} = \frac{60}{\pi}$$

$$\Rightarrow \quad \frac{(60-d)}{(60+d)\dfrac{\pi}{180}} = \frac{60}{\pi} \Rightarrow 3(60-d) = (60+d) \Rightarrow 120 = 4d \Rightarrow d = 30.$$

Hence, measures of the angles are $(60-30)°$, $60°$, $(60+30)°$ i.e. $30°$, $60°$, $90°$.

EXAMPLE 9 *The angles of a triangle are in A.P. The number of grades in the least, is to the number of radians in the greatest as* $40 : \pi$. *Find the angles in degrees.*

SOLUTION Let measures of the angles of the triangle in degrees be $(a-d)°$, $a°$ and $(a+d)°$. Then,

$$(a-d)+a+(a+d) = 180 \Rightarrow 3a = 180 \Rightarrow a = 60$$

So, measures of the angles are $(60-d)°$, $60°$ and $(60+d)°$. Clearly, measure of the least angle is $(60°-d)°$ and that of the greatest angle is $(60+d)°$.

Now,

$$\text{Measure of the least angle} = (60-d)° = \left\{(60-d)\times\frac{100}{90}\right\}^g = \left\{(60-d)\times\frac{10}{9}\right\}^g \qquad [\because 90° = 100^g]$$

$$\text{Measure of the greatest angle} = (60+d)° = \left\{(60+d)\times\frac{\pi}{180}\right\}^c$$

It is given that:

$$\frac{\text{Number of grades in the least angle}}{\text{Number of radians in the greatest angle}} = \frac{40}{\pi}$$

$$\Rightarrow \quad \frac{(60-d)\times\dfrac{10}{9}}{(60+d)\dfrac{\pi}{180}} = \frac{40}{\pi} \Rightarrow \frac{600-10d}{9}\times\frac{180}{(60+d)\pi} = \frac{40}{\pi}$$

$$\Rightarrow \quad 600-10d = 120+2d \Rightarrow 12d = 480 \Rightarrow d = 40.$$

Hence, measures the angles of the triangle are $20°$, $60°$ and $100°$.

EXAMPLE 10 *Express the angular measurement of the angle of a regular decagon in degrees, grades and radians.*

SOLUTION We know that the angle of an n sided regular polygon is equal to $\left(\dfrac{2n-4}{n}\right)$ right angles. Let θ be the angle of a regular decagon. Then,

$$\theta = \left(\frac{2\times10-4}{10}\right) = \frac{8}{5}\text{ right angles} = \left(\frac{8}{5}\times 90\right)^\circ = 144° \qquad [\because 1 \text{ right angle} = 90°]$$

$$\text{Again, } \theta = \frac{8}{5}\text{ right angles} = \left(\frac{8}{5}\times100\right)^g = 160^g \qquad [\because 1 \text{ right angle} = 100^g]$$

$$\text{And, } \quad \theta = \frac{8}{5}\text{ right angles} = \left(\frac{8}{5}\times\frac{\pi}{2}\right)^c = \left(\frac{4\pi}{5}\right)^c \qquad \left[\because 1 \text{ right angle} = \frac{\pi^c}{2}\right]$$

EXAMPLE 11 *If the arcs of same length in two circles subtend angles of 60° and 75° at their centres. Find the ratio of their radii.*

SOLUTION Let r_1 and r_2 be the radii of the given circles and let their arcs of same length s subtend angles of 60° and 75° at their centres.

Now, $60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$ and, $75° = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$

$\therefore \quad \frac{\pi}{3} = \frac{s}{r_1}$, and $\frac{5\pi}{12} = \frac{s}{r_2}$ $\quad \left[\because \theta = \left(\frac{s}{r}\right)^c\right]$

$\Rightarrow \quad \frac{\pi}{3} r_1 = s$, and $\frac{5\pi}{12} r_2 = s \Rightarrow \frac{\pi}{3} r_1 = \frac{5\pi}{12} r_2 \Rightarrow 4r_1 = 5r_2 \Rightarrow r_1 : r_2 = 5:4$

Hence, $r_1 : r_2 = 5:4$

EXAMPLE 12 *A circular wire of radius 3 cm is cut and bent so as to lie along the circumference of a hoop whose radius is 48 cm. Find in degrees the angle which is subtended at the centre of the hoop.* [NCERT EXEMPLAR]

SOLUTION It is given that the radius of the circular wire is 3 cm.

$\therefore$ Length of the circular wire $= 2\pi \times 3 = 6\pi$ cm $\quad [\because$ Circumference $= 2\pi r]$

Radius of the hoop = 48 cm.

Let θ be the angle subtended by the wire at the centre of the hoop. Then,

$$\theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \theta = \left(\frac{6\pi}{48}\right)^c = \left(\frac{\pi}{8}\right)^c = \left(\frac{\pi}{8} \times \frac{180}{\pi}\right)^\circ = 22°\ 30'$$

EXAMPLE 13 *The perimeter of a certain sector of a circle is equal to the length of the arc of a semi-circle having the same radius. Express the angle of the sector in degrees, minutes and seconds.*

SOLUTION Let r be the radius of the circle and θ be the sector angle. Then,

Perimeter of the sector $= 2r + r\theta$, Length of the arc of a semi-circle of radius $r = \pi r$

It is given that

$$2r + r\theta = \pi r \Rightarrow 2 + \theta = \pi$$

$$\Rightarrow \quad \theta = (\pi - 2) \text{ radians} = \left\{(\pi - 2) \times \frac{180}{\pi}\right\}^\circ = 180° - \left(\frac{360}{\pi}\right)^\circ = 180° - 114°\ 32'\ 44'' = 65°\ 27'\ 16''$$

EXAMPLE 14 *The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes?* (Use $\pi = 3.14$)

SOLUTION In 60 minutes, the minute hand of a watch completes one rotation i.e., it rotates through 360°.

$\therefore$ Angle traced by the minute hand in 1 minute $= \left(\frac{360}{60}\right)^\circ = 6^\circ$

$\Rightarrow$ Angle traced by the minute hand in 40 minutes $= (40 \times 6)° = 240° = \left(240 \times \frac{\pi}{180}\right)^c = \left(\frac{4\pi}{3}\right)^c$

Now,

$$\theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{4\pi}{3} = \frac{\text{arc}}{1.5} \Rightarrow \text{arc} = \left(\frac{4\pi}{3} \times 1.5\right) \text{cm} = 2\pi \text{ cm} = 2 \times 3.14 \text{ cm} = 6.28 \text{ cm}$$

Hence, the tip of the minute hand travels 6.28 cm in 40 minutes.

EXAMPLE 15 *Find the angle between the minute hand of a clock and the hour hand when the time is 7 : 20 AM.*

SOLUTION We know that the hour hand completes one rotation in 12 hours while the minute hand completes one rotation in 60 minutes.

$\therefore$ Angle traced by the hour hand in 12 hours = 360°

$\Rightarrow$ Angle traced by the hour hand in 7 hrs 20 min. i.e $\frac{22}{3}$ hrs $= \left(\frac{360}{12} \times \frac{22}{3}\right)^{\circ} = 220°$.

Also, the angle traced by the minute hand in 60 min = 360°.

$\Rightarrow$ Angle traced by the minute hand in 20 min = $\left(\frac{360}{60} \times 20\right)^{\circ} = 120°$

Hence, the required angle between two hands = $220° - 120° = 100°$.

EXAMPLE 16 *Find in degrees and radians the angle between the hour hand and the minute-hand of a clock at half past three.*

SOLUTION The angle traced by the hour hand in 12 hours = 360°

$\therefore$ The angle traced by the hour hand in 3 hrs 30 min. i.e. $\frac{7}{2}$ hrs $= \left(\frac{360}{12} \times \frac{7}{2}\right)^{\circ} = 105°$

The angle traced by the minute hand in 60 min = 360°

$\Rightarrow$ The angle traced by the minute hand in 30 min $= \left(\frac{360}{60} \times 30\right) = 180°$

Hence, the required angle between two hands $= 180° - 105° = 75° = \left(75 \times \frac{\pi}{180}\right) = \frac{5\pi}{12}$ radians.

BASED ON HOTS

EXAMPLE 17 *The moon's distance from the earth is 360,000 kms and its diameter subtends an angle of 31′ at the eye of the observer. Find the diameter of the moon.*

SOLUTION Let AB be the diameter of the moon and let E be the eye of the observer. Since the distance between the earth and the moon is quite large, so we take diameter AB as arc AB. Let d be the diameter of the moon. Then, d = arc AB.

We have,

$$\theta = 31' = \left(\frac{31}{60}\right)^{\circ} = \left(\frac{31}{60} \times \frac{\pi}{180}\right)^{c} \text{ and, } r = 360000 \text{ kms}$$

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$$

$$\Rightarrow \quad \frac{31}{60} \times \frac{\pi}{180} = \frac{d}{360000}$$

$$\Rightarrow \quad d = \left(\frac{31}{60} \times \frac{\pi}{180} \times 360000\right) \text{km} = 3247.62 \text{ kms}$$

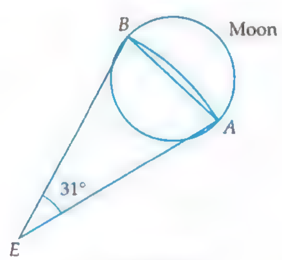


Fig. 4.16

Hence, the diameter of the moon is 3247.62 km.

EXAMPLE 18 *If the angular diameter of the moon be 30′, how far from the eye a coin of diameter 2.2 cm be kept to hide the moon ?*

SOLUTION Suppose the coin is kept at a distance r from the eye to hide the moon completely. Let E be the eye of the observer and let AB be the diameter of the coin. Then, arc AB = diameter AB = 2.2 cm.

We have, $\theta = 30' = \left(\frac{30}{60}\right)^\circ = \left(\frac{1}{2} \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{360}\right)^c$

$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$

$\Rightarrow \quad \frac{\pi}{360} = \frac{2.2}{r}$

$\Rightarrow \quad r = \frac{2.2 \times 360}{\pi}$ cm $\Rightarrow r = \frac{2.2 \times 360 \times 7}{22} = 252$ cm.

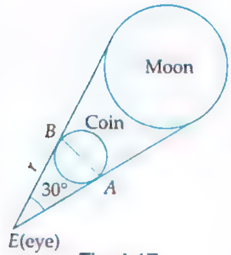


Fig. 4.17

EXAMPLE 19 *Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5′ at his eye, find what is the height of the letters that he can read at a distance of 12 metres.*

SOLUTION Let h be the required height in metres. Here h can be considered as the arc of a circle of radius 12 m, which subtends an angle of 5′ at its centre.

Here, $\theta = 5' = \left(\frac{5}{60}\right)^\circ = \left(\frac{1}{12} \times \frac{\pi}{180}\right)^c$ and, $r = 12$ m.

$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{\pi}{12 \times 180} = \frac{h}{12} \Rightarrow h = \left(\frac{\pi}{180}\right)$ metre $= 1.7$ cm.

EXAMPLE 20 *For each natural number k, let C_k denote the circle with radius k centimetres and centre at the origin. On the circle C_k, a particle moves k centimetres in the counter-clockwise direction. After completing its motion on C_k, the particle moves on C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of the x-axis for the first time on the circle C_n, then find the value of n.*

SOLUTION The path of the particle is shown by bold line segments and arcs. It is given that on the circle C_k of radius k centimetres the particle moves k centimeters. Therefore, angular displacement on kth circle is given by

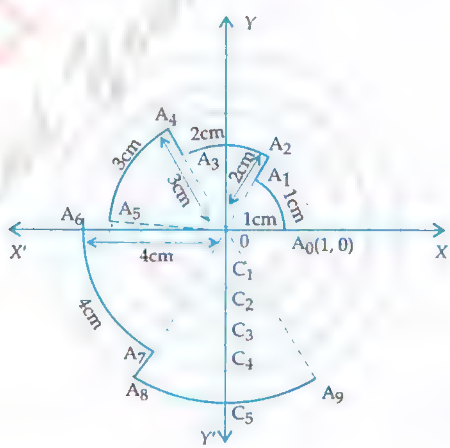


Fig. 4.18

$$\theta = \frac{k}{k} \text{ radian} = 1 \text{ radian}.$$

Thus, angular displacement on each circle is 1 radian.

If the particle crosses the x-axis for the first time on circle C_n then

Total angular displacement $= n$ radians.

As the particle crosses the positive direction of the x-axis for the first time on the n^{th} circle C_n.

Total angular displacement $> 2\pi$ radians

$\Rightarrow \quad n > 2\pi$

$\Rightarrow \quad n = 7 \qquad$ [$\because$ n is the natural number such that $n > 2\pi$]

EXERCISE 4.1

BASIC

1. Find the degree measure corresponding to the following radian measures (Use $\pi = 22/7$:
   (i) $\frac{9\pi}{5}$ (ii) $-\frac{5\pi}{6}$ (iii) $\left(\frac{18\pi}{5}\right)^c$ (iv) $(-3)^c$ (v) 11^c (vi) 1^c
2. Find the radian measure corresponding to the following degree measures:
   (i) $300°$ (ii) $35°$ (iii) $-56°$ (iv) $135°$ (v) $-300°$
   (vi) $7° 30'$ (vii) $125° 30'$ (viii) $-47° 30'$
3. The difference between the two acute angles of a right-angled triangle is $\frac{2\pi}{5}$ radians. Express the angles in degrees.
4. One angle of a triangle is $\frac{2}{3}x$ grades and another is $\frac{3}{2}x$ degrees while the third is $\frac{\pi x}{75}$ radians. Express all the angles in degrees.
5. A rail road curve is to be laid out on a circle. What radius should be used if the track is to change direction by 25° in a distance of 40 metres?
6. Find the angle in radians through which a pendulum swings, if its length is 75 cm and the tip describes an arc of length (i) 10 cm (ii) 15 cm (iii) 21 cm.
7. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm (Use $\pi = 22/7$).

BASED ON LOTS

8. The angle in one regular polygon is to that in another as 3 : 2 and the number of sides in first is twice that in the second. Determine the number of sides of two polygons.
9. The angles of a triangle are in A.P. such that the greatest is 5 times the least. Find the angles in radians.
10. The number of sides of two regular polygons are as 5 : 4 and the difference between their angles is 9°. Find the number of sides of the polygons.
11. If the arcs of the same length in two circles subtend angles 65° and 110° at the centre, find the ratio of their radii.
12. Find the length which at a distance of 5280 m will subtend an angle of 1′ at the eye.
13. A wheel makes 360 revolutions per minute. Through how many radians does it turn in 1 second?
14. A railway train is travelling on a circular curve of 1500 metres radius at the rate of 66 km/hr. Through what angle has it turned in 10 seconds?
15. The radius of a circle is 30 cm. Find the length of an arc of this circle, if the length of the chord of the arc is 30 cm.
16. The angles of a triangle are in *A.P.* and the number of degrees in the least angle is to the number of degrees in the mean angle as 1 : 120. Find the angles in radians.
17. Find the magnitude, in radians and degrees, of the interior angle of a regular
    (i) pentagon (ii) octagon (iii) heptagon (iv) duodecagon.

18. The angles of a quadrilateral are in A.P. and the greatest angle is 120°. Express the angles in radians.

BASED ON HOTS

19. Find the distance from the eye at which a coin of 2 cm diameter should be held so as to conceal the full moon whose angular diameter is 31′.
20. Find the diameter of the sun in km supposing that it subtends an angle of 32′ at the eye of an observer. Given that the distance of the sun is 151.92×10^6 km.

ANSWERS

1. (i) 324° (ii) −150° (iii) 648° (iv) −171° 49′ 5″ (v) 630° (vi) 57° 16′ 21″
2. (i) $\frac{5\pi}{3}$ (ii) $\frac{7\pi}{36}$ (iii) $-\frac{14\pi}{45}$ (iv) $\frac{3\pi}{4}$ (v) $-\frac{5\pi}{3}$ (vi) $\frac{\pi}{24}$ (vii) $\frac{251\pi}{360}$ (viii) $-\frac{19\pi}{72}$
3. 81°, 9°
4. 24°, 60°, 96°
5. 91.64 m
6. $\left(\frac{2}{15}\right)^c$ (ii) $\left(\frac{1}{5}\right)^c$ (iii) $\left(\frac{7}{25}\right)^c$
7. 12°, 36′
8. 8, 4
9. $\frac{\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{9}$
10. 10, 8
11. 22 : 13
12. 1.5365
13. 12π
14. $\left(\frac{11}{90}\right)^c$
15. 10π cm
16. $\frac{\pi}{360}, \frac{\pi}{3}, \frac{239\pi}{360}$
17. (i) $\left(\frac{3\pi}{5}\right)^c$; 108° (ii) $\left(\frac{3\pi}{4}\right)^c$; 135° (iii) $\left(\frac{5\pi}{7}\right)^c$, 128° 34′ 17″ (iv) $\left(\frac{5\pi}{6}\right)^c$; 150°
18. $\frac{\pi}{3}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{2\pi}{3}$
19. 2.217m
20. 1.4147×10^6 km

HINTS TO SELECTED PROBLEMS

4. $$\left(\frac{2}{3}x\right)^g = \left(\frac{2}{3}x \times \frac{90}{100}\right)^\circ = \left(\frac{3x}{5}\right)^\circ \text{ and, } \left(\frac{\pi x}{75}\right)^c = \left(\frac{\pi}{75} \times \frac{180}{\pi}\right)^c = \left(\frac{12x}{5}\right)^\circ$$

$$\therefore \quad \left(\frac{3}{5}x\right)^\circ + \left(\frac{3}{2}x\right)^\circ + \left(\frac{12x}{5}\right)^\circ = 180^\circ \Rightarrow x = 40^\circ$$

5. Here, $\theta = 25^\circ = \left(25 \times \frac{\pi}{180}\right)^c$ and arc = 40 meters.

17. A heptagon has seven sides and the number of sides of a dudecagon is twelve .

18. Let the measures of angles in degrees be $a - 3d, a - d, a + d, a + 3d$. Then,
Sum of the angles = $360^\circ \Rightarrow 4a = 360^\circ \Rightarrow a = 90^\circ$.
Also, Greatest angle = $120^\circ \Rightarrow a + 3d = 120^\circ \Rightarrow d = 10^\circ$.

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If D, G and R denote respectively the number of degrees, grades and radians in an angle, then

   (a) $\frac{D}{100} = \frac{G}{90} = \frac{2R}{\pi}$ (b) $\frac{D}{90} = \frac{G}{100} = \frac{R}{\pi}$

   (c) $\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$ (d) $\frac{D}{90} = \frac{G}{100} = \frac{R}{2\pi}$

2. If the angles of a triangle are in A.P., then the measure of one of the angles in radians is

   (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

3. The angle between the minute and hour hands of a clock at 8:30 is

   (a) 80° (b) 75° (c) 60° (d) 105°

4. At 3:40, the hour and minute hands of a clock are inclined at

   (a) $\frac{2\pi^c}{3}$ (b) $\frac{7\pi^c}{12}$ (c) $\frac{13\pi^c}{18}$ (d) $\frac{3\pi^c}{4}$

5. If the arcs of the same length in two circles subtend angles 65° and 110° at the centre, then the ratio of the radii of the circles is

   (a) 22 : 13 (b) 11 : 13 (c) 22 : 15 (d) 21 : 13

6. If OP makes 4 revolutions in one second, the angular velocity in radians per second is

   (a) π (b) 2π (c) 4π (d) 8π

7. A circular wire of radius 7 cm is cut and bent again into an arc of a circle of radius 12 cm. The angle subtended by the arc at the centre is

   (a) 50° (b) 210° (c) 100° (d) 60° (e) 195°

8. The radius of the circle whose arc of length 15π cm makes an angle of $3\pi/4$ radian at the centre is

   (a) 10 cm (b) 20 cm (c) $11\frac{1}{4}$ cm (d) $22\frac{1}{2}$ cm

ANSWERS

1. (c) 2. (b) 3. (b) 4. (c) 5. (a) 6. (d) 7. (b) 8. (b)

SUMMARY

1. The measure of an angle is the amount of rotation from the initial side to the terminal side.
2. The sense of an angle is positive or negative according as the initial side rotates in anti-clockwise or clockwise direction to get the terminal side.
3. Three systems of measuring angles are:

   (i) Sexagesimal system (ii) Centesimal system (iii) Circular system

   In sexagesimal system:

   1 right angle = 90 degrees (= 90°)

   1° = 60 minutes (= 60′)

   1′ = 60 seconds (= 60″)

In centesimal system:

$$1 \text{ right angle} = 100 \text{ grades } (=100^g)$$

$$1^g = 100 \text{ minutes } (= 100')$$

$$1' = 100 \text{ seconds } (= 100'')$$

In circular system, the unit of measurement is radian. One radian is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.

$$\pi \text{ radians} = 180°$$

4. The relation between three systems of measurement of an angle is

$$\frac{D}{90°} = \frac{G}{100} = \frac{2R}{\pi}$$

CHAPTER 5

TRIGONOMETRIC FUNCTIONS

5.1 INTRODUCTION

In earlier classes, we have studied trigonometric ratios for acute angles as the ratio of the sides of a right angled triangle. In this chapter, we will extend the definitions of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions.

5.2 TRIGONOMETRIC FUNCTIONS OF A REAL NUMBER

In the previous chapter, we have learnt that the radian measures of angles and real numbers can be considered as one and the same. In other words, every real number can be considered as the radian measure of an angle and radian measures of angles are real numbers. In fact, we have learnt that corresponding to every point P, representing a real number x, on the real line there is a point P' on the unit circle centred at the origin such that the radian measure of $\angle AOP'$ is x (see Fig. 4.13) and the radian measure of every angle determines a point on the real line representing a real number on the real line. So, let x be a real number represented by a point on the real line. Then there is a point P on the unit circle with centre at the origin of the coordinate axes such that the radian measure of $\angle AOP$ is x and so arc $AP = x$.

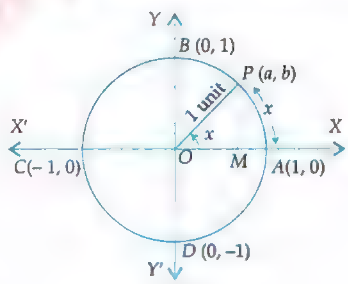


Fig. 5.1

Let the coordinates of point P be (a, b). Then, we define cosine and sine functions of radian measure (or real number) x as follows :

$$\cos x = a \quad \text{and} \quad \sin x = b$$

Thus, if x is any real number then the consine of x i.e. $\cos x$ is the x-coordinate of the point P on the unit circle such that arc $AP = x$. Similarly, sine of x i.e. $\sin x$ is the y-coordinate of point P.

REMARK 1 *In Fig. 5.1, x is the length of arc AP of the unit circle. Therefore, $\cos x$ and $\sin x$ are also known as circular functions of the real variable x.*

REMARK 2 *In Fig. 5.1, ΔOMP is a right triangle right angled at M. The trigonometric ratios $\angle MOP$ are*

$$\cos \angle MOP = \frac{OM}{OP} = \frac{a}{1} = a, \ \sin \angle MOP = \frac{PM}{OP} = \frac{b}{1} = b$$

$$\Rightarrow \quad \cos \angle AOP = a \quad \text{and} \quad \sin \angle AOP = b \qquad [\because \ \angle MOP = \angle AOP]$$

$$\Rightarrow \quad \cos \angle AOP = \cos x \quad \text{and} \quad \sin \angle AOP = \sin x$$

Thus, the trigonometric ratios sine and cosine of an acute angle of radian measure x are same as the corresponding trigonometric function of a real number x.

REMARK 3 *From the above definition it follows that if P is a point on the unit circle such that length of arc $AP = x$ or equivalently P is a point where the terminal side of the angle with radian measure x meets the unit circle, then the coordinates of the point P are* $(\cos x, \sin x)$.

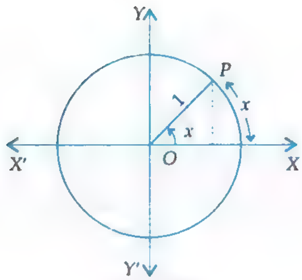


Fig. 5.2

5.2.1 VALUES OF SINE AND COSINE FUNCTIONS

Consider a unit circle with centre at the origin of the coordinate axes. Suppose the circle cuts the coordinate axes at A, B, C and D. The coordinates of these points are $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Clearly, $\angle AOB = \frac{\pi}{2}$, $\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$.

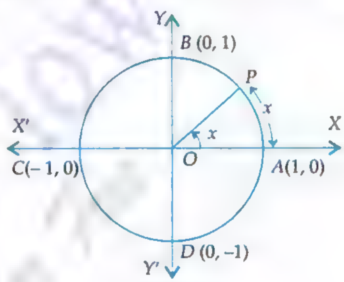


Fig. 5.3

We shall now find the values of sine and cosine functions at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π.

Values of sine and cosine functions at $x = 0$: When $x = 0$, point P coincides with A and the terminal side OP coincides with OA. The coordinates of A are $(1, 0)$.

$\therefore \quad \cos 0 = 1$ and $\sin 0 = 0$

Values of sine and cosine functions at $x = \frac{\pi}{2}$: We observe that $\angle AOB = \frac{\pi}{2}$ and the coordinates of B are $(0, 1)$.

$\therefore \quad \cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$.

Values of sine and cosine functions at $x = \pi$: Clearly, $\angle AOC = \pi$ and the coordinates of C are $(-1, 0)$.

$\therefore \quad \cos \pi = -1$ and $\sin \pi = 0$

Values of sine and cosine functions at $x = \frac{3\pi}{2}$: The coordinates of point D are $(0, -1)$ and $\angle AOC = \frac{3\pi}{2}$.

$\therefore \quad \cos \frac{3\pi}{2} = 0$ and $\sin \frac{3\pi}{2} = -1$

Values of sine and cosine functions at $x = 2\pi$: The coordinates of point A are $(1, 0)$ and one complete revolution subtends an angle of measure 2π at the centre O.

$\therefore \quad \cos 2\pi = 1$ and $\sin 2\pi = 0$

If the terminal side OP of $\angle AOP$ takes one complete revolution from the position OP, it again comes back to the same position.

$\therefore \quad \cos(2\pi + x) = \cos x$ and $\sin(2\pi + x) = \sin x$ for all $x \in R$

We also observe that if the terminal side OP of $\angle AOP$ takes any number of complete revolutions in anticlockwise or clockwise directions, it again comes back to the same position.

$\therefore \quad \cos(2n\pi + x) = \cos x$ and $\sin(2n\pi + x) = \sin x$ for all $x \in R$ and $n \in Z$.

It is evident from Fig. 5.2 that

$$\sin 0 = 0,\ \sin \pi = 0,\ \sin 2\pi = 0,\ \sin 3\pi = 0 \ldots\ldots$$

Also, $\sin(-\pi) = 0,\ \sin(-2\pi) = 0,\ \sin(-3\pi) = 0 \ldots\ldots$

$\therefore \quad \sin n\pi = 0$ for all $n \in Z$

and, $\cos\frac{\pi}{2} = 0,\ \cos\frac{3\pi}{2} = 0,\ \cos\frac{5\pi}{2} = 0 \ldots\ldots$

$$\cos\left(-\frac{\pi}{2}\right) = 0,\ \cos\left(-\frac{3\pi}{2}\right) = 0,\ \cos\left(-\frac{5\pi}{2}\right) = 0 \ldots\ldots$$

$\therefore \quad \cos(2n+1)\frac{\pi}{2} = 0$, for all $n \in Z$.

Thus, $\sin x = 0 \Rightarrow x = n\pi, n \in Z$ and, $\cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}, n \in Z$

5.2.2 OTHER TRIGONOMETRIC FUNCTIONS

In the previous subsections, we have defined sine and cosine functions. In this section, we shall define other four trigonometric functions in terms of these two functions.

We define

$$\operatorname{cosec} x = \frac{1}{\sin x}, \text{ where } x \neq n\pi, n \in Z;\ \sec x = \frac{1}{\cos x}, \text{ where } x \neq (2n+1)\frac{\pi}{2}, n \in Z$$

$$\tan x = \frac{\sin x}{\cos x}, \text{ where } x \neq (2n+1)\frac{\pi}{2}, n \in Z;\ \cot x = \frac{\cos x}{\sin x}, \text{ where } x \neq n\pi, n \in Z$$

5.3 VALUES OF TRIGONOMETRIC FUNCTIONS

In this section, we will find the values of trigonometric functions for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ and $\frac{\pi}{2}$. In section 5.2.1, we have learnt that $\sin 0 = 0, \cos 0 = 1, \sin\frac{\pi}{2} = 1, \cos\frac{\pi}{2} = 0, \sin \pi = 0, \cos \pi = -1, \sin\frac{3\pi}{2} = -1, \sin 2\pi = 0$ and $\cos 2\pi = 1$.

We have also learnt that $\operatorname{cosec} x = \frac{1}{\sin x}$, where $n \neq n\pi, n \in Z$. Therefore, cosec 0, cosec π and cosec 2π are not defined.

$$\operatorname{cosec}\frac{\pi}{2} = \frac{1}{\sin\frac{\pi}{2}} = 1 \text{ and } \operatorname{cosec}\frac{3\pi}{2} = \frac{1}{\sin\frac{3\pi}{2}} = -1$$

Similarly, $\sec x = \frac{1}{\cos x}$ implies that

$$\sec 0 = \frac{1}{\cos 0} = 1,\ \sec \pi = \frac{1}{\cos \pi} = -1,\ \sec 2\pi = \frac{1}{\cos 2\pi} = 1 \text{ and, } \sec\frac{\pi}{2} = \sec\frac{3\pi}{2} \text{ are not defined.}$$

$\tan 0 = \frac{\sin x}{\cos x}$ implies that

$$\tan 0 = \frac{\sin 0}{\cos 0} = 0,\ \tan \pi = \frac{\sin \pi}{\cos \pi} = 0,\ \tan 2\pi = \frac{\sin 2\pi}{\cos 2\pi} = 0 \text{ and } \tan\frac{\pi}{2}, \tan\frac{3\pi}{2} \text{ are undefined.}$$

$\cot x = \frac{\cos x}{\sin x}$ implies that

$$\cot\frac{\pi}{2}=\frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}}=0,\ \cot\frac{3\pi}{2}=\frac{\cos\frac{3\pi}{2}}{\sin\frac{3\pi}{2}}=0 \text{ and } \cot 0, \cot\pi, \cot 2\pi \text{ are not defined.}$$

Let us now find the values of all trigonometric functions at $\frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$.

Values of trigonometric functions at $\frac{\pi}{4}$: Consider a unit circle with centre at the origin of the coordinate axes. Let P be a point on the circle such that $\angle XOP=\frac{\pi}{4}$. Draw PM perpendicular from P on OX. In right triangle OMP, we have $\angle POM=\frac{\pi}{4}$. Therefore, $\angle OPM=\frac{\pi}{4}$. Thus, we obtain

$$\angle POM=\angle OPM=\frac{\pi}{4} \Rightarrow OM=PM$$

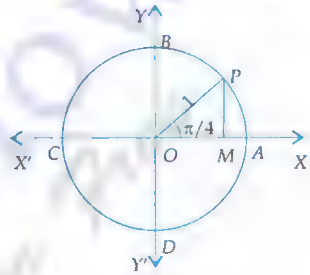


Fig. 5.4

Applying Pythagoras theorem in ΔOMP, we obtain

$$OM^2+PM^2=OP^2$$

$$\Rightarrow \quad 2\,OM^2=1 \qquad [\because OM=PM \text{ and } OP=1]$$

$$\Rightarrow \quad OM=\frac{1}{\sqrt{2}} \Rightarrow OM=PM=\frac{1}{\sqrt{2}}$$

So, the coordinates of P are $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$. Hence, $\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ and $\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$.

Now,

$$\operatorname{cosec} x=\frac{1}{\sin x} \Rightarrow \operatorname{cosec}\frac{\pi}{4}=\frac{1}{\sin\frac{\pi}{4}}=\sqrt{2}\,;\ \sec x=\frac{1}{\cos x} \Rightarrow \sec\frac{\pi}{4}=\frac{1}{\cos\frac{\pi}{4}}=\sqrt{2}$$

$$\tan x=\frac{\sin x}{\cos x} \Rightarrow \tan\frac{\pi}{4}=\frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}}=1\,;\ \cot x=\frac{\cos x}{\sin x} \Rightarrow \cot\frac{\pi}{4}=\frac{\cos\frac{\pi}{4}}{\cos\frac{\pi}{4}}=1$$

Values of trigonometric functions at $\frac{\pi}{3}$: Consider a unit circle with centre at the origin of the coordinate axes. Let P be a point on the circle such that $\angle XOP=\frac{\pi}{3}$. Join PA. In ΔOAP, we have $OA=OP=1$ Unit. Therefore, $\angle OPA=\angle OAP$. But, $\angle AOP=\frac{\pi}{3}$.

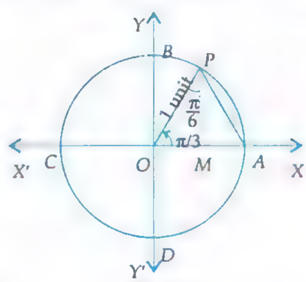


Fig. 5.5

Using angle sum property in ΔOAP, we obtain

$$\angle OAP+\angle OPA+\angle AOP=\pi \Rightarrow \frac{\pi}{3}+2\,\angle OPA=\pi \Rightarrow \angle OPA=\frac{\pi}{3}$$

Thus, in ΔOAP, we obtain: $\angle OAP=\angle OPA=\angle AOP=\frac{\pi}{3}$

So, ΔOAP is an equilateral triangle and hence perpendicular PM drawn from vertex P to the opposite side OA bisects it.

$$\therefore \quad OM=AM=\frac{1}{2} \text{ unit}$$

Applying Pythagoras theorem in ΔOMP, we obtain

$$OP^2 = OM^2 + MP^2 \Rightarrow 1^2 = \left(\frac{1}{2}\right)^2 + MP^2 \Rightarrow PM = \frac{\sqrt{3}}{2}$$

So, the coordinates of P are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. But, $\angle AOP = \frac{\pi}{3}$. So, the coordinates of P are $\left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right)$.

$\therefore \quad \cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

Now, $\quad \operatorname{cosec} x = \frac{1}{\sin x} \Rightarrow \operatorname{cosec}\frac{\pi}{3} = \frac{1}{\sin\frac{\pi}{3}} = \frac{2}{\sqrt{3}}$; $\sec x = \frac{1}{\cos x} \Rightarrow \sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = 2$

$$\tan x = \frac{\sin x}{\cos x} \Rightarrow \tan\frac{\pi}{3} = \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \sqrt{3}; \ \cot x = \frac{\cos x}{\sin x} \Rightarrow \cot\frac{\pi}{3} = \frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} = \frac{1}{\sqrt{3}}$$

Values of trigonometric functions at $\frac{\pi}{6}$: Consider a unit circle with centre at the origin of the coordinate axes. Let P be a point on the circle such that $\angle XOP = \frac{\pi}{6}$. Draw PM perpendicular from P on OX.

In right triangle OMP right angled at M, we have $\angle POM = \frac{\pi}{6}$.

Therefore, $\angle OPM = \frac{\pi}{3}$.

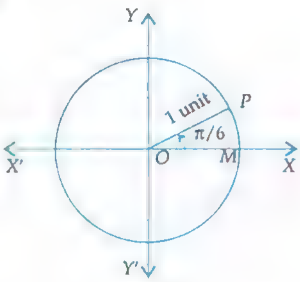


Fig. 5.6

In Fig. 5.6, we have seen that in a right triangle if the measures of angles other than the right angle are $\frac{\pi}{3}$ and $\frac{\pi}{6}$, then the sides opposite to them are of length $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$ respectively. Thus, in ΔOMP, we obtain: $OM = \frac{\sqrt{3}}{2}$ and $PM = \frac{1}{2}$

So, the coordinates of P are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. But, the coordinates of P are $\left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$.

$\therefore \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$

Now,

$$\operatorname{cosec} x = \frac{1}{\sin x} \Rightarrow \operatorname{cosec}\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = 2; \ \sec x = \frac{1}{\cos x} \Rightarrow \sec\frac{\pi}{6} = \frac{1}{\cos\frac{\pi}{6}} = \frac{2}{\sqrt{3}}$$

$$\tan x = \frac{\sin x}{\cos x} \Rightarrow \tan\frac{\pi}{6} = \frac{\sin\frac{\pi}{6}}{\sin\frac{\pi}{6}} = \frac{1}{\sqrt{3}}; \ \cot x = \frac{\cos x}{\sin x} \Rightarrow \cot\frac{\pi}{6} = \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} = \sqrt{3}$$

In the above discussion, we have obtained the values of various trigonometric functions for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π. These values are listed below for ready reference.

Angle / Trigonometric Function	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0
cosec	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	not defined	-1	not defined
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined	-1	not defined	1
cot	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	not defined	0	not defined

5.4 TRIGONOMETRIC IDENTITIES

IDENTITY *An equation involving trigonometric functions which is true for all these values of the variable for which the functions are defined is called a trigonometric identity.*

For example, $\sin^2 x = 1 - \cos^2 x$, $\sec^2 x - 1 = \tan^2 x$ are trigonometric identities as they hold for all values of variable x except those values for which $\sec x$ and $\tan x$ are not defined. But, $\sin x = \cos x$ is a trigonometric equation not a trigonometric identity because it does not hold for all values of x.

5.4.1 FUNDAMENTAL TRIGONOMETRIC IDENTITIES

In this section, we shall state and prove three fundamental trigonometric identities as a theorem.

THEOREM *Prove that:*

(i) $\cos^2 x + \sin^2 x = 1$ *for all* $x \in R$

(ii) $1 + \tan^2 x = \sec^2 x$ *for all* $x \in R - \left\{(2n-1)\frac{\pi}{2} : n \in Z\right\}$

(iii) $1 + \cot^2 x = \text{cosec}^2 x$ *for all* $x \in R - \{n\pi : n \in Z\}$

PROOF (i) Consider a unit circle with centre at the origin O of coordinates axes. Let $P(a, b)$ be a point on the circle such that arc $AP = x$. Then, $\angle AOP = x$. Using the definition of trigonometric functions $\cos x$ and $\sin x$, we obtain: $a = \cos x$ and $b = \sin x$

Now,

$$OP = 1$$

$$\Rightarrow \quad \sqrt{(a-0)^2 + (b-0)^2} = 1$$

$$\Rightarrow \quad a^2 + b^2 = 1 \Rightarrow \cos^2 x + \sin^2 x = 1$$

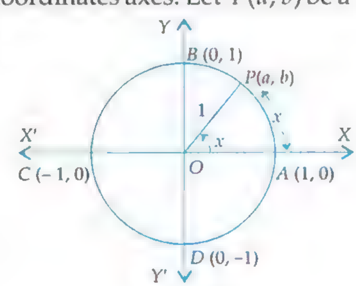


Fig. 5.7

(ii) If $x \neq (2n-1)\frac{\pi}{2}$, then $\cos x \neq 0$. So, dividing the identity $\cos^2 x + \sin^2 x = 1$ throughout by $\cos^2 x$, we obtain

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \quad \text{for all } x \neq (2n-1)\frac{\pi}{2}$$

$$\Rightarrow \quad \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow 1 + \tan^2 x = \sec^2 x \quad \text{for all } x \neq (2n-1)\frac{\pi}{2}$$

(iii) If $x \neq n\pi$, then $\sin x \neq 0$. So, dividing both sides of the identity $\cos^2 x + \sin^2 x = 1$ by $\sin^2 x$, we obtain

$$\frac{\cos^2 x + \sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \quad \text{for all } x \neq n\pi$$

$$\Rightarrow \quad \frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow \cot^2 x + 1 = \text{cosec}^2 x \Rightarrow 1 + \cot^2 x = \text{cosec}^2 x \quad \text{for all } x \neq n\pi$$

Q.E.D

REMARK 1 *The identity* $1 + \tan^2 x = \sec^2 x$ *is also written in the following forms :*

$$\sec^2 x - 1 = \tan^2 x \quad \text{and} \quad \sec^2 x - \tan^2 x = 1$$

REMARK 2 *The identity* $1 + \cot^2 x = \text{cosec}^2 x$ *is also written in the following forms :*

$$\text{cosec}^2 x - 1 = \cot^2 x \quad \text{and,} \quad \text{cosec}^2 x - \cot^2 x = 1$$

REMARK 3 *We have,* $\sec^2 x - \tan^2 x = 1$

$$\Rightarrow \quad (\sec x + \tan x)(\sec x - \tan x) = 1 \Rightarrow \sec x - \tan x = \frac{1}{\sec x + \tan x} \text{ and, } \sec x + \tan x = \frac{1}{\sec x - \tan x}$$

i.e. $\sec x + \tan x$ and $\sec x - \tan x$ are reciprocal of each other.

REMARK 4 *We have,* $\text{cosec}^2 x - \cot^2 x = 1$

$$\Rightarrow (\text{cosec } x - \cot x)(\text{cosec } x + \cot x) = 1$$

$$\Rightarrow \text{cosec } x - \cot x = \frac{1}{\text{cosec } x + \cot x} \quad \text{and} \quad \text{cosec } x + \cot x = \frac{1}{\text{cosec } x - \cot x}$$

i.e. $\text{cosec } x + \cot x$ and $\text{cosec } x - \cot x$ are reciprocal of each other.

We shall now discuss more identities involving trigonometric functions in the following examples.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Prove the following identities:*

(i) $\sin^8 x - \cos^8 x = (\sin^2 x - \cos^2 x)(1 - 2\sin^2 x \cos^2 x)$

(ii) $\cot^4 x + \cot^2 x = \text{cosec}^4 x - \text{cosec}^2 x$

(iii) $2\sec^2 x - \sec^4 x - 2\,\text{cosec}^2 x + \text{cosec}^4 x = \cot^4 x - \tan^4 x$

(iv) $(\sin x + \text{cosec } x)^2 + (\cos x + \sec x)^2 = \tan^2 x + \cot^2 x + 7$

SOLUTION (i) LHS $= (\sin^8 x - \cos^8 x) = (\sin^4 x)^2 - (\cos^4 x)^2$

$= (\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)$

$= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x)$

$$= (\sin^2 x - \cos^2 x)(\sin^4 x + \cos^4 x)$$
$$= (\sin^2 x - \cos^2 x)(\sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x - 2\sin^2 x \cos^2 x)$$
$$= (\sin^2 x - \cos^2 x)\left\{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x\right\}$$
$$= (\sin^2 x - \cos^2 x)(1 - 2\sin^2 x\cos^2 x) = \text{RHS}$$

(ii) $\text{LHS} = \cot^4 x + \cot^2 x = (\cot^2 x)^2 + \cot^2 x$

$= (\text{cosec}^2 x - 1)^2 + (\text{cosec}^2 x - 1)$ $\quad [\because 1 + \cot^2 x = \text{cosec}^2 x]$

$= \text{cosec}^4 x - 2\,\text{cosec}^2 x + 1 + \text{cosec}^2 x - 1 = \text{cosec}^4 x - \text{cosec}^2 x = \text{RHS}$

(iii) $\text{LHS} = 2\sec^2 x - \sec^4 x - 2\,\text{cosec}^2 x + \text{cosec}^4 x$

$= 2\sec^2 x - (\sec^2 x)^2 - 2\,\text{cosec}^2 x + (\text{cosec}^2 x)^2$

$= 2(1 + \tan^2 x) - (1 + \tan^2 x)^2 - 2(1 + \cot^2 x) + (\cot^2 x + 1)^2$

$= 2 + 2\tan^2 x - (1 + \tan^4 x + 2\tan^2 x) - 2 - 2\cot^2 x + (\cot^4 x + 2\cot^2 x + 1)$

$= \cot^4 x - \tan^4 x = \text{RHS}$

(iv) $\text{LHS} = (\sin x + \text{cosec}\, x)^2 + (\cos x + \sec x)^2$

$= \sin^2 x + \text{cosec}^2 x + 2\sin x\,\text{cosec}\, x + \cos^2 x + \sec^2 x + 2\cos x\sec x$

$= (\sin^2 x + \cos^2 x) + (\text{cosec}^2 x + \sec^2 x) + 2 + 2$

$= 1 + (1 + \cot^2 x) + (1 + \tan^2 x) + 4 = \tan^2 x + \cot^2 x + 7 = \text{RHS}$

EXAMPLE 2 *Prove the following identities:*

(i) $(1 + \cot x - \text{cosec}\, x)(1 + \tan x + \sec x) = 2$ (ii) $\dfrac{\tan x + \sec x - 1}{\tan x - \sec x + 1} = \dfrac{1 + \sin x}{\cos x}$

[NCERT EXEMPLAR]

SOLUTION (i) $\text{LHS} = (1 + \cot x - \text{cosec}\, x)(1 + \tan x + \sec x)$

$$= \left(1 + \frac{\cos x}{\sin x} - \frac{1}{\sin x}\right)\left(1 + \frac{\sin x}{\cos x} + \frac{1}{\cos x}\right) = \frac{(\sin x + \cos x - 1)}{\sin x}\,\frac{(\sin x + \cos x + 1)}{\cos x}$$

$$= \frac{(\sin x + \cos x)^2 - 1}{\sin x \cos x} = \frac{\sin^2 x + \cos^2 x + 2\sin x\cos x - 1}{\sin x\cos x}$$

$$= \frac{2\sin x\cos x}{\sin x\cos x} = 2 = \text{RHS}$$

(ii) $\text{LHS} = \dfrac{\tan x + \sec x - 1}{\tan x - \sec x + 1} = \dfrac{(\tan x + \sec x) - (\sec^2 x - \tan^2 x)}{\tan x - \sec x + 1}$ $\quad [\because \sec^2 x - \tan^2 x = 1]$

$$= \frac{(\sec x + \tan x)\{1 - (\sec x - \tan x)\}}{\tan x - \sec x + 1} = \frac{(\sec x + \tan x)(\tan x - \sec x + 1)}{\tan x - \sec x + 1}$$

$$= \sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \frac{1 + \sin x}{\cos x} = \text{RHS}$$

EXAMPLE 3 *If* $\tan x + \sin x = m$ *and* $\tan x - \sin x = n$, *show that* $m^2 - n^2 = 4\sqrt{mn}$.

[NCERT EXEMPLAR]

SOLUTION We have, $\tan x + \sin x = m$ and $\tan x - \sin x = n$.

$\therefore \quad m^2 - n^2 = (\tan x + \sin x)^2 - (\tan x - \sin x)^2 = 4\tan x\sin x \quad$ ...(i)

and, $4\sqrt{mn} = 4\sqrt{(\tan x + \sin x)(\tan x - \sin x)} = 4\sqrt{\tan^2 x - \sin^2 x}$

$$= 4\sqrt{\frac{\sin^2 x}{\cos^2 x} - \sin^2 x} = 4\sqrt{\frac{\sin^2 x - \sin^2 x \cos^2 x}{\cos^2 x}} = 4\sqrt{\frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x}}$$

$$= 4\sqrt{\frac{\sin^4 x}{\cos^2 x}} = 4\frac{\sin^2 x}{\cos x} = 4\tan x \sin x \qquad ...(ii)$$

From (i) and (ii), we obtain: $m^2 - n^2 = 4\sqrt{mn}$.

EXAMPLE 4 *If* $\cos x + \sin x = \sqrt{2}\cos x$, *prove that* $\cos x - \sin x = \sqrt{2}\sin x$.

SOLUTION We have,

$$\cos x + \sin x = \sqrt{2}\cos x$$

$\Rightarrow \quad (\cos x + \sin x)^2 = (\sqrt{2}\cos x)^2 \Rightarrow \cos^2 x + \sin^2 x + 2\sin x\cos x = 2\cos^2 x$

$\Rightarrow \quad \cos^2 x - \sin^2 x = 2\sin x\cos x \Rightarrow (\cos x + \sin x)(\cos x - \sin x) = 2\sin x\cos x$

$\Rightarrow \quad \cos x - \sin x = \dfrac{2\sin x\cos x}{\cos x + \sin x} \Rightarrow \cos x - \sin x = \dfrac{2\sin x\cos x}{\sqrt{2}\cos x} \quad [\because \cos x + \sin x = \sqrt{2}\cos x]$

$\Rightarrow \quad \cos x - \sin x = \sqrt{2}\sin x$

ALITER We know that

$$(\cos x + \sin x)^2 + (\cos x - \sin x)^2 = 2$$

$\Rightarrow \quad (\sqrt{2}\cos x)^2 + (\cos x - \sin x)^2 = 2 \qquad [\because \cos x + \sin x = \sqrt{2}\cos x]$

$\Rightarrow \quad (\cos x - \sin x)^2 = 2 - 2\cos^2 x \Rightarrow (\cos x - \sin x)^2 = 2\sin^2 x \Rightarrow \cos x - \sin x = \sqrt{2}\sin x$.

EXAMPLE 5 *If* $a\cos x + b\sin x = m$ *and* $a\sin x - b\cos x = n$, *prove that* $a^2 + b^2 = m^2 + n^2$.

[NCERT EXEMPLAR]

SOLUTION We have, $m = a\cos x + b\sin x$ and $n = a\sin x - b\cos x$.

$\therefore \quad m^2 + n^2 = (a\cos x + b\sin x)^2 + (a\sin x - b\cos x)^2$

$$= (a^2\cos^2 x + b^2\sin^2 x + 2ab\sin x\cos x) + (a^2\sin^2 x + b^2\cos^2 x - 2ab\sin x\cos x)$$

$$= a^2(\cos^2 x + \sin^2 x) + b^2(\sin^2 x + \cos^2 x) = a^2 + b^2$$

EXAMPLE 6 *If* $a\cos x - b\sin x = c$, *show that* $a\sin x + b\cos x = \pm\sqrt{a^2 + b^2 - c^2}$.

SOLUTION Clearly,

$$(a\cos x - b\sin x)^2 + (a\sin x + b\cos x)^2$$

$$= a^2(\cos^2 x + \sin^2 x) + b^2(\sin^2 x + \cos^2 x) - 2ab\sin x\cos x + 2ab\sin x\cos x = a^2 + b^2$$

$\therefore \quad (a\sin x + b\cos x)^2 = a^2 + b^2 - (a\cos x - b\sin x)^2$

$\Rightarrow \quad (a\sin x + b\cos x)^2 = a^2 + b^2 - c^2 \qquad [\because a\cos x - b\sin x = c]$

$\Rightarrow \quad a\sin x + b\cos x = \pm\sqrt{a^2 + b^2 - c^2}$

EXAMPLE 7 *If* $\sec x + \tan x = p$, *obtain the values of* $\sec x$, $\tan x$ *and* $\sin x$ *in terms of* p.

SOLUTION We know that : $\sec x + \tan x$ and $\sec x - \tan x$ are reciprocal of each other.

$\therefore \quad \sec x + \tan x = p \Rightarrow \sec x - \tan x = \dfrac{1}{p}$

$\Rightarrow \quad (\sec x + \tan x) + (\sec x - \tan x) = p + \dfrac{1}{p}$ and, $(\sec x + \tan x) - (\sec x - \tan x) = p - \dfrac{1}{p}$

$$\Rightarrow \quad 2\sec x = p + \frac{1}{p} \text{ and, } 2\tan x = p - \frac{1}{p} \Rightarrow \sec x = \frac{p^2+1}{2p} \text{ and, } \tan x = \frac{p^2-1}{2p}$$

$$\therefore \quad \sin x = \frac{\tan x}{\sec x} \Rightarrow \sin x = \frac{p^2-1}{p^2+1}$$

EXAMPLE 8 *Prove that:* $2\sec^2 x - \sec^4 x - 2\operatorname{cosec}^2 x + \operatorname{cosec}^4 x = \dfrac{1-\tan^8 x}{\tan^4 x}$.

SOLUTION $2\sec^2 x - \sec^4 x - 2\operatorname{cosec}^2 x + \operatorname{cosec}^4 x$

$$= 2(1+\tan^2 x) - (1+\tan^2 x)^2 - 2(1+\cot^2 x) + (1+\cot^2 x)^2$$

$$= 2(1+\tan^2 x - 1 - \cot^2 x) + (1 + 2\cot^2 x + \cot^4 x) - (1 + 2\tan^2 x + \tan^4 x)$$

$$= 2(\tan^2 x - \cot^2 x) + (2\cot^2 x - 2\tan^2 x) + \cot^4 x - \tan^4 x$$

$$= \cot^4 x - \tan^4 x = \frac{1}{\tan^4 x} - \tan^4 x = \frac{1-\tan^8 x}{\tan^4 x}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 9 *Prove that:* $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) - 13 = 0$.

SOLUTION We have,

$$3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) - 13$$

$$= 3\left\{(\sin x - \cos x)^2\right\}^2 + 6(\sin x + \cos x)^2$$

$$+ 4\left\{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)\right\} - 13$$

$$= 3(1 - 2\sin x\cos x)^2 + 6(1 + 2\sin x\cos x) + 4(1 - 3\sin^2 x\cos^2 x) - 13$$

$$= 3(1 - 4\sin x\cos x + 4\sin^2 x\cos^2 x) + 6(1 + 2\sin x\cos x) + 4(1 - 3\sin^2 x\cos^2 x) - 13$$

$$= 3 + 6 + 4 - 13 = 0$$

EXAMPLE 10 *Given that:* $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma) = (1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$. *Show that one of the values of each member of this equality is* $\sin\alpha\sin\beta\sin\gamma$.

SOLUTION We have,

$$(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma) = (1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$$

Multiplying both sides by $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)$, we get

$$(1+\cos\alpha)^2(1+\cos\beta)^2(1+\cos\gamma)^2 = (1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)$$

$$\Rightarrow \quad (1+\cos\alpha)^2(1+\cos\beta)^2(1+\cos\gamma)^2 = (1-\cos^2\alpha)(1-\cos^2\beta)(1-\cos^2\gamma)$$

$$\Rightarrow \quad (1+\cos\alpha)^2(1+\cos\beta)^2(1+\cos\gamma)^2 = \sin^2\alpha\sin^2\beta\sin^2\gamma$$

$$\Rightarrow \quad (1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma) = \pm\sin\alpha\sin\beta\sin\gamma$$

Hence, one of the values of $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)$ is $\sin\alpha\sin\beta\sin\gamma$.

Similarly, by multiplying both sides by $(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$, we find that one of the values of $(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$ is also $\sin\alpha\sin\beta\sin\gamma$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 11 *Prove that:* $\sec^2 x + \text{cosec}^2 x \geq 4$.

SOLUTION $\sec^2 x + \text{cosec}^2 x = (1 + \tan^2 x) + 1 + (\cot^2 x) = 2 + \tan^2 x + \cot^2 x$

$$= 2 + \tan^2 x + \cot^2 x - 2\tan x \cot x + 2 \tan x \cot x$$

$$= 2 + (\tan x - \cot x)^2 + 2$$

$$= 4 + (\tan x - \cot x)^2 \geq 4 \qquad [\because (\tan x - \cot x)^2 \geq 0]$$

EXAMPLE 12 *If* $10 \sin^4 \alpha + 15 \cos^4 \alpha = 6$, *find the value of* $27 \text{ cosec}^6 \alpha + 8 \sec^6 \alpha$.

SOLUTION We have,

$$10 \sin^4 \alpha + 15 \cos^4 \alpha = 6$$

$$\Rightarrow \quad 10 \sin^4 \alpha + 15 \cos^4 \alpha = 6 (\sin^2 \alpha + \cos^2 \alpha)^2$$

$$\Rightarrow \quad 10 \tan^4 \alpha + 15 = 6 (\tan^2 \alpha + 1)^2 \qquad [\text{Dividing boths sides by } \cos^4 \alpha]$$

$$\Rightarrow \quad (2 \tan^2 \alpha - 3)^2 = 0 \Rightarrow \tan^2 \alpha = \frac{3}{2}$$

$$\therefore \quad 27 \text{ cosec}^6 \alpha + 8 \sec^6 \alpha = 27 (1 + \cot^2 \alpha)^3 + 8 (1 + \tan^2 \alpha)^3$$

$$= 27 \left(1 + \frac{2}{3}\right)^3 + 8 \left(1 + \frac{3}{2}\right)^3 = 27 \times \frac{125}{27} + 8 \times \frac{125}{8} = 250.$$

EXAMPLE 13 *If* $\dfrac{\sin A}{\sin B} = p$ *and* $\dfrac{\cos A}{\cos B} = q$, *find* $\tan A$ *and* $\tan B$.

SOLUTION We have,

$$\frac{\sin A}{\sin B} = p \text{ and } \frac{\cos A}{\cos B} = q \Rightarrow \frac{\sin A}{\sin B} \cdot \frac{\cos B}{\cos A} = \frac{p}{q}$$

$$\Rightarrow \quad \frac{\tan A}{\tan B} = \frac{p}{q} \Rightarrow \frac{\tan A}{p} = \frac{\tan B}{q} = \lambda \text{ (say)} \Rightarrow \tan A = p\lambda \text{ and } \tan B = q\lambda \qquad \text{...(i)}$$

Now, $\sin A = p \sin B$

$$\Rightarrow \quad \frac{\tan A}{\sqrt{1 + \tan^2 A}} = p \frac{\tan B}{\sqrt{1 + \tan^2 B}}$$

$$\Rightarrow \quad \frac{p\lambda}{\sqrt{1 + p^2 \lambda^2}} = p \frac{q\lambda}{\sqrt{1 + q^2 \lambda^2}} \Rightarrow p^2 (1 + q^2 \lambda^2) = p^2 q^2 (1 + p^2 \lambda^2)$$

$$\Rightarrow \quad \lambda^2 (q^2 - p^2 q^2) = q^2 - 1 \Rightarrow \lambda^2 = \frac{q^2 - 1}{q^2 (1 - p^2)} \Rightarrow \lambda = \pm \frac{1}{q} \sqrt{\frac{q^2 - 1}{1 - p^2}}$$

$$\therefore \quad \tan A = \pm \frac{p}{q} \sqrt{\frac{q^2 - 1}{1 - p^2}} \text{ and, } \tan B = \pm \sqrt{\frac{q^2 - 1}{1 - p^2}} \qquad [\text{Using (i)}]$$

EXAMPLE 14 *If* $\tan^2 x = 1 - a^2$, *prove that* $\sec x + \tan^3 x \text{ cosec } x = (2 - a^2)^{3/2}$. *Also, find the values of* a *for which the above result holds true.*

SOLUTION We have,

$$\sec x + \tan^3 x \operatorname{cosec} x = \sec x \left\{1 + \tan^3 x \frac{\operatorname{cosec} x}{\sec x}\right\}$$
$$= \sqrt{1+\tan^2 x}\left\{1 + \tan^3 x \times \cot x\right\}$$
$$= (1+\tan^2 x)^{3/2} = (1+1-a^2)^{3/2} = (2-a^2)^{3/2} \quad [\because \tan^2 x = 1-a^2]$$

Now,
$$\tan^2 x \geq 0 \text{ for all } x \Rightarrow 1-a^2 \geq 0 \Rightarrow a^2-1 \leq 0 \Rightarrow -1 \leq a \leq 1 \quad \text{...(i)}$$

Since LHS of $\sec x + \tan^3 x \operatorname{cosec} x = (2-a^2)^{3/2}$ is real for all $x \in R$. So, RHS must also be real.

$$\therefore \quad 2-a^2 \geq 0 \Rightarrow a^2-2 \leq 0 \Rightarrow -\sqrt{2} \leq a \leq \sqrt{2} \quad \text{...(ii)}$$

From (i) and (ii), we find that the given relation holds true for all $a \in [-1, 1]$.

EXAMPLE 15 *If* $a\cos^3 x + 3a\cos x \sin^2 x = m$ *and* $a\sin^3 x + 3a\cos^2 x \sin x = n$, *then prove that:* $(m+n)^{2/3} + (m-n)^{2/3} = 2a^{2/3}$.

SOLUTION We have,
$$a\cos^3 x + 3a\cos x\sin^2 x = m \quad \text{and} \quad a\sin^3 x + 3a\cos^2 x \sin x = n$$
$$\Rightarrow \quad a\cos^3 x + 3a\cos x\sin^2 x + a\sin^3 x + 3a\cos^2 x\sin x = m+n$$
and, $$a\cos^3 x + 3a\cos x\sin^2 x - a\sin^3 x - 3a\cos^2 x \sin x = m-n$$
$$\Rightarrow \quad a(\cos x + \sin x)^3 = m+n \quad \text{and,} \quad a(\cos x - \sin x)^3 = m-n$$
$$\Rightarrow \quad \cos x + \sin x = \left(\frac{m+n}{a}\right)^{1/3} \quad \text{and,} \quad \cos x - \sin x = \left(\frac{m-n}{a}\right)^{1/3}$$
$$\Rightarrow \quad (\cos x + \sin x)^2 + (\cos x - \sin x)^2 = \left(\frac{m+n}{a}\right)^{2/3} + \left(\frac{m-n}{a}\right)^{2/3}$$
$$\Rightarrow \quad 2(\cos^2 x + \sin^2 x) = \frac{(m+n)^{2/3}}{a^{2/3}} + \frac{(m-n)^{2/3}}{a^{2/3}}$$
$$\Rightarrow \quad (m+n)^{2/3} + (m-n)^{2/3} = 2a^{2/3}$$

EXAMPLE 16 *If* $2\tan^2\alpha\tan^2\beta\tan^2\gamma + \tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha = 1$, *prove that* $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$.

SOLUTION We have,
$$2\tan^2\alpha\tan^2\beta\tan^2\gamma + \tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha = 1$$

Dividing throughout by $\tan^2\alpha\tan^2\beta\tan^2\gamma$, we get
$$\Rightarrow \quad 2 + \cot^2\gamma + \cot^2\alpha + \cot^2\beta = \cot^2\alpha\cot^2\beta\cot^2\gamma$$
$$\Rightarrow \quad 2 + \operatorname{cosec}^2\gamma - 1 + \operatorname{cosec}^2\alpha - 1 + \operatorname{cosec}^2\beta - 1 = (\operatorname{cosec}^2\alpha - 1)(\operatorname{cosec}^2\beta - 1)(\operatorname{cosec}^2\gamma - 1)$$
$$\Rightarrow \quad \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma - 1 = \operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma - \operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta$$
$$- \operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma - \operatorname{cosec}^2\gamma\operatorname{cosec}^2\alpha + \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma - 1$$
$$\Rightarrow \quad \operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma = \operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta + \operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma + \operatorname{cosec}^2\gamma\operatorname{cosec}^2\alpha$$
$$\Rightarrow \quad 1 = \sin^2\gamma + \sin^2\alpha + \sin^2\beta \quad [\text{Multiplying throughout by } \sin^2\alpha\sin^2\beta\sin^2\gamma]$$
$$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$$

EXAMPLE 17 *If* $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2-b^2$ *and,* $\frac{ax\sin\theta}{\cos^2\theta}-\frac{by\cos\theta}{\sin^2\theta}=0$, *prove that* $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$.

SOLUTION We have,

$$\frac{ax\sin\theta}{\cos^2\theta}-\frac{by\cos\theta}{\sin^2\theta}=0 \Rightarrow ax\sin^3\theta-by\cos^3\theta=0 \Rightarrow \frac{\sin^3\theta}{by}=\frac{\cos^3\theta}{ax}$$

$$\Rightarrow \left(\frac{\sin^3\theta}{by}\right)^{2/3}=\left(\frac{\cos^3\theta}{ax}\right)^{2/3} \Rightarrow \frac{\sin^2\theta}{(by)^{2/3}}=\frac{\cos^2\theta}{(ax)^{2/3}}$$

$$\Rightarrow \frac{\sin^2\theta}{(by)^{2/3}}=\frac{\cos^2\theta}{(ax)^{2/3}}=\frac{\sin^2\theta+\cos^2\theta}{(by)^{2/3}+(ax)^{2/3}} \quad \text{[Using ratio and proportions]}$$

$$\Rightarrow \frac{\sin^2\theta}{(by)^{2/3}}=\frac{\cos^2\theta}{(ax)^{2/3}}=\frac{1}{(ax)^{2/3}+(by)^{2/3}}$$

$$\Rightarrow \sin^2\theta=\frac{(by)^{2/3}}{(ax)^{2/3}+(by)^{2/3}} \text{ and, } \cos^2\theta=\frac{(ax)^{2/3}}{(ax)^{2/3}+(by)^{2/3}}$$

$$\Rightarrow \sin\theta=\frac{(by)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}} \text{ and, } \cos\theta=\frac{(ax)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}}$$

Substituting these values in $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2-b^2$, we get

$$(ax)^{2/3}\sqrt{(ax)^{2/3}+(by)^{2/3}}+(by)^{2/3}\sqrt{(ax)^{2/3}+(by)^{2/3}}=a^2-b^2$$

$$\Rightarrow \left\{\sqrt{(ax)^{2/3}+(by)^{2/3}}\right\}\left\{(ax)^{2/3}+(by)^{2/3}\right\}=a^2-b^2$$

$$\Rightarrow \left\{(ax)^{2/3}+(by)^{2/3}\right\}^{3/2}=a^2-b^2 \Rightarrow (ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$$

EXAMPLE 18 *If* $m^2+m'^2+2mm'\cos x=1$, $n^2+n'^2+2nn'\cos x=1$, *and* $mn+m'n'+(mn'+m'n)\cos x=0$, *prove that* (i) $m^2+n^2=\operatorname{cosec}^2 x$ (ii) $m'^2+n'^2=\operatorname{cosec}^2 x$.

SOLUTION (i) We have,

$$m^2+m'^2+2mm'\cos x=1 \text{ and } n^2+n'^2+2nn'\cos x=1$$

$$\Rightarrow m'^2+2mm'\cos x+m^2\cos^2 x-m^2\cos^2 x+m^2=1$$

and, $$n'^2+2nn'\cos x+n^2\cos^2 x-n^2\cos^2 x+n^2=1$$

$$\Rightarrow (m'+m\cos x)^2+m^2(1-\cos^2 x)=1 \text{ and, } (n'+n\cos x)^2+n^2(1-\cos^2 x)=1$$

$$\Rightarrow (m'+m\cos x)^2=1-m^2\sin^2 x \quad ...(i) \quad \text{and,} \quad (n'+n\cos x)^2=1-n^2\sin^2 x \quad ...(ii)$$

Now, $$(m'+m\cos x)(n'+n\cos x)=m'n'+(mn'+m'n)\cos x+mn\cos^2 x$$

$$\Rightarrow (m'+m\cos x)(n'+n\cos x)=-mn+mn\cos^2 x \quad [\because mn+m'n'+(mn'+m'n)\cos x=0]$$

$$\Rightarrow (m'+m\cos x)(n'+n\cos x)=-mn(1-\cos^2 x)$$

$$\Rightarrow (m'+m\cos x)(n'+n\cos x)=-mn\sin^2 x$$

$$\Rightarrow (m'+m\cos x)^2(n'+n\cos x)^2=m^2n^2\sin^4 x \quad \text{[On squaring both sides]}$$

$$\Rightarrow (1-m^2\sin^2 x)(1-n^2\sin^2 x)=m^2n^2\sin^4 x \quad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad 1-(m^2+n^2)\sin^2 x+m^2 n^2 \sin^4 x = m^2 n^2 \sin^4 x$$

$$\Rightarrow \quad 1 = (m^2+n^2)\sin^2 x \Rightarrow m^2+n^2 = \operatorname{cosec}^2 x$$

(ii) As the given relations do not alter by replacing m by m' and n by n'. Therefore, on replacing m by m' and n by n' in $m^2+n^2=\operatorname{cosec}^2 x$, we get $m'+n'^2=\operatorname{cosec}^2 x$.

EXAMPLE 19 *If* $\dfrac{\sin^4 x}{a}+\dfrac{\cos^4 x}{b}=\dfrac{1}{a+b}$, *prove that*

(i) $\dfrac{\sin^8 x}{a^3}+\dfrac{\cos^8 x}{b^3}=\dfrac{1}{(a+b)^3}$ (ii) $\dfrac{\sin^{4n} x}{a^{2n-1}}+\dfrac{\cos^{4n} x}{b^{2n-1}}=\dfrac{1}{(a+b)^{2n-1}}$, $n\in N$

SOLUTION We have,

$$\frac{\sin^4 x}{a}+\frac{\cos^4 x}{b}=\frac{1}{a+b}$$

$$\Rightarrow \quad (a+b)\left(\frac{\sin^4 x}{a}+\frac{\cos^4 x}{b}\right)=1$$

$$\Rightarrow \quad (a+b)\left(\frac{\sin^4 x}{a}+\frac{\cos^4 x}{b}\right)=(\sin^2 x+\cos^2 x)^2$$

$$\Rightarrow \quad \sin^4 x+\cos^4 x+\frac{b}{a}\sin^4 x+\frac{a}{b}\cos^4 x=\sin^4 x+\cos^4 x+2\sin^2 x\cos^2 x$$

$$\Rightarrow \quad \frac{b}{a}\sin^4 x+\frac{a}{b}\cos^4 x-2\sin^2 x\cos^2 x=0$$

$$\Rightarrow \quad \left\{\sqrt{\frac{b}{a}}\sin^2 x-\sqrt{\frac{a}{b}}\cos^2 x\right\}^2=0 \Rightarrow \sqrt{\frac{b}{a}}\sin^2 x=\sqrt{\frac{a}{b}}\cos^2 x \Rightarrow \tan^2 x=\frac{a}{b} \Rightarrow \frac{\sin^2 x}{\cos^2 x}=\frac{a}{b}$$

$$\Rightarrow \quad \frac{\sin^2 x}{a}=\frac{\cos^2 x}{b}$$

$$\Rightarrow \quad \frac{\sin^2 x}{a}=\frac{\cos^2 x}{b}=\frac{\sin^2 x+\cos^2 x}{a+b}$$

$$\Rightarrow \quad \frac{\sin^2 x}{a}=\frac{\cos^2 x}{b}=\frac{1}{a+b} \Rightarrow \sin^2 x=\frac{a}{a+b},\ \cos^2 x=\frac{b}{a+b} \quad \text{...(i)}$$

(i)
$$\frac{\sin^8 x}{a^3}+\frac{\cos^8 x}{b^3}=\frac{1}{a^3}(\sin^2 x)^4+\frac{1}{b^3}(\cos^2 x)^4$$

$$=\frac{1}{a^3}\left(\frac{a}{a+b}\right)^4+\frac{1}{b^3}\left(\frac{b}{a+b}\right)^4 \quad \text{[Using (i)]}$$

$$=\frac{a}{(a+b)^4}+\frac{b}{(a+b)^4}=\frac{a+b}{(a+b)^4}=\frac{1}{(a+b)^3}$$

(ii)
$$\frac{\sin^{4n} x}{a^{2n-1}}+\frac{\cos^{4n} x}{b^{2n-1}}=\frac{(\sin^2 x)^{2n}}{a^{2n-1}}+\frac{(\cos^2 x)^{2n}}{b^{2n-1}}=\frac{1}{a^{2n-1}}\left(\frac{a}{a+b}\right)^{2n}+\frac{1}{b^{2n-1}}\left(\frac{b}{a+b}\right)^{2n}$$

$$=\frac{a}{(a+b)^{2n}}+\frac{b}{(a+b)^{2n}}=\frac{a+b}{(a+b)^{2n}}=\frac{1}{(a+b)^{2n-1}}$$

EXAMPLE 20 *If* $\dfrac{\cos^4\alpha}{\cos^2\beta}+\dfrac{\sin^4\alpha}{\sin^2\beta}=1$, *prove that*

(i) $\sin^4\alpha+\sin^4\beta=2\sin^2\alpha\sin^2\beta$ (ii) $\dfrac{\cos^4\beta}{\cos^2\alpha}+\dfrac{\sin^4\beta}{\sin^2\alpha}=1$

SOLUTION We have,

$$\frac{\cos^4\alpha}{\cos^2\beta}+\frac{\sin^4\alpha}{\sin^2\beta}=1$$

$\Rightarrow\quad \cos^4\alpha\sin^2\beta+\sin^4\alpha\cos^2\beta=\cos^2\beta\sin^2\beta$

$\Rightarrow\quad \cos^4\alpha(1-\cos^2\beta)+\cos^2\beta(1-\cos^2\alpha)^2=\cos^2\beta(1-\cos^2\beta)$

$\Rightarrow\quad \cos^4\alpha-\cos^4\alpha\cos^2\beta+\cos^2\beta-2\cos^2\alpha\cos^2\beta+\cos^4\alpha\cos^2\beta=\cos^2\beta-\cos^4\beta$

$\Rightarrow\quad \cos^4\alpha-2\cos^2\alpha\cos^2\beta+\cos^4\beta=0$

$\Rightarrow\quad (\cos^2\alpha-\cos^2\beta)^2=0\Rightarrow\cos^2\alpha-\cos^2\beta=0\Rightarrow\cos^2\alpha=\cos^2\beta$ ...(i)

$\Rightarrow\quad 1-\sin^2\alpha=1-\sin^2\beta\Rightarrow\sin^2\alpha=\sin^2\beta$ ... (ii)

(i) $\sin^4\alpha+\sin^4\beta=(\sin^2\alpha-\sin^2\beta)^2+2\sin^2\alpha\sin^2\beta=2\sin^2\alpha\sin^2\beta$ $[\because\ \sin^2\alpha=\sin^2\beta]$

(ii) $\dfrac{\cos^4\beta}{\cos^2\alpha}+\dfrac{\sin^4\beta}{\sin^2\alpha}=\dfrac{\cos^2\beta\cos^2\beta}{\cos^2\alpha}+\dfrac{\sin^2\beta\sin^2\beta}{\sin^2\alpha}$

$$=\frac{\cos^2\beta\ \cos^2\alpha}{\cos^2\alpha}+\frac{\sin^2\beta\ \sin^2\alpha}{\sin^2\alpha}=\cos^2\beta+\sin^2\beta \quad \text{[Using (i)and (ii)]}$$

EXAMPLE 21 *If a is any non-zero real number, show that* $\cos x$ *and* $\sin x$ *can never be equal to* $a+\dfrac{1}{a}$.

SOLUTION We have following cases:

Case I *When* $a>0$: In this case, we have

$$a+\frac{1}{a}=(\sqrt{a})^2+\left(\frac{1}{\sqrt{a}}\right)^2-2\times\sqrt{a}\times\frac{1}{\sqrt{a}}+2\times\sqrt{a}\times\frac{1}{\sqrt{a}}=\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2+2\geq 2\left[\because\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2\geq 0\right]$$

Case II *When* $x<0$: Let $a=-b$. Then, $b>0$

$\therefore\quad a+\dfrac{1}{a}=-b-\dfrac{1}{b}=-\left(b+\dfrac{1}{b}\right)$

But, $b+\dfrac{1}{b}\geq 2\Rightarrow-\left(b+\dfrac{1}{b}\right)\leq-2\Rightarrow a+\dfrac{1}{a}\leq-2$ [From Case I]

$\therefore\quad a+\dfrac{1}{a}\geq 2$ for $a>0$ and, $a+\dfrac{1}{a}\leq-2$ for $a<0$. But, $-1\leq\sin x\leq 1$ and $-1\leq\cos x\leq 1$ for all x.

Hence, $\sin x$ and $\cos x$ cannot be equal to $a+\dfrac{1}{a}$ for any non-zero a.

EXAMPLE 22 *If* $A=\cos^2 x+\sin^4 x$, *prove that* $\dfrac{3}{4}\leq A\leq 1$ *for all values of* x.

SOLUTION We have, $A=\cos^2 x+\sin^4 x=\cos^2 x+(\sin^2 x)^2$

Now, $\quad -1\leq\sin x\leq 1$ for all x

$\Rightarrow\quad 0\leq\sin^2 x\leq 1$ for all x

$\Rightarrow\quad (\sin^2 x)^2\leq\sin^2 x$ [For $0<x<1$, $x^n<x$ for all $n\in N-\{1\}$]

$\Rightarrow\quad \cos^2 x+(\sin^2 x)^2\leq\cos^2 x+\sin^2 x$ for all x

$\Rightarrow \quad A \le 1 \quad \text{for all } x$ ...(i)

Again,

$$A = \cos^2 x + \sin^4 x = 1 - \sin^2 x + (\sin^2 x)^2 = 1 - \frac{1}{4} + \left\{\frac{1}{4} - \sin^2 x + (\sin^2 x)^2\right\} = \frac{3}{4} + \left(\frac{1}{2} - \sin^2 x\right)^2$$

Now, $\left(\frac{1}{2} - \sin^2 x\right)^2 \ge 0$ for all $x \Rightarrow \frac{3}{4} + \left(\frac{1}{2} - \sin^2 x\right)^2 \ge \frac{3}{4}$ for all $x \Rightarrow A \ge \frac{3}{4}$ for all x ...(ii)

From (i) and (ii), we obtain $\frac{3}{4} \le A \le 1$ for all x.

EXERCISE 5.1

BASIC

Prove the following identities (1-16)

1. $\sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$
2. $\sin^6 x + \cos^6 x = 1 - 3 \sin^2 x \cos^2 x$
3. $(\text{cosec } x - \sin x)(\sec x - \cos x)(\tan x + \cot x) = 1$
4. $\text{cosec } x (\sec x - 1) - \cot x (1 - \cos x) = \tan x - \sin x$
5. $\dfrac{1 - \sin x \cos x}{\cos x (\sec x - \text{cosec } x)} \cdot \dfrac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x} = \sin x$
6. $\dfrac{\tan x}{1 - \cot x} + \dfrac{\cot x}{1 - \tan x} = (\sec x \text{ cosec } x + 1)$
7. $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} + \dfrac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 2$
8. $(\sec x \sec y + \tan x \tan y)^2 - (\sec x \tan y + \tan x \sec y)^2 = 1$
9. $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \cos x + \sin x}{1 + \cos x - \sin x}$

BASED ON LOTS

10. $\dfrac{\tan^3 x}{1 + \tan^2 x} + \dfrac{\cot^3 x}{1 + \cot^2 x} = \dfrac{1 - 2 \sin^2 x \cos^2 x}{\sin x \cos x}$
11. $1 - \dfrac{\sin^2 x}{1 + \cot x} - \dfrac{\cos^2 x}{1 + \tan x} = \sin x \cos x$
12. $\left(\dfrac{1}{\sec^2 x - \cos^2 x} + \dfrac{1}{\text{cosec}^2 x - \sin^2 x}\right) \sin^2 x \cos^2 x = \dfrac{1 - \sin^2 x \cos^2 x}{2 + \sin^2 x \cos^2 x}$
13. $(1 + \tan \alpha \tan \beta)^2 + (\tan \alpha - \tan \beta)^2 = \sec^2 \alpha \sec^2 \beta$
14. $\dfrac{(1 + \cot x + \tan x)(\sin x - \cos x)}{\sec^3 x - \text{cosec}^3 x} = \sin^2 x \cos^2 x$
15. $\dfrac{2 \sin x \cos x - \cos x}{1 - \sin x + \sin^2 x - \cos^2 x} = \cot x$
16. $\cos x (\tan x + 2)(2 \tan x + 1) = 2 \sec x + 5 \sin x$
17. If $a = \dfrac{2 \sin x}{1 + \cos x + \sin x}$, then prove that $\dfrac{1 - \cos x + \sin x}{1 + \sin x}$ is also equal to a.

[NCERT EXEMPLAR]

18. If $\sin x = \dfrac{a^2 - b^2}{a^2 + b^2}$, find the values of $\tan x$, $\sec x$ and $\text{cosec } x$

19. If $\tan x = \frac{b}{a}$, then find the value of $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$. [NCERT EXEMPLAR]

20. If $\tan x = \frac{a}{b}$, show that $\frac{a \sin x - b \cos x}{a \sin x + b \cos x} = \frac{a^2 - b^2}{a^2 + b^2}$.

BASED ON HOTS

21. If $\operatorname{cosec} x - \sin x = a^3$, $\sec x - \cos x = b^3$, then prove that $a^2 b^2 (a^2 + b^2) = 1$.

22. If $\cot x (1 + \sin x) = 4m$ and $\cot x (1 - \sin x) = 4n$, prove that $(m^2 - n^2)^2 = mn$.

23. If $\sin x + \cos x = m$, then prove that $\sin^6 x + \cos^6 x = \frac{4 - 3(m^2 - 1)^2}{4}$, where $m^2 \le 2$

24. If $a = \sec x - \tan x$ and $b = \operatorname{cosec} x + \cot x$, then show that $ab + a - b + 1 = 0$. [NCERT EXEMPLAR]

25. Prove that: $\left| \sqrt{\frac{1 - \sin x}{1 + \sin x}} + \sqrt{\frac{1 + \sin x}{1 - \sin x}} \right| = -\frac{2}{\cos x}$, where $\frac{\pi}{2} < x < \pi$

26. If $T_n = \sin^n x + \cos^n x$, prove that

(i) $\frac{T_3 - T_5}{T_1} = \frac{T_5 - T_7}{T_3}$ (ii) $2T_6 - 3T_4 + 1 = 0$ (iii) $6T_{10} - 15T_8 + 10T_6 - 1 = 0$

ANSWERS

18. $\tan x = \frac{a^2 - b^2}{2ab}$, $\sec x = \frac{a^2 + b^2}{2ab}$, $\operatorname{cosec} x = \frac{a^2 + b^2}{a^2 - b^2}$ 19. $\frac{2 \cos x}{\sqrt{\cos^2 x - \sin^2 x}}$.

HINTS TO SELECTED PROBLEMS

17. We have,

$$a = \frac{2 \sin x}{1 + \cos x + \sin x} = \frac{2 \sin x (1 - \cos x + \sin x)}{(1 + \cos x + \sin x)(1 - \cos x + \sin x)}$$

$$\Rightarrow a = \frac{2 \sin x (1 - \cos x + \sin x)}{(1 + \sin x)^2 - \cos^2 x} = \frac{2 \sin x (1 - \cos x + \sin x)}{1 + 2 \sin x + \sin^2 x - \cos^2 x}$$

$$\Rightarrow a = \frac{2 \sin x (1 - \cos x + \sin x)}{2 \sin x + \sin^2 x + \sin^2 x} = \frac{1 - \cos x + \sin x}{1 + \sin x}$$

19. We have, $\tan x = \frac{b}{a}$

$$\therefore \sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}} = \sqrt{\frac{1 + \frac{b}{a}}{1 - \frac{b}{a}}} + \sqrt{\frac{1 - \frac{b}{a}}{1 + \frac{b}{a}}} = \sqrt{\frac{1 + \tan x}{1 - \tan x}} + \sqrt{\frac{1 - \tan x}{1 + \tan x}} = \sqrt{\frac{\cos x + \sin x}{\cos x - \sin x}} + \sqrt{\frac{\cos x - \sin x}{\cos x + \sin x}}$$

$$= \frac{\cos x + \sin x + \cos x - \sin x}{\sqrt{\cos^2 x - \sin^2 x}} = \frac{2 \cos x}{\sqrt{\cos^2 x - \sin^2 x}}$$

21. We have, $\operatorname{cosec} x - \sin x = a^3$, $\sec x - \cos x = b^3$

$$\Rightarrow \frac{1 - \sin^2 x}{\sin x} = a^3, \frac{1 - \cos^2 x}{\cos x} = b^3 \Rightarrow \frac{\cos^2 x}{\sin x} = a^3, \frac{\sin^2 x}{\cos x} = b^3$$

$$\Rightarrow \quad \frac{\sin^2 x}{\cos x} \div \frac{\cos^2 x}{\sin x} = \frac{b^3}{a^3} \Rightarrow \tan^3 x = \frac{b^3}{a^3} \Rightarrow \tan x = \frac{b}{a} \Rightarrow \sin x = \frac{b}{\sqrt{a^2+b^2}} \text{ and } \cos x = \frac{a}{\sqrt{a^2+b^2}}$$

Substituting these values of $\sin x$ and $\cos x$ in $\frac{\cos^2 x}{\sin x} = a^3$, we obtain

$$\frac{a^2}{b\sqrt{(a^2+b^2)}} = a^3 \Rightarrow ab\sqrt{a^2+b^2} = 1 \Rightarrow a^2 b^2 (a^2+b^2) = 1$$

22. We have, $\cot x(1+\sin x) = 4m$ and $\cot x(1-\sin x) = 4n$

$$\Rightarrow \quad \cot x + \cos x = 4m \text{ and } \cot x - \cos x = 4n$$

$$\Rightarrow \quad (\cot x + \cos x)^2 - (\cot x - \cos x)^2 = 16m^2 - 16n^2 \text{ and } (\cot x + \cos x)(\cot x - \cos x) = 16\,mn$$

$$\Rightarrow \quad 4\cot x \cos x = 16(m^2 - n^2) \text{ and } \cot^2 x - \cos^2 x = 16\,mn$$

$$\Rightarrow \quad \frac{\cos^2 x}{\sin x} = 4(m^2-n^2) \text{ and } \frac{\cos^4 x}{\sin^2 x} = 16\,mn$$

$$\Rightarrow \quad \frac{\cos^4 x}{\sin^2 x} = 16(m^2-n^2)^2 \text{ and } \frac{\cos^4 x}{\sin^2 x} = 16\,mn \Rightarrow 16(m^2-n^2)^2 = 16\,mn \Rightarrow (m^2-n^2)^2 = mn$$

24. We have, $a = \sec x - \tan x$ and $b = \operatorname{cosec} x + \cot x \Rightarrow a = \frac{1-\sin x}{\cos x}$ and $b = \frac{1+\cos x}{\sin x}$

$$\therefore \quad ab + a - b + 1 = \frac{(1-\sin x)(1+\cos x)}{\sin x \cos x} + \frac{1-\sin x}{\cos x} - \frac{1+\cos x}{\sin x} + 1$$

$$= \frac{1 - \sin x + \cos x - \sin x \cos x + \sin x - \sin^2 x - \cos x - \cos^2 x + \sin x \cos x}{\sin x \cos x}$$

$$= \frac{1-(\cos^2 x + \sin^2 x)}{\sin x \cos x} = \frac{1-1}{\sin x \cos x} = 0$$

5.5 SIGNS OF TRIGONOMETRIC FUNCTIONS

Consider a unit circle with centre at the origin O of the coordinate axes. Clearly, this circle meets the coordinates axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ be a point on the circle such that length of arc $AP = x$ or equivalently, let $P(a, b)$ be the point where the terminal side of the angle $\angle AOP$ with radian measure x meets the unit circle. Then,

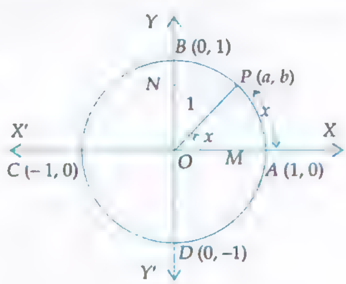


Fig. 5.8

(i) $\cos x = a$ for all $x \in R$

(ii) $\sin x = b$ for all $x \in R$

(iii) $\tan x = \frac{\sin x}{\cos x} = \frac{b}{a}$ for all $x \neq (2n+1)\frac{\pi}{2}, n \in Z$

(iv) $\cot x = \frac{\cos x}{\sin x} = \frac{a}{b}$ for all $x \neq n\pi, n \in Z$

(v) $\sec x = \frac{1}{a}$ for all $x \neq (2n+1)\frac{\pi}{2}, n \in Z$

(vi) $\operatorname{cosec} x = \frac{1}{b}$ for all $x \neq n\pi, n \in Z$

We observe that as the point $P(a, b)$ moves on the unit circle, point M moves between C and A and N moves between D and B. Consequently, $OM = a$ varies between -1 and 1 and $ON = PM = b$ also varies between -1 and 1 i.e. $-1 \le a \le 1$ and $-1 \le b \le 1$. Therefore, $-1 \le \cos x \le 1$ and $-1 \le \sin x \le 1$ for all x.

Also,

$a > 0, b > 0$ in I quadrant ; $a < 0, b > 0$ in II quadrant

$a < 0, b < 0$ in III quadrant ; $a > 0, b < 0$ in IV quadrant.

Thus, the signs of trigonometric functions in various quadrants are as discussed below :

In the first quadrant : We have, $a > 0$ and $b > 0$

$$\therefore \quad \cos x = a > 0,\ \sin x = b > 0,\ \tan x = \frac{b}{a} > 0,\ \cot x = \frac{a}{b} > 0,\ \sec x = \frac{1}{a} > 0 \text{ and } \operatorname{cosec} x = \frac{1}{b} > 0$$

Consequently, all the six trigonometric functions are positive in the first quadrant.

In the second quadrant : We have, $a < 0$ and $b > 0$

$$\therefore \quad \cos x = a < 0,\ \sin x = b > 0,\ \tan x = \frac{b}{a} < 0,\ \cot x = \frac{a}{b} < 0,\ \sec x = \frac{1}{a} < 0 \text{ and } \operatorname{cosec} x = \frac{1}{b} > 0$$

Consequently, in the second quadrant $\sin x$ and $\operatorname{cosec} x$ both are positive and all other trigonometric functions are negative.

In the third quadrant : We have, $a < 0$ and $b < 0$.

$$\therefore \quad \cos x = a < 0,\ \sin x = b < 0,\ \tan x = \frac{b}{a} > 0,\ \cot x = \frac{a}{b} > 0,\ \sec x = \frac{1}{a} < 0 \text{ and } \operatorname{cosec} x = \frac{1}{b} < 0$$

Thus, in the third quadrant $\tan x$ and $\cot x$ are positive and all other trigonometric functions are negative.

In the fourth quadrant : We have, $a > 0$ and $b < 0$.

$$\therefore \quad \cos x = a > 0,\ \sin x = b < 0,\ \tan x = \frac{b}{a} < 0,\ \cot x = \frac{a}{b} < 0,\ \sec x = \frac{1}{a} > 0 \text{ and } \operatorname{cosec} x = \frac{1}{b} < 0$$

Thus, in the fourth quadrant $\cos x$ and $\sec x$ both are positive and all other trigonometric functions are negative.

The signs of trigonometric functions in different quadrants can be summarised as under :

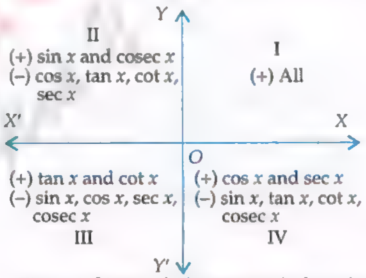


Fig. 5.9 Signs of trigonometric functions

SIMPLE RULE TO REMEMBER *A crude aid to memorise the signs of trigonometrical ratios in different quadrants is the four-word phrase "ALL SCHOOL TO COLLEGE". The first letter of the first word in this phrase is 'A'. This may be taken to indicate that all trigonometric ratios are positive in the first quadrant. The first letter of the second word is 'S'. This indicates that sine and its reciprocal are positive in the second quadrant. The first letter of third word is 'T'. This may be taken as to indicate that tangent and its reciprocal are positive in the third quadrant. The first letter of the fourth word in the phrase is 'C' which may be taken as to indicate that only cosine and its reciprocal are positive in the fourth quadrant.*

5.6 VARIATIONS IN VALUES OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS

Consider a unit circle centred at the origin O of the coordinate axes. The circle cuts the coordinates axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ be a point on the circle whose equation is $x^2 + y^2 = 1$ such that arc $AP = x$ or equivalently radian measure of $\angle AOP$ is x. Then, $a = \cos x$ and $b = \sin x$.

It is evident from Fig. 5.10 that

$$-1 \leq a \leq 1 \text{ and } -1 \leq b \leq 1 \Rightarrow -1 \leq \cos x \leq 1 \text{ and} -1 \leq \sin x \leq 1 \text{ for all } x.$$

Further, we observe that in the first quadrant, as x increases from 0 to $\frac{\pi}{2}$, b increases from 0 to 1 and as x increases from $\frac{\pi}{2}$ to π, b decreases from 1 to 0. But, $b = \sin x$. Thus, in the first quadrant $\sin x$ increases from 0 to 1 and in the second quadrant it decreases from 1 to 0. In the third quadrant as x increases from π to $\frac{3\pi}{2}$ the values of b decrease from 0 to -1 and in the fourth quadrant as x increases from $\frac{3\pi}{2}$ to 2π the values b of increase from -1 to 0. Thus, in the third quadrant as x increases from π to $\frac{3\pi}{2}$, $\sin x$ decreases from 0 to -1 and finally in the fourth quadrant as x increases from $\frac{3\pi}{2}$ to 2π, $\sin x$ increases from -1 to 0.

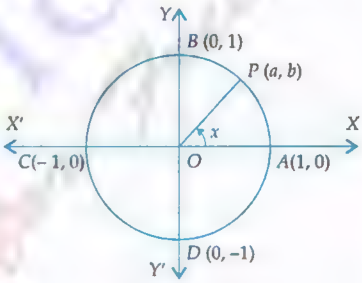


Fig. 5.10

Similarly, we can observe the variations in the values of other trigonometric functions. The following table exhibits the same.

Trigonometric function	*I quadrant*	*II quadrant*	*III quadrant*	*IV quadrant*
sine	increase from 0 to 1	decreases from 1 to 0	decreases from 0 to -1	increases from -1 to 0
cosine	decreases from 1 to 0	decreases from 0 to -1	increases from -1 to 0	increases from 0 to 1
tangent	increases from 0 to ∞	increases from $-\infty$ to 0	increases from 0 to ∞	increases from $-\infty$ to 0
cotangent	decreases from ∞ to 0	decreases from 0 to $-\infty$	decreases from ∞ to 0	decreases from 0 to $-\infty$
secant	increases from 1 to ∞	increases from $-\infty$ to -1	decreases from -1 to $-\infty$	decreases from ∞ to 1
cosecant	decreases from ∞ to 1	increases from 1 to ∞	increases from $-\infty$ to -1	decreases from -1 to $-\infty$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find* $\sin x$ and $\tan x$, *if* $\cos x = -\frac{12}{13}$ *and x lies in the third quadrant.*

SOLUTION We know that: $\cos^2 x + \sin^2 x = 1 \Rightarrow \sin x = \pm\sqrt{1-\cos^2 x}$

In third quadrant $\sin x$ is negative.

$$\therefore \quad \sin x = -\sqrt{1-\cos^2 x} \Rightarrow \sin x = -\sqrt{1-\left(-\frac{12}{13}\right)^2} = -\frac{5}{13}$$

$$\text{and,} \quad \tan x = \frac{\sin x}{\cos x} \Rightarrow \tan x = -\frac{5}{13} \times \frac{13}{-12} = \frac{5}{12}$$

EXAMPLE 2 *Find the values of* $\cos x$ *and* $\tan x$, *if* $\sin x = -\frac{3}{5}$ *and* $\pi < x < \frac{3\pi}{2}$.

SOLUTION We know that: $\cos^2 x + \sin^2 x = 1 \Rightarrow \cos x = \pm\sqrt{1-\sin^2 x}$

In the third quadrant $\cos x$ is negative and $\tan x$ is positive.

$$\therefore \quad \cos x = -\sqrt{1-\sin^2 x} \Rightarrow \cos x = -\sqrt{1-\frac{9}{25}} = -\frac{4}{5}$$

$$\text{and,} \quad \tan x = \frac{\sin x}{\cos x} \Rightarrow \tan x = -\frac{3}{5} \times -\frac{5}{4} = \frac{3}{4}$$

EXAMPLE 3 *Find all other trigonometrical ratios, if* $\sin x = -\frac{2\sqrt{6}}{5}$ *and x lies in quadrant III.*

SOLUTION We know that: $\cos^2 x + \sin^2 x = 1 \Rightarrow \cos x = \pm\sqrt{1-\sin^2 x}$

In the third quadrant $\cos x$ is negative.

$$\therefore \quad \cos x = -\sqrt{1-\sin^2 x} \Rightarrow \cos x = -\sqrt{1-\frac{24}{25}} = -\frac{1}{5}$$

In the third quadrant $\tan x$ is positive.

$$\therefore \quad \tan x = \frac{\sin x}{\cos x} \Rightarrow \tan x = -\frac{2\sqrt{6}}{5} \times -\frac{5}{1} = 2\sqrt{6}$$

$$\text{Now,} \quad \operatorname{cosec} x = \frac{1}{\sin x} \Rightarrow \operatorname{cosec} x = -\frac{5}{2\sqrt{6}}; \ \sec x = \frac{1}{\cos x} \Rightarrow \sec x = -5$$

$$\text{and,} \quad \cot x = \frac{1}{\tan x} \Rightarrow \cot x = \frac{1}{2\sqrt{6}}.$$

EXAMPLE 4 *If* $\cos x = -\frac{1}{2}$ *and* $\pi < x < \frac{3\pi}{2}$, *find the value of* $4\tan^2 x - 3\operatorname{cosec}^2 x$.

SOLUTION It is given that x lies in the third quadrant. Therefore, $\sin x$ is negative and $\tan x$ is positive. Thus,

$$\sin x = \pm\sqrt{1-\cos^2 x} \Rightarrow \sin x = -\sqrt{1-\frac{1}{4}} = -\frac{\sqrt{3}}{2} \Rightarrow \operatorname{cosec} x = \frac{-2}{\sqrt{3}}$$

And, $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{-\sqrt{3}/2}{-1/2} = \sqrt{3}.$

Hence, $4\tan^2 x - 3\,\text{cosec}^2\, x = 4 \times 3 - 3 \times \dfrac{4}{3} = 8.$

EXAMPLE 5 *If* $\sec x = \sqrt{2}$ *and* $\dfrac{3\pi}{2} < x < 2\pi$, *find the value of* $\dfrac{1 + \tan x + \text{cosec}\, x}{1 + \cot x - \text{cosec}\, x}$.

SOLUTION We have, $\sec x = \sqrt{2}$. Therefore, $\cos x = \dfrac{1}{\sec x} \Rightarrow \cos x = \dfrac{1}{\sqrt{2}}$

It is given that x lies in the fourth quadrant in which $\sin x$ is negative.

$$\therefore \quad \sin x = -\sqrt{1 - \cos^2 x} = -\sqrt{1 - \frac{1}{2}} = -\frac{1}{\sqrt{2}}, \quad \text{cosec}\, x = \frac{1}{\sin x} \Rightarrow \text{cosec}\, x = -\sqrt{2}$$

$$\text{and,} \quad \tan x = \frac{\sin x}{\cos x} \Rightarrow \tan x = -\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{1} = -1 \Rightarrow \cot x = -1$$

$$\therefore \quad \frac{1 + \tan x + \text{cosec}\, x}{1 + \cot x - \text{cosec}\, x} = \frac{1 - 1 - \sqrt{2}}{1 - 1 + \sqrt{2}} = -1$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 6 *Prove that :* $\sqrt{\dfrac{1 - \sin x}{1 + \sin x}} = \begin{cases} \sec x - \tan x, & \text{if } -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ -\sec x + \tan x, & \text{if } \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \end{cases}$

SOLUTION We have,

$$\sqrt{\frac{1 - \sin x}{1 + \sin x}} = \sqrt{\frac{(1 - \sin x)^2}{1 - \sin^2 x}} = \frac{1 - \sin x}{\sqrt{\cos^2 x}} = \frac{1 - \sin x}{|\cos x|} = \begin{cases} \dfrac{1 - \sin x}{\cos x}, & \text{if } -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ \dfrac{1 - \sin x}{-\cos x}, & \text{if } \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \end{cases}$$

$$= \begin{cases} \sec x - \tan x, & \text{if } -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ -\sec x + \tan x, & \text{if } \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \end{cases}$$

EXAMPLE 7 *Prove that :* $\sqrt{\dfrac{1 + \cos x}{1 - \cos x}} = \begin{cases} \text{cosec}\, x + \cot x, & \text{if } 0 < x < \pi \\ -\text{cosec}\, x - \cot x, & \text{if } \pi < x < 2\pi \end{cases}$

SOLUTION We have,

$$\sqrt{\frac{1 + \cos x}{1 - \cos x}} = \sqrt{\frac{(1 + \cos x)^2}{1 - \cos^2 x}} = \frac{1 + \cos x}{\sqrt{\sin^2 x}} = \frac{1 + \cos x}{|\sin x|} = \begin{cases} \dfrac{1 + \cos x}{\sin x}, & \text{if } 0 < x < \pi \\ \dfrac{1 + \cos x}{-\sin x}, & \text{if } \pi < x < 2\pi \end{cases}$$

$$= \begin{cases} \text{cosec}\, x + \cot x, & \text{if } 0 < x < \pi \\ -\text{cosec}\, x - \cot x, & \text{if } \pi < x < 2\pi \end{cases}$$

EXAMPLE 8 *Prove that :* $\sqrt{\dfrac{1 - \sin x}{1 + \sin x}} + \sqrt{\dfrac{1 + \sin x}{1 - \sin x}} = \begin{cases} \dfrac{2}{\cos x}, & \text{if } 0 \le x < \dfrac{\pi}{2} \\ -\dfrac{2}{\cos x}, & \text{if } \dfrac{\pi}{2} < x \le \pi \end{cases}$

SOLUTION We have,

$$\sqrt{\frac{1-\sin x}{1+\sin x}} + \sqrt{\frac{1+\sin x}{1-\sin x}} = \frac{(1-\sin x)+(1+\sin x)}{\sqrt{1-\sin^2 x}} = \frac{2}{\sqrt{\cos^2 x}} = \frac{2}{|\cos x|} \quad [\because \sqrt{x^2} = |x|]$$

$$= \begin{cases} \dfrac{2}{\cos x}, \text{ if } 0 \le x < \dfrac{\pi}{2} \\ -\dfrac{2}{\cos x}, \text{ if } \dfrac{\pi}{2} < x \le \pi \end{cases}$$

EXERCISE 5.2

BASIC

1. Find the values of the other five trigonometric functions in each of the following:
   (i) $\cot x = \frac{12}{5}$, x in quadrant III (ii) $\cos x = -\frac{1}{2}$, x in quadrant II
   (iii) $\tan x = \frac{3}{4}$, x in quadrant III (iv) $\sin x = \frac{3}{5}$, x in quadrant I
2. If $\sin x = \frac{12}{13}$ and x lies in the second quadrant, find the value of $\sec x + \tan x$.
3. If $\sin x = \frac{3}{5}$, $\tan y = \frac{1}{2}$ and $\frac{\pi}{2} < x < \pi < y < \frac{3\pi}{2}$, find the value of $8 \tan x - \sqrt{5} \sec y$.

BASED ON LOTS

4. If $\sin x + \cos x = 0$ and x lies in the fourth quadrant, find $\sin x$ and $\cos x$.
5. If $\cos x = -\frac{3}{5}$ and $\pi < x < \frac{3\pi}{2}$, find the values of other five trigonometric functions and hence evaluate $\frac{\operatorname{cosec} x + \cot x}{\sec x - \tan x}$.

ANSWERS

1. (i) $\sin x = -\frac{5}{13}$, $\cos x = -\frac{12}{13}$, $\tan x = \frac{5}{12}$, $\operatorname{cosec} x = -\frac{13}{5}$, $\sec x = -\frac{13}{12}$
   (ii) $\sin x = \frac{\sqrt{3}}{2}$, $\tan x = -\sqrt{3}$, $\operatorname{cosec} x = \frac{2}{\sqrt{3}}$, $\cot x = \frac{-1}{\sqrt{3}}$, $\sec x = -2$
   (iii) $\sin x = -\frac{3}{5}$, $\cos x = -\frac{4}{5}$, $\operatorname{cosec} x = -\frac{5}{3}$, $\sec x = -\frac{5}{4}$, $\cot x = \frac{4}{3}$
   (iv) $\cos x = \frac{4}{5}$, $\tan x = \frac{3}{4}$, $\sec x = \frac{5}{4}$, $\cot x = \frac{4}{3}$, $\operatorname{cosec} x = \frac{5}{3}$
2. -5 3. $-\frac{7}{2}$ 4. $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ 5. $\frac{1}{6}$

HINTS TO SELECTED PROBLEM

4. We have, $\sin x + \cos x = 0 \Rightarrow \sin x = -\cos x \Rightarrow \tan x = -1$
   $\therefore \sec^2 x = 1 + \tan^2 x \Rightarrow \sec^2 x = 1 + (-1)^2 = 2 \Rightarrow \sec x = \sqrt{2} \Rightarrow \cos x = \frac{1}{\sqrt{2}}$

5.7 VALUES OF TRIGONOMETRIC FUNCTIONS AT ALLIED ANGLES

Two angles are said to be allied when their sum or difference is either zero or a multiple of $\frac{\pi}{2}$.

The angles allied to x are $-x$, $\frac{\pi}{2} \pm x$, $\pi \pm x$, $\frac{3\pi}{2} \pm x$, $2\pi \pm x$ etc. In this section, we will express the values of trigonometric functions at allied angles of an angle x in terms of the values at x.

5.7.1 VALUES OF TRIGONOMETRIC FUNCTIONS AT $-x$

Consider a circle of unit radius centred at the origin of coordinate axes which cuts the coordinates axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ be a point on the circle such that arc $AP = x$ so that the measure of $\angle AOP$ is x. Then, $a = \cos x$ and $b = \sin x$. Let Q be the image of P in x-axis. Then, $\angle AOQ = -x$ and the coordinates of Q are $(a, -b)$.

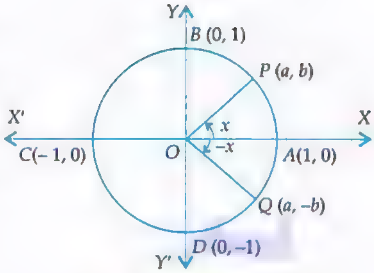


Fig. 5.11

$\therefore \quad a = \cos(-x)$ and $-b = \sin(-x)$

$\Rightarrow \quad \cos x = \cos(-x)$ and $-\sin x = \sin(-x)$ $\quad [\because\ a = \cos x, b = \sin x]$

$\Rightarrow \quad \cos(-x) = \cos x$ and $\sin(-x) = -\sin x$

$$\therefore \quad \tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x,\ \cot(-x) = \frac{\cos(-x)}{\sin(-x)} = \frac{\cos x}{-\sin x} = -\frac{\cos x}{\sin x} = -\cot x$$

$$\sec(-x) = \frac{1}{\cos(-x)} = \frac{1}{\cos x} = \sec x \text{ and, } \operatorname{cosec}(-x) = \frac{1}{\sin(-x)} = -\frac{1}{\sin x} = -\operatorname{cosec} x$$

5.7.2 VALUES OF TRIGONOMETRIC FUNCTIONS AT $\left(\frac{\pi}{2} - x\right)$

Consider a unit circle centred at the origin of coordinates axes which cuts the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ be a point on the circle such that are $AP = x$ consequently $\angle AOP = x$ and hence $a = \cos x$ and $b = \sin x$. Let Q be a point on the circle such that arc $AQ = \frac{\pi}{2} - x$ and so $\angle AOQ = \frac{\pi}{2} - x$. Consequently, measure of $\angle BOQ$ is x.

Draw perpendiculars PM and QN from P and Q respectively on OX. In triangle ONQ, we have $\angle QON = \frac{\pi}{2} - x$, $\angle ONQ = \frac{\pi}{2}$. Therefore, by using angle sum property, we obtain $\angle OQN = x$.

In triangles OMP and ONQ, we have

$\angle OMP = \angle ONQ$, $\angle POM = \angle OQN$ and, $OP = OQ$

So, by using RHS criterion of congruence, we obtain

$\Delta OMP \cong \Delta QNO$

$\Rightarrow \quad OM = QN$ and $ON = PM$

$\Rightarrow \quad QN = a$ and $ON = b$

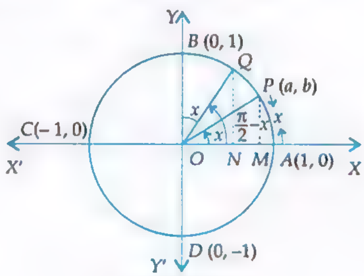


Fig. 5.12

So, the coordinates of Q are (b, a). Since $\angle QON = \frac{\pi}{2} - x$

$$\therefore \quad b = \cos\left(\frac{\pi}{2} - x\right) \text{ and } a = \sin\left(\frac{\pi}{2} - x\right)$$

$$\Rightarrow \quad \sin x = \cos\left(\frac{\pi}{2} - x\right) \text{ and } \cos x = \sin\left(\frac{\pi}{2} - x\right) \qquad [\because\ a = \cos x \text{ and } b = \sin x]$$

$$\Rightarrow \quad \cos\left(\frac{\pi}{2} - x\right) = \sin x \text{ and } \sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\therefore \quad \tan\left(\frac{\pi}{2}-x\right)=\frac{\sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)}=\frac{\cos x}{\sin x}=\cot x,\ \cot\left(\frac{\pi}{2}-x\right)=\frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)}=\frac{\sin x}{\cos x}=\tan x$$

$$\operatorname{cosec}\left(\frac{\pi}{2}-x\right)=\frac{1}{\sin\left(\frac{\pi}{2}-x\right)}=\frac{1}{\cos x}=\sec x,\ \sec\left(\frac{\pi}{2}-x\right)=\frac{1}{\cos\left(\frac{\pi}{2}-x\right)}=\frac{1}{\sin x}=\operatorname{cosec} x$$

5.7.3 VALUES OF TRIGONOMETRIC FUNCTIONS AT $(\pi - x)$

Consider a unit circle with centre at the origin O of the coordinate axes and cutting the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ be a point on the circle such that arc $AP = x$ consequently, the measure of $\angle AOP$ is x. Further, let Q be a point on the unit circle such that $\angle AOQ = \pi - x$ and so measure of arc $AQ = \pi - x$.

$\therefore \quad$ arc $QC = \pi - (\pi - x) = x \Rightarrow \angle QOC = x \qquad$ [$\because$ arc $AC = \pi$]

In triangles OMP and ONQ, we have

$\angle POM = \angle QON, OP = OQ = 1$ unit and $\angle OMP = \angle ONQ$

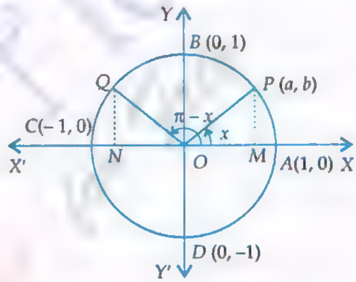


Fig. 5.13

So, $\quad \Delta OMP \cong \Delta ONQ$

$\Rightarrow \quad OM = ON$ and $PM = QN$

$\Rightarrow \quad ON = a$ and $QN = b$

So, the coordinates of Q are $(-a, b)$. Since, $\angle QOM = (\pi - x)$.

$\therefore \quad -a = \cos(\pi - x)$ and $b = \sin(\pi - x)$

$\Rightarrow \quad \cos(\pi - x) = -\cos x$ and $\sin(\pi - x) = \sin x$

[$\because a = \cos x, b = \sin x$]

Thus, we obtain

$\cos(\pi - x) = -\cos x$ and $\sin(\pi - x) = \sin x$

$$\therefore \quad \tan(\pi-x)=\frac{\sin(\pi-x)}{\cos(\pi-x)}=\frac{\sin x}{-\cos x}=-\frac{\sin x}{\cos x}=-\tan x,$$

$$\cot(\pi-x)=\frac{\cos(\pi-x)}{\sin(\pi-x)}=\frac{-\cos x}{\sin x}=-\frac{\cos x}{\sin x}=-\cot x.$$

$$\operatorname{cosec}(\pi-x)=\frac{1}{\sin(\pi-x)}=\frac{1}{\sin x}=\operatorname{cosec} x,\ \sec(\pi-x)=\frac{1}{\cos(\pi-x)}=\frac{1}{-\cos x}=-\sec x$$

5.7.4 VALUES OF TRIGONOMETRIC FUNCTIONS AT $\left(\frac{\pi}{2}+x\right)$

Consider a unit circle centred at the origin of the coordinate axes which cuts the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $(0, -1)$. Let $P(a, b)$ a point on the circle such that $\angle AOP = x$ equivalently arc $AP = x$. Let Q be a point on the circle such that $\angle AOQ = \frac{\pi}{2} + x$ equivalently arc $AQ = \frac{\pi}{2} + x$.

$$\therefore \quad \text{arc } QC = \pi - \left(\frac{\pi}{2}+x\right) = \frac{\pi}{2} - x. \Rightarrow \angle QON = \frac{\pi}{2} - x.$$

In triangles OMP and OQN, we have

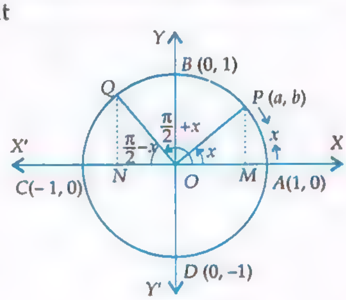


Fig. 5.14

$\angle POM = \angle OQN$

$\angle OMP = \angle ONQ$ and, $OP = OQ$

$\therefore \quad \Delta POM \cong \Delta OQN$

$\Rightarrow \quad OM = QN$ and $PM = ON$

$\Rightarrow \quad a = QN$ and $b = ON \Rightarrow ON = b$ and $QN = a$

So, the coordinates of Q are $(-b, a)$. Since $\angle QOM = \left(\frac{\pi}{2}+x\right)$.

$\therefore \quad -b = \cos\left(\frac{\pi}{2}+x\right)$ and $a = \sin\left(\frac{\pi}{2}+x\right)$

$\Rightarrow \cos\left(\frac{\pi}{2}+x\right) = -\sin x$ and $\sin\left(\frac{\pi}{2}+x\right) = \cos x$ $\quad [\because a = \cos x, b = \sin x]$

$$\therefore \tan\left(\frac{\pi}{2}+x\right) = \frac{\sin\left(\frac{\pi}{2}+x\right)}{\cos\left(\frac{\pi}{2}+x\right)} = \frac{\cos x}{-\sin x} = -\cot x,\ \cot\left(\frac{\pi}{2}+x\right) = \frac{\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)} = \frac{-\sin x}{\cos x} = -\tan x$$

$$\operatorname{cosec}\left(\frac{\pi}{2}+x\right) = \frac{1}{\sin\left(\frac{\pi}{2}+x\right)} = \frac{1}{\cos x} = \sec x,\ \sec\left(\frac{\pi}{2}+x\right) = \frac{1}{\cos\left(\frac{\pi}{2}+x\right)} = \frac{1}{-\sin x} = -\operatorname{cosec} x$$

5.7.5 VALUES OF TRIGONOMETRIC FUNCTIONS AT $(\pi + x)$

Consider a unit circle centred at the origin of the coordinate axes cutting the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ a point on the circle such that arc $AP = x$ consequently $\angle AOP = x$.

$\therefore \quad a = \cos x$ and $b = \sin x$.

Let Q be a point on the circle such that $\angle AOQ = \pi + x$ or equivalently, arc $AQ = \pi + x$.

$\therefore \quad$ Arc $CQ = x$ and $\angle COQ = x$

Clearly, $\Delta OMP \cong \Delta ONQ$

$\Rightarrow \quad OM = ON$ and $PM = QN \Rightarrow a = ON$ and $b = QN$

$\Rightarrow \quad ON = a$ and $QN = b$

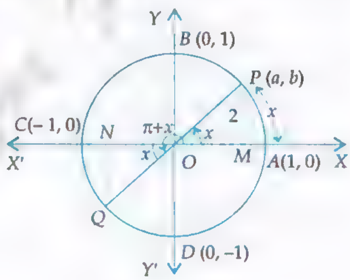


Fig. 5.15

So, the coordinates of Q are $(-a, -b)$. Since $\angle QOM = (\pi + x)$.

$\therefore \quad -a = \cos(\pi + x)$ and $-b = \sin(\pi + x) \Rightarrow$

$\cos(\pi + x) = -\cos x$ and $\sin(\pi + x) = -\sin x$

$$\therefore \quad \tan(\pi + x) = \frac{\sin(\pi + x)}{\cos(\pi + x)} = \frac{-\sin x}{-\cos x} = \tan x,\ \cot(\pi + x) = \frac{\cos(\pi + x)}{\sin(\pi + x)} = \frac{-\cos x}{-\sin x} = \cot x$$

$$\operatorname{cosec}(\pi + x) = \frac{1}{\sin(\pi + x)} = \frac{1}{-\sin x} = -\operatorname{cosec} x,\ \sec(\pi + x) = \frac{1}{\cos(\pi + x)} = \frac{1}{-\cos x} = -\sec x$$

5.7.6 VALUES OF TRIGONOMETRIC FUNCTIONS AT $(2\pi - x)$

Consider a unit circle centred at the origin of the coordinate axes. Suppose the circle cuts the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ a point on the unit circle such that $\angle AOP = x$. Then, $a = \cos x$ and $b = \sin x$.

Let Q be a point on the unit circle such that $\angle AOQ = 2\pi - x$. Then, the measure of $\angle AOQ = x$. Consequently, Q is image of $P(a, b)$ in the line mirror along OX. So, the coordinates of Q are $(a, -b)$.

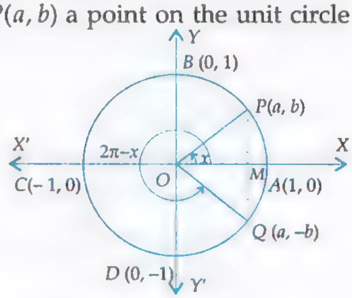


Fig. 5.16

$\therefore \quad a = \cos(2\pi - x)$ and $-b = \sin(2\pi - x)$

$\Rightarrow \cos x = \cos(2\pi - x)$ and $-\sin x = \sin(2\pi + x)$ $\quad [\because a = \cos x, b = \sin x]$

$\Rightarrow \cos(2\pi - x) = \cos x$ and $\sin(2\pi - x) = -\sin x$

$$\therefore \quad \tan(2\pi - x) = \frac{\sin(2\pi - x)}{\cos(2\pi - x)} = \frac{-\sin x}{\cos x} = -\tan x, \quad \cot(2\pi - x) = \frac{\cos(2\pi - x)}{\sin(2\pi - x)} = \frac{\cos x}{-\sin x} = -\cot x$$

$$\operatorname{cosec}(2\pi - x) = \frac{1}{\sin(2\pi - x)} = \frac{1}{\sin x} = \operatorname{cosec} x, \quad \sec(2\pi - x) = \frac{1}{\cos(2\pi - x)} = \frac{1}{\cos x} = \sec x$$

5.7.7 VALUES OF TRIGONOMETRIC FUNCTIONS AT $(2\pi + x)$

Consider a unit circle centred at the origin of the coordinates axes. The circle cuts the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let P be a point on the circle such that arc AP or equivalently $\angle AOP = x$. Therefore, $a = \cos x$ and $b = \sin x$.

The circumference of the unit circle is 2π. Therefore, if we begin from P and travel distance 2π along the circle, we return to the same point P.

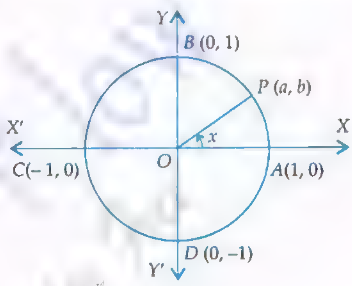


Fig. 5.17

$\therefore \quad a = \cos(2\pi + x)$ and $b = \sin(2\pi + x)$

$\Rightarrow \quad \cos x = \cos(2\pi + x)$ and $\sin x = \sin(2\pi + x)$

$\Rightarrow \quad \cos(2\pi + x) = \cos x$ and $\sin(2\pi + x)$

$$\therefore \quad \tan(2\pi + x) = \frac{\sin(2\pi + x)}{\cos(2\pi + x)} = \frac{\sin x}{\cos x} = \tan x,$$

$$\cot(2\pi + x) = \frac{\cos(2\pi + x)}{\sin(2\pi + x)} = \frac{\cos x}{\sin x} = \cot x$$

$$\operatorname{cosec}(2\pi + x) = \frac{1}{\sin(2\pi + x)} = \frac{1}{\sin x} = \operatorname{cosec} x, \quad \sec(2\pi + x) = \frac{1}{\cos(2\pi + x)} = \frac{1}{\cos x} = \sec x$$

5.8 PERIODIC FUNCTIONS

A function $f(x)$ is said to be a periodic function, if there exits a positive real number T such that $f(x+T) = f(x)$ for all x.

If T is the smallest positive real number such that $f(x+T) = f(x)$ for all x, then T is called the fundamental period of $f(x)$.

We observe that if $f(x)$ is periodic with period T, then

$f(x+2T) = f((x+T)+T) = f(x+T) = f(x)$, $f(x+3T) = f\left((x+2T) + T\right) = f(x+2T) = f(x)$ and so on.

In general, $f(x+nT) = f(x)$ for all x and $n \in N$.

5.8.1 PERIODICITY OF TRIGONOMETRIC FUNCTIONS

Consider a unit circle centred at the origin of the coordinate axes. The circle cuts the coordinate axes at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$. Let $P(a, b)$ be a point on the circle such that arc $AP = x$ or equivalently $\angle AOP = x$.

$\therefore \quad a = \cos x$ and $b = \sin x$.

Now, if we take one complete revolution from the point P along the circumference of the circle, we again come back to the same point P. In other words, if x increases or decreases by 2π, we return to the same point.

$\therefore \quad a = \cos(2\pi + x)$ and $b = \sin(2\pi + x)$

$\Rightarrow \quad \cos x = \cos(2\pi + x)$ and $\sin x = \sin(2\pi + x)$

Also, $a = \cos(-2\pi + x)$ and $b = \sin(-2\pi + x)$

$\therefore$ $\cos(-2\pi + x) = \cos x$ and $\sin(-2\pi + x) = \sin x$

We also observe that if x increases or decreases by any integral multiple of 2π, we come back to the same point P.

$\therefore$ $\cos(2n\pi + x) = \cos x$ and $\sin(2n\pi + x) = \sin x, n \in Z$

Thus, cosine and sine functions are periodic functions.

It is evident from the above discussion that 2π is the smallest positive number such that

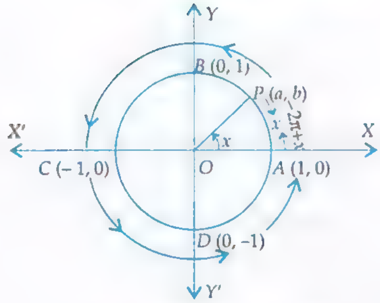


Fig. 5.18

$\cos(2\pi + x) = \cos x$ and $\sin(2\pi + x) = \sin x$ for all x

Hence, cosine and sine functions are periodic functions with period 2π.

In sub-sections 5.7.5 and 5.7.7, we have learnt that

$\tan(\pi + x) = x$ and $\tan(2\pi + x) = x$ for all $x \in R$

$\Rightarrow$ $\tan(n\pi + x) = x$ for all $x \in R$ and $n \in Z$

Also, π is the smallest positive real number such that $\tan(\pi + x) = \tan x$ for all $x \in R$. So, tangent function is also periodic with period π.

Using the definition of cotangent, cosecants and secant functions, we obtain

$$\cot(\pi + x) = \frac{1}{\tan(\pi + x)} = \frac{1}{\tan x} = \cot x \text{ for all } x\ (\neq n\pi) \in R$$

$$\operatorname{cosec}(2\pi + x) = \frac{1}{\sin(2\pi + x)} = \frac{1}{\sin x} = \operatorname{cosec} x \text{ for all } x\ (\neq n\pi) \in R$$

$$\sec(2\pi + x) = \frac{1}{\cos(2\pi + x)} = \frac{1}{\cos x} = \sec x \text{ for all } x\left(\neq (2n+1)\frac{\pi}{2}\right) \in R$$

Thus, consecant and secant functions are periodic with period 2π and cotangent is periodic with period π.

5.9 EVEN AND ODD FUNCTIONS

EVEN FUNCTION *A function $f(x)$ is said to be an even function, if $f(-x) = f(x)$ for all x in its domain.*

ODD FUNCTION *A function $f(x)$ is said to be an odd function, if $f(-x) = -f(x)$ for all x in its domain.*

ILLUSTRATION 1 *Determine whether the following functions are even or odd or neither :*

(i) $f(x) = x^3 + x$ (ii) $g(x) = 3x^2 + 1$ (iii) $h(x) = x^2 + x + 4$

SOLUTION (i) We have, $f(x) = x^3 + x$

$\therefore$ $f(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -f(x)$ for all $x \in R$.

So, $f(x)$ is an odd function.

(ii) We have, $g(x) = 3x^2 + 1$

$\therefore$ $g(-x) = 3(-x)^2 + 1 = 3x^2 + 1 = g(x)$ for all $x \in R$

So, $g(x)$ is an even function.

(iii) We have, $h(x) = x^2 + x + 4$

$\therefore$ $h(-x) = (-x)^2 + (-x) + 4 = x^2 - x + 4$

Clearly, $h(-x)$ is neither equal to $h(x)$ nor to $-h(x)$. So, $h(x)$ is neither even nor odd function.

We have learnt that

$$\sin(-x) = -\sin x,\ \tan(-x) = -\tan x,\ \text{cosec}(-x) = -\text{cosec}\, x \text{ and } \cot(-x) = -\cot x$$

So, sine, tangent, cosecant and cotangent functions are odd functions.

We have also learnt that $\cos(-x) = \cos x$ and $\sec(-x) = \sec x$. So, cosine and secant functions are even functions.

The values of trigonometric functions at $-x$, $\frac{\pi}{2} \pm x$, $\pi \pm x$, $\frac{3\pi}{2} \pm x$ and $2\pi \pm x$ are given in terms of values at x in the following tabular form for ready reference.

Trigonometric Function / Point /Angle	sin	cos	tan	cot	cosec	sec
$-x$	$-\sin x$	$\cos x$	$-\tan x$	$-\cot x$	$-\text{cosec}\, x$	$\sec x$
$\frac{\pi}{2} - x$	$\cos x$	$\sin x$	$\cot x$	$\tan x$	$\sec x$	$\text{cosec}\, x$
$\frac{\pi}{2} + x$	$\cos x$	$-\sin x$	$-\cot x$	$-\tan x$	$\sec x$	$-\text{cosec}\, x$
$\pi - x$	$\sin x$	$-\cos x$	$-\tan x$	$-\cot x$	$\text{cosec}\, x$	$-\sec x$
$\pi + x$	$-\sin x$	$-\cos x$	$\tan x$	$\cot x$	$-\text{cosec}\, x$	$-\sec x$
$\frac{3\pi}{2} - x$	$-\cos x$	$-\sin x$	$\cot x$	$\tan x$	$-\sec x$	$-\text{cosec}\, x$
$\frac{3\pi}{2} + x$	$-\cos x$	$\sin x$	$-\cot x$	$-\tan x$	$-\sec x$	$\text{cosec}\, x$
$2\pi - x$	$-\sin x$	$\cos x$	$-\tan x$	$-\cot x$	$-\text{cosec}\, x$	$\sec x$
$2\pi + x$	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\text{cosec}\, x$	$\sec x$

From the above table we observe that the values of sine function at $\frac{\pi}{2} \pm x = 1 \times \frac{\pi}{2} \pm x$ and $\frac{3\pi}{2} \pm x = 3 \times \frac{\pi}{2} \pm x$ are $\cos x$ or $-\cos x$ depending upon the quadrant in which the terminating ray of the angle lies. Also, sine function is periodic with period 2π. So, the values of sine function at $2n\pi + \frac{\pi}{2} \pm x$ and $2n\pi + \frac{3\pi}{2} \pm x$ are $\cos x$ or $-\cos x$. But, $2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$ and $2n\pi + \frac{3\pi}{2} = (4n+3)\frac{\pi}{2}$ are odd multiples of $\frac{\pi}{2}$. Therefore, the value of sine function at $(2n-1)\frac{\pi}{2} \pm x$ is $\cos x$ or $-\cos x$ depending upon the position of the terminating ray of the angle. We also note from the table that the values of sine function at $\pi \pm x = 2 \times \frac{\pi}{2} \pm x$ and $2\pi \pm x = 4 \times \frac{\pi}{2} \pm x$ are sin or $-\sin x$ depending upon the position of the terminating ray of the angle. The periodicity of the sine function gives that the values of sine function at $n\pi \pm x = 2n \times \frac{\pi}{2} \pm x$ are $\sin x$ or $-\sin x$ depending upon the quadrant in which the terminating ray of the angle lies.

We also note that at point expressible in the form $(2n-1)\frac{\pi}{2} \pm x$ the values of cosine, tangent, cotangent, secant and cosecant functions are $\pm \sin x$, $\pm \cot x$, $\pm \tan x$, $\pm \text{cosec}\, x$ and $\pm \sec x$ respectively. At a point expressible in the form $2n\left(\frac{\pi}{2}\right) \pm x$ the values of cosine, tangent,

cotangent, secant and cosecant functions are $\pm \cos x$, $\pm \tan x$, $\pm \cot x$, $\pm \sec x$ and $\pm \text{cosec}\, x$ respectively.

The above discussion suggests us the following algorithm to find the value of a trigonometric function at a point.

ALGORITHM

Step I *Obtain the point x at which the value of a trigonometric function is to be determined.*

Step II *Check whether x is positive or negative. If x is negative, say $x = -y$, then write $f(x) = -f(y)$, if f is an odd function. Otherwise, write $f(x) = f(y)$. Here, f is the given trigonometric function.*

Step III *Express the positive value of x in step II, in the form $x = \frac{n\pi}{2} \pm \alpha$, where $\alpha \in \left(0, \frac{\pi}{2}\right)$.*

Step IV *Determine the quadrant in which the terminating ray of the angle x lies and determine the sign of the trigonometric function in that quadrant.*

Step V *If n is step III is an odd positive integer, then* $\sin x = \pm \cos \alpha$, $\cos x = \pm \sin \alpha$, $\sec x = \pm \text{cosec}\, \alpha$, $\text{cosec}\, x = \pm \sec \alpha$, *where the sign on RHS of these values will be the sign obtained in step IV.*

If n in step III is an even positive integer, then $\sin x = \pm \sin \alpha$, $\cos x = \pm \cos \alpha$, $\tan x = \pm \tan \alpha$, $\cot x = \pm \cot \alpha$, $\text{cosec}\, x = \pm \text{cosec}\, \alpha$, $\sec x = \pm \sec \alpha$, *where the sign on RHS of these values will be the sign obtained in step IV.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate the following :*

(i) $\sin \frac{7\pi}{4}$ (ii) $\cos \frac{7\pi}{6}$ (iii) $\cos\left(-\frac{8\pi}{3}\right)$ (iv) $\sin\left(-\frac{25\pi}{4}\right)$

SOLUTION (i) Clearly, $\sin \frac{7\pi}{4} = \sin\left(3 \times \frac{\pi}{2} + \frac{\pi}{4}\right)$. Since $3 \times \frac{\pi}{2} + \frac{\pi}{4}$ lies in the IVth quadrant in which sine function is negative and 3 is an odd integer.

$$\therefore \quad \sin \frac{7\pi}{4} = \sin\left(3 \times \frac{\pi}{2} + \frac{\pi}{4}\right) = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

(ii) Clearly, $\cos \frac{7\pi}{6} = \cos\left(2 \times \frac{\pi}{2} + \frac{\pi}{6}\right)$. Since $\frac{7\pi}{6}$ is in the III quadrant in which cosine function is negative. Also the multiple of $\frac{\pi}{2}$ is even.

$$\therefore \quad \cos \frac{7\pi}{6} = \cos\left(2 \times \frac{\pi}{2} + \frac{\pi}{6}\right) = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

(iii) We know that cosine is an even function. Therefore, $\cos\left(-\frac{8\pi}{3}\right) = \cos \frac{8\pi}{3}$.

Also, $\frac{8\pi}{3} = 5 \times \frac{\pi}{2} + \frac{\pi}{6}$. So, $\frac{8\pi}{3}$ is in the II quadrant in which cosine function is negative. Also, the multiple of $\frac{\pi}{2}$ is odd.

$$\therefore \quad \cos \frac{8\pi}{3} = \cos\left(5 \times \frac{\pi}{2} + \frac{\pi}{6}\right) = -\sin \frac{\pi}{6} = -\frac{1}{2}. \text{ Hence, } \cos\left(-\frac{8\pi}{3}\right) = \cos \frac{8\pi}{3} = -\frac{1}{2}$$

(iv) The sine function is an odd function. Therefore, $\sin\left(-\frac{25\pi}{4}\right) = -\sin\frac{25\pi}{4}$.

Now, $\frac{25\pi}{4} = \left(12 \times \frac{\pi}{2} + \frac{\pi}{4}\right) \Rightarrow \frac{25\pi}{4}$ lies in the I quadrant and multiple of $\frac{\pi}{2}$ in this expression is even.

$\therefore \quad \sin\frac{25\pi}{4} = \sin\left(12 \times \frac{\pi}{2} + \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Hence, $\sin\left(-\frac{25\pi}{4}\right) = -\sin\frac{25\pi}{4} = -\frac{1}{\sqrt{2}}$.

EXAMPLE 2 *Evaluate the following :*

(i) $\operatorname{cosec} 390°$ (ii) $\cot 570°$ (iii) $\tan 480°$ (iv) $\cos 270°$

(v) $\tan\frac{19\pi}{3}$ (vi) $\sin\left(-\frac{11\pi}{3}\right)$ (vii) $\cot\left(-\frac{15\pi}{4}\right)$

SOLUTION (i) We have, $390° = \frac{13\pi}{6} = 4 \times \frac{\pi}{2} + \frac{\pi}{6}$. This shows that $\frac{13\pi}{6}$ is in I quadrant in which cosecant function is positive and the multiple of $\frac{\pi}{2}$ is even.

$\therefore \quad \operatorname{cosec} 390° = \operatorname{cosec}\frac{13\pi}{6} = \operatorname{cosec}\left(4 \times \frac{\pi}{2} + \frac{\pi}{6}\right) = \operatorname{cosec}\frac{\pi}{6} = 2.$

(ii) We have, $570° = \frac{19\pi}{6} = 6 \times \frac{\pi}{2} + \frac{\pi}{6}$. It shows that $\frac{19\pi}{6}$ is in the IIIrd quadrant in which cotangent function is positive and the multiple of $\frac{\pi}{2}$ is even.

$\therefore \quad \cot 570° = \cot\frac{19\pi}{6} = \cot\left(6 \times \frac{\pi}{2} + \frac{\pi}{6}\right) = \cot\frac{\pi}{6} = \sqrt{3}$

(iii) We have, $480° = \frac{8\pi}{3} = 5 \times \frac{\pi}{2} + \frac{\pi}{6}$. Clearly, $\frac{8\pi}{3}$ is in the IInd quadrant in which tangent function is negative and the multiple of $\frac{\pi}{2}$ is odd.

$\therefore \quad \tan 480° = \tan\frac{8\pi}{3} = \tan\left(5 \times \frac{\pi}{2} + \frac{\pi}{6}\right) = -\cot\frac{\pi}{6} = -\sqrt{3}$

(iv) We have, $270° = \frac{3\pi}{2} = \left(3 \times \frac{\pi}{2} + 0\right)$. Clearly, 270° is in the negative direction of y-axis i.e. on the boundary line of II and III quadrant. Also, the multiple of $\frac{\pi}{2}$ is an odd integer.

$\therefore \quad \cos 270° = \cos\left(3 \times \frac{\pi}{2} + 0\right) = \pm \sin 0 = 0$

ALITER We know that $\cos(2n-1)\frac{\pi}{2} = 0$. Therefore, $\cos\frac{3\pi}{2} = 0$.

(v) We have, $\frac{19\pi}{3} = \left(12 \times \frac{\pi}{2} + \frac{\pi}{3}\right)$. Clearly, this angle lies in I quadrant in which tangent function is positive and the multiple of $\frac{\pi}{2}$ is even.

$\therefore \quad \tan\frac{19\pi}{3} = \tan\left(12 \times \frac{\pi}{2} + \frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$

(vi) We have, $\frac{11\pi}{3}=7\times\frac{\pi}{2}+\frac{\pi}{6}$. This angle lies in the IV quadrant in which sine function is negative and the multiple of $\frac{\pi}{2}$ is odd.

$\therefore\quad \sin\frac{11\pi}{3}=\sin\left(7\times\frac{\pi}{2}\times\frac{\pi}{6}\right)=-\cos\frac{\pi}{6}=-\frac{\sqrt{3}}{2}$. Hence, $\sin\left(-\frac{11\pi}{3}\right)=-\sin\frac{11\pi}{3}=-\left(-\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{2}$

(vii) We have, $\frac{15\pi}{4}=\left(7\times\frac{\pi}{2}+\frac{\pi}{4}\right)$. This means that $\frac{15\pi}{4}$ is in the IVth quadrant in which cotangent function is negative and the multiple of $\frac{\pi}{2}$ is odd.

$\therefore\quad \cot\frac{15\pi}{4}=\cot\left(7\times\frac{\pi}{2}+\frac{\pi}{4}\right)=-\tan\frac{\pi}{4}=-1$. Hence, $\cot\left(-\frac{15\pi}{4}\right)=-\cot\left(\frac{15\pi}{4}\right)=-(-1)=1.$

EXAMPLE 3 *Prove that:* $\cos 510^\circ\cos 330^\circ+\sin 390^\circ\cos 120^\circ=-1.$

SOLUTION LHS $=\cos 510^\circ\cos 330^\circ+\sin 390^\circ\cos 120^\circ=\cos\frac{17\pi}{6}\cos\frac{11\pi}{6}+\sin\frac{13\pi}{6}\cos\frac{2\pi}{3}$

$$=\cos\left(5\times\frac{\pi}{2}+\frac{\pi}{3}\right)\cos\left(3\times\frac{\pi}{2}+\frac{\pi}{3}\right)+\sin\left(4\times\frac{\pi}{2}+\frac{\pi}{6}\right)\cos\left(1\times\frac{\pi}{2}+\frac{\pi}{6}\right)$$

$$=\left(-\sin\frac{\pi}{3}\right)\left(\sin\frac{\pi}{3}\right)+\left(\sin\frac{\pi}{6}\right)\left(-\sin\frac{\pi}{6}\right)$$

$$=-\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)=-\frac{3}{4}-\frac{1}{4}=-1=\text{RHS}$$

EXAMPLE 4 *Prove that :* $\sin(-420^\circ)(\cos 390^\circ)+\cos(-660^\circ)(\sin 330^\circ)=-1.$

SOLUTION We know that $\sin(-x)=-\sin x$ and $\cos(-x)=\cos x$

$\therefore\quad$ LHS $=\sin(-420^\circ)(\cos 390^\circ)+\cos(-660^\circ)(\sin 330^\circ)$

$$=-\sin 420^\circ\cos 390^\circ+\cos 660^\circ\sin 330^\circ=-\sin\frac{7\pi}{3}\cos\frac{13\pi}{6}+\cos\frac{11\pi}{3}\sin\frac{11\pi}{6}$$

$$=-\sin\left(4\times\frac{\pi}{2}+\frac{\pi}{3}\right)\cos\left(4\times\frac{\pi}{2}+\frac{\pi}{6}\right)+\cos\left(7\times\frac{\pi}{2}+\frac{\pi}{6}\right)\sin\left(3\times\frac{\pi}{2}+\frac{\pi}{3}\right)$$

$$=-\sin\frac{\pi}{3}\cos\frac{\pi}{6}+\left(\sin\frac{\pi}{6}\right)\left(-\cos\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)=-\frac{3}{4}-\frac{1}{4}=-1=\text{RHS}.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Prove that:*

(i) $\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{3}-\tan^2\frac{\pi}{4}=-\frac{1}{2}$

(ii) $2\sin^2\frac{\pi}{6}+\text{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}=\frac{3}{2}$

(iii) $\cot^2\frac{\pi}{6}+\text{cosec}\frac{5\pi}{6}+3\tan^2\frac{\pi}{6}=6$

(iv) $2\sin^2\frac{3\pi}{4}+2\cos^2\frac{\pi}{4}+2\sec^2\frac{\pi}{3}=10$

SOLUTION (i) LHS $=\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{3}-\tan^2\frac{\pi}{4}$

$$=\left(\sin\frac{\pi}{6}\right)^2+\left(\cos\frac{\pi}{3}\right)^2-\left(\tan\frac{\pi}{4}\right)^2=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2-(1)^2=\frac{1}{4}+\frac{1}{4}-1=-\frac{1}{2}$$

(ii) LHS $=2\sin^2\frac{\pi}{6}+\text{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}=2\left(\sin\frac{\pi}{6}\right)^2+\left(\text{cosec}\frac{7\pi}{6}\right)^2\left(\cos\frac{\pi}{3}\right)^2$

$$= 2\left(\sin\frac{\pi}{6}\right)^2 + \left\{\operatorname{cosec}\left(\pi+\frac{\pi}{6}\right)\right\}^2\left(\cos\frac{\pi}{3}\right)^2$$

$$= 2\left(\sin\frac{\pi}{6}\right)^2 + \left\{-\operatorname{cosec}\frac{\pi}{6}\right\}^2\left(\cos\frac{\pi}{3}\right)^2 \qquad [\because \operatorname{cosec}(\pi+x) = -\operatorname{cosec} x]$$

$$= 2\left(\frac{1}{2}\right)^2 + (-2)^2 \times \left(\frac{1}{2}\right)^2 = \frac{1}{2}+1 = \frac{3}{2} = \text{RHS}$$

(iii) $\text{LHS} = \cot^2\frac{\pi}{6} + \operatorname{cosec}\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6} = \left(\cot\frac{\pi}{6}\right)^2 + \operatorname{cosec}\left(\pi-\frac{\pi}{6}\right) + 3\left(\tan\frac{\pi}{6}\right)^2$

$$= \left(\cot\frac{\pi}{6}\right)^2 + \operatorname{cosec}\frac{\pi}{6} + 3\left(\tan\frac{\pi}{6}\right)^2 = (\sqrt{3})^2 + 2 + 3\left(\frac{1}{\sqrt{3}}\right)^2 = 3+2+1 = 6 = \text{RHS}$$

(iv) $\text{LHS} = 2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3} = 2\left(\sin\frac{3\pi}{4}\right)^2 + 2\left(\cos\frac{\pi}{4}\right)^2 + 2\left(\sec\frac{\pi}{3}\right)^2$

$$= 2\left(\sin\frac{\pi}{4}\right)^2 + 2\left(\cos\frac{\pi}{4}\right)^2 + 2\left(\sec\frac{\pi}{3}\right)^2 \qquad \left[\because \sin\frac{3\pi}{4} = \sin\left(\pi-\frac{\pi}{4}\right) = \sin\frac{\pi}{4}\right]$$

$$= 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2(2)^2 = 1+1+8 = 10 = \text{RHS}$$

EXAMPLE 6 *Prove that:* $\dfrac{\cos\left(\frac{\pi}{2}+x\right)\sec(-x)\tan(\pi-x)}{\sec(2\pi-x)\sin(\pi+x)\cot\left(\frac{\pi}{2}-x\right)} = -1.$

SOLUTION We have,

$$\text{LHS} = \frac{\cos\left(\frac{\pi}{2}+x\right)\sec(-x)\tan(\pi-x)}{\sec(2\pi-x)\sin(\pi+x)\cot\left(\frac{\pi}{2}-x\right)} = \frac{(-\sin x)(\sec x)(-\tan x)}{(\sec x)(-\sin x)(\tan x)} = -1 = \text{RHS}$$

EXAMPLE 7 *Prove that :*

(i) $\dfrac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)} = \cot^2 x$

(ii) $\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left\{\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right\} = 1$

SOLUTION (i) $\text{LHS} = \dfrac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)} = \dfrac{(-\cos x)\times(\cos x)}{(\sin x)(-\sin x)} = \dfrac{-\cos^2 x}{-\sin^2 x} = \cot^2 x = \text{RHS}$

(ii) We know that

$\cos\left(\frac{3\pi}{2}+x\right) = \sin x$, $\cos(2\pi+x) = \cos x$, $\cot\left(\frac{3\pi}{2}-x\right) = \tan x$ and $\cot(2\pi+x) = \cot x$

$$\therefore \text{LHS} = \cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left\{\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right\}$$

$$= (\sin x)(\cos x)(\tan x + \cot x) = \sin x\cos x\left\{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right\}$$

$$= \sin x \cos x \left\{ \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \right\} = \sin x \cos x \times \frac{1}{\sin x \cos x} = 1$$

EXAMPLE 8 *If A, B, C, D are angles of a cyclic quadrilateral, prove that* $\cos A + \cos B + \cos C + \cos D = 0$.

SOLUTION We know that the opposite angles of a cyclic quadrilateral are supplementary i.e. $A + C = \pi$ and $B + D = \pi$. Therefore, $A = \pi - C$ and $B = \pi - D$.

$\therefore \quad \cos A = \cos(\pi - C) = -\cos C$ and, $\cos B = \cos(\pi - D) = -\cos D$

Hence, $\cos A + \cos B + \cos C + \cos D = -\cos C - \cos D + \cos C + \cos D = 0$

EXAMPLE 9 *In any quadrilateral ABCD, prove that*

(i) $\sin(A + B) + \sin(C + D) = 0$ (ii) $\cos(A + B) = \cos(C + D)$

SOLUTION (i) $A + B + C + D = 2\pi$

$\Rightarrow \quad A + B = 2\pi - (C + D)$

$\Rightarrow \quad \sin(A + B) = \sin\{2\pi - (C + D)\}$

$\Rightarrow \quad \sin(A + B) = -\sin(C + D) \Rightarrow \sin(A + B) + \sin(C + D) = 0$ [$\because \sin(2\pi - x) = -\sin x$]

(ii) $\quad A + B + C + D = 2\pi$

$\Rightarrow \quad A + B = 2\pi - (C + D)$

$\Rightarrow \quad \cos(A + B) = \cos\{2\pi - (C + D)\} \Rightarrow \cos(A + B) = \cos(C + D)$ [$\because \cos(2\pi - x) = \cos x$]

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 10 *Find the value of the expression*

$$3\left\{\sin^4\left(\frac{3\pi}{2} - x\right) + \sin^4(3\pi + x)\right\} - 2\left\{\sin^6\left(\frac{\pi}{2} + x\right) + \sin^6(5\pi - x)\right\}$$ [NCERT EXEMPLAR]

SOLUTION The given expression is

$$3\left\{\sin^4\left(\frac{3\pi}{2} - x\right) + \sin^4(3\pi + x)\right\} - 2\left\{\sin^6\left(\frac{\pi}{2} + x\right) + \sin^6(5\pi - x)\right\}$$

$$= 3\left\{(-\cos x)^4 + (-\sin x)^4\right\} - 2\left\{(\cos x)^6 + (\sin x)^6\right\} = 3(\cos^4 x + \sin^4 x) - 2(\cos^6 x + \sin^6 x)$$

$$= 3\left\{(\cos^2 x + \sin^2 x)^2 - 2\sin^2 x \cos^2 x\right\} - 2\left\{(\cos^2 x + \sin^2 x)^3 - 3\cos^2 x \sin^2 x(\cos^2 x + \sin^2 x)\right\}$$

$$= 3(1 - 2\sin^2 x \cos^2 x) - 2(1 - 3\cos^2 x \sin^2 x) = 3 - 6\sin^2 x \cos^2 x - 2 + 6\sin^2 x \cos^2 x = 1.$$

EXERCISE 5.3

BASIC

1. Find the values of the following trigonometric ratios:

(i) $\sin\frac{5\pi}{3}$ (ii) $\sin 17\pi$ (iii) $\tan\frac{11\pi}{6}$ (iv) $\cos\left(-\frac{25\pi}{4}\right)$

(v) $\tan\frac{7\pi}{4}$ (vi) $\sin\frac{17\pi}{6}$ (vii) $\cos\frac{19\pi}{6}$ (viii) $\sin\left(-\frac{11\pi}{6}\right)$

(ix) $\operatorname{cosec}\left(-\frac{20\pi}{3}\right)$ (x) $\tan\left(-\frac{13\pi}{4}\right)$ (xi) $\cos\frac{19\pi}{4}$ (xii) $\sin\frac{41\pi}{4}$

(xiii) $\cos\frac{39\pi}{4}$ (xiv) $\sin\frac{151\pi}{6}$

2. Prove that:

(i) $\tan 225° \cot 405° + \tan 765° \cot 675° = 0$

(ii) $\sin \frac{8\pi}{3} \cos \frac{23\pi}{6} + \cos \frac{13\pi}{3} \sin \frac{35\pi}{6} = \frac{1}{2}$

(iii) $\cos 24° + \cos 55° + \cos 125° + \cos 204° + \cos 300° = \frac{1}{2}$

(iv) $\tan(-225°) \cot(-405°) - \tan(-765°) \cot(675°) = 0$

(v) $\cos 570° \sin 510° + \sin(-330°) \cos(-390°) = 0$

(vi) $\tan \frac{11\pi}{3} - 2 \sin \frac{4\pi}{6} - \frac{3}{4} \operatorname{cosec}^2 \frac{\pi}{4} + 4 \cos^2 \frac{17\pi}{6} = \frac{3-4\sqrt{3}}{2}$

(vii) $3 \sin \frac{\pi}{6} \sec \frac{\pi}{3} - 4 \sin \frac{5\pi}{6} \cot \frac{\pi}{4} = 1$

3. Prove that:

(i) $\dfrac{\cos(2\pi + x) \operatorname{cosec}(2\pi + x) \tan(\pi/2 + x)}{\sec(\pi/2 + x) \cos x \cot(\pi + x)} = 1$

(ii) $\dfrac{\operatorname{cosec}(90° + x) + \cot(450° + x)}{\operatorname{cosec}(90° - x) + \tan(180° - x)} + \dfrac{\tan(180° + x) + \sec(180° - x)}{\tan(360° + x) - \sec(-x)} = 2$

(iii) $\dfrac{\sin(\pi + x) \cos\left(\frac{\pi}{2} + x\right) \tan\left(\frac{3\pi}{2} - x\right) \cot(2\pi - x)}{\sin(2\pi - x) \cos(2\pi + x) \operatorname{cosec}(-x) \sin\left(\frac{3\pi}{2} - x\right)} = 1$

(iv) $\left\{1 + \cot x - \sec\left(\frac{\pi}{2} + x\right)\right\} \left\{1 + \cot x + \sec\left(\frac{\pi}{2} + x\right)\right\} = 2 \cot x$

(v) $\dfrac{\tan\left(\frac{\pi}{2} - x\right) \sec(\pi - x) \sin(-x)}{\sin(\pi + x) \cot(2\pi - x) \operatorname{cosec}\left(\frac{\pi}{2} - x\right)} = 1$

BASED ON LOTS

4. Prove that : $\sin^2 \frac{\pi}{18} + \sin^2 \frac{\pi}{9} + \sin^2 \frac{7\pi}{18} + \sin^2 \frac{4\pi}{9} = 2$

5. Prove that : $\sec\left(\frac{3\pi}{2} - x\right) \sec\left(x - \frac{5\pi}{2}\right) + \tan\left(\frac{5\pi}{2} + x\right) \tan\left(x - \frac{3\pi}{2}\right) = -1.$

6. In a ΔABC, prove that :

(i) $\cos(A + B) + \cos C = 0$ (ii) $\cos\left(\frac{A+B}{2}\right) = \sin \frac{C}{2}$ (iii) $\tan \frac{A+B}{2} = \cot \frac{C}{2}$

7. If A, B, C, D be the angles of a cyclic quadrilateral, taken in order, prove that :

$$\cos(180° - A) + \cos(180° + B) + \cos(180° + C) - \sin(90° + D) = 0$$

8. Find x from the following equations :

(i) $\operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) + x \cos \theta \cot\left(\frac{\pi}{2} + \theta\right) = \sin\left(\frac{\pi}{2} + \theta\right)$

(ii) $x \cot\left(\frac{\pi}{2} + \theta\right) + \tan\left(\frac{\pi}{2} + \theta\right) \sin \theta + \operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) = 0$

9. Prove that:

(i) $\tan 4\pi - \cos \frac{3\pi}{2} - \sin \frac{5\pi}{6} \cos \frac{2\pi}{3} = \frac{1}{4}$ (ii) $\sin \frac{13\pi}{3} \sin \frac{8\pi}{3} + \cos \frac{2\pi}{3} \sin \frac{5\pi}{6} = \frac{1}{2}$

(iii) $\sin\frac{13\pi}{3}\sin\frac{2\pi}{3}+\cos\frac{4\pi}{3}\sin\frac{13\pi}{6}=\frac{1}{2}$

(iv) $\sin\frac{10\pi}{3}\cos\frac{13\pi}{6}+\cos\frac{8\pi}{3}\sin\frac{5\pi}{6}=-1$

(v) $\tan\frac{5\pi}{4}\cot\frac{9\pi}{4}+\tan\frac{17\pi}{4}\cot\frac{15\pi}{4}=0$

ANSWERS

1. (i) $-\frac{\sqrt{3}}{2}$ (ii) 0 (iii) $-\frac{1}{\sqrt{3}}$ (iv) $\frac{1}{\sqrt{2}}$ (v) -1 (vi) $\frac{1}{2}$
(vii) $-\frac{\sqrt{3}}{2}$ (viii) $\frac{1}{2}$ (ix) $-\frac{2}{\sqrt{3}}$ (x) -1 (xi) $-\frac{1}{\sqrt{2}}$ (xii) $\frac{1}{\sqrt{2}}$
(xiii) $1/\sqrt{2}$ (xiv) $-1/2$

8. (i) $\tan\theta$ (ii) $\sin\theta$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The value of $\frac{\sin 70°}{\sin 110°}$ is ....
2. The value of $\frac{\cos 50°}{\cos 130°}$ is ....
3. The values of $f(x)=2\sin\sqrt{x^2+x+1}$ lie in the interval ....
4. If $\sin x+\cos x=a$, then $\sin x-\cos x=$ ....
5. If $-\frac{\pi}{2}<x<\frac{\pi}{2}$, then $\sqrt{\frac{1-\sin x}{1+\sin x}}$ is equal to ....
6. If $\sec x+\tan x=\sqrt{3}$, then $\sec x-\tan x=$ ....
7. If $\sec x-\tan x=\frac{2}{3}$, the $\tan x=$ ....
8. If $\operatorname{cosec} x+\cot x=\alpha$, then $\sin x=$ ....
9. If $\operatorname{cosec} x+\cot x=\frac{11}{2}$, then the value of $\tan x$ is ....
10. If $\sin x=\frac{-24}{25}$, then the value of $\tan x$ is ....
11. If $\sin x+\operatorname{cosec} x=2$, then $\sin^2 x+\operatorname{cosec}^2 x=$ ....
12. The value of $\tan x+\cot(\pi+x)+\cot\left(\frac{\pi}{2}+x\right)+\cot(2\pi-x)$ is ....
13. The value of $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x)$ is ....
14. Given $x>0$, the value of $f(x)=-3\cos\sqrt{3+x+x^2}$ lie in the interval ....
15. If $\sin x+\cos x=a$, then $\sin^6 x+\cos^6 x=$ ....
16. The value of $\cos 1°\cos 2°\cos 3°\ldots\ldots\cos 179°$ is ....
17. The value of $\tan 1°\tan 2°\tan 3°\ldots\ldots\tan 89°$ is ....
18. If $\frac{\pi}{2}<x<\pi$ and $\sqrt{\frac{1+\sin x}{1-\sin x}}=k\sec x$, then $k=$ ....
19. If $\frac{\pi}{2}<x<\pi$ and $\sqrt{\frac{1+\sin x}{1-\sin x}}+\sqrt{\frac{1-\sin x}{1+\sin x}}=k\sec x$, then $k=$ ....

20. If $\pi < x < 2\pi$ and $\sqrt{\dfrac{1+\cos x}{1-\cos x}} + \sqrt{\dfrac{1-\cos x}{1+\cos x}} = k \operatorname{cosec} x$, then $k =$ ....

21. The minimum value of $9\tan^2\theta + 4\cot^2\theta$ is ....

22. If $\sec x = m$ and $\tan x = n$, then $\dfrac{1}{m}\left\{(m+n) + \dfrac{1}{m+n}\right\}$ is equal to ....

23. If $\cos^2 x + \sin x + 1 = 0$, and $0 < x < 2\pi$ then $x =$ ....

24. If $\sin x = \dfrac{2t}{1+t^2}$ and x lies in the second quadrant, then $\cos x =$ ....

25. If $\sec x = t + \dfrac{1}{4t}$, then the value of $\sec x + \tan x$ is ....

26. If $\tan x + \cot x = 4$, then $\tan^4 x + \cot^4 x =$ ....

27. If $\dfrac{3\pi}{4} < x < \pi$, then $\sqrt{\operatorname{cosec}^2 x + 2\cot x}$ is equal to ....

ANSWERS

1. 1 2. -1 3. $[-2, 2]$ 4. $\sqrt{2-a^2}$ 5. $\sec x - \tan x$ 6. $\dfrac{1}{\sqrt{3}}$ 7. $\dfrac{5}{12}$

8. $\dfrac{2\alpha}{\alpha^2+1}$ 9. $\dfrac{44}{117}$ 10. $-\dfrac{24}{7}$ 11. 2 12. 0 13. 13 14. $[-3, 3]$

15. $\dfrac{1}{4}[4-3(a^2-1)^2]$ 16. 0 17. 1 18. -1 19. -2 20. -2 21. 12

22. 2 23. $\dfrac{3\pi}{2}$ 24. $\dfrac{|1-t^2|}{1+t^2}$ 25. $2t$ 26. 194 27. $-1-\cot x$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the maximum and minimum values of $\cos(\cos x)$.
2. Write the maximum and minimum values of $\sin(\sin x)$.
3. Write the maximum value of $\sin(\cos x)$.
4. If $\sin x = \cos^2 x$, then write the value of $\cos^2 x(1+\cos^2 x)$.
5. If $\sin x + \operatorname{cosec} x = 2$, then write the value of $\sin^n x + \operatorname{cosec}^n x$.
6. If $\sin x + \sin^2 x = 1$, then write the value of $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x$.
7. If $\sin x + \sin^2 x = 1$, then write the value of $\cos^8 x + 2\cos^6 x + \cos^4 x$.
8. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then write the value of $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$.
9. Write the value of $\sin 10° + \sin 20° + \sin 30° + \ldots + \sin 360°$.
10. A circular wire of radius 15 cm is cut and bent so as to lie along the circumference of a loop of radius 120 cm. Write the measure of the angle subtended by it at the centre of the loop.
11. Write the value of $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1$.
12. Write the value of $\cos 1° + \cos 2° + \cos 3° + \ldots + \cos 180°$.
13. If $\cot(\alpha+\beta) = 0$, then write the value of $\sin(\alpha + 2\beta)$.

14. If $\tan A + \cot A = 4$, then write the value of $\tan^4 A + \cot^4 A$.

15. Write the least value of $\cos^2 x + \sec^2 x$.

16. If $x = \sin^{14} x + \cos^{20} x$, then write the smallest interval in which the value of x lie.

17. If $3 \sin x + 5 \cos x = 5$, then write the value of $5 \sin x - 3 \cos x$.

ANSWERS

1. 1, cos 1 2. sin 1, – sin 1 3. sin 1 4. 1 5. 2 6. 1 7. 1 8. 0
9. 0 10. 45° 11. 0 12. –1 13. sin α or cos β 14. 194 15. 2 16. $(0, 1]$
17. 3 or – 3

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $\tan x = x - \frac{1}{4x}$, then $\sec x - \tan x$ is equal to

(a) $-2x, \frac{1}{2x}$ (b) $-\frac{1}{2x}, 2x$ (c) $2x$ (d) $2x, \frac{1}{2x}$

2. If $\sec x = x + \frac{1}{4x}$, then $\sec x + \tan x =$

(a) $x, \frac{1}{x}$ (b) $2x, \frac{1}{2x}$ (c) $-2x, \frac{1}{2x}$ (d) $-\frac{1}{x}, x$

3. If $\frac{\pi}{2} < x < \frac{3\pi}{2}$, then $\sqrt{\frac{1 - \sin x}{1 + \sin x}}$ is equal to

(a) $\sec x - \tan x$ (b) $\sec x + \tan x$ (c) $\tan x - \sec x$ (d) none of these

4. If $\pi < x < 2\pi$, then $\sqrt{\frac{1 + \cos x}{1 - \cos x}}$ is equal to

(a) $\operatorname{cosec} x + \cot x$ (b) $\operatorname{cosec} x - \cot x$ (c) $-\operatorname{cosec} x + \cot x$ (d) $-\operatorname{cosec} x - \cot x$

5. If $0 < x < \frac{\pi}{2}$, and if $\frac{y+1}{1-y} = \sqrt{\frac{1 + \sin x}{1 - \sin x}}$, then y is equal to

(a) $\cot \frac{x}{2}$ (b) $\tan \frac{x}{2}$ (c) $\cot \frac{x}{2} + \tan \frac{x}{2}$ (d) $\cot \frac{x}{2} - \tan \frac{x}{2}$

6. If $\frac{\pi}{2} < x < \pi$, then $\sqrt{\frac{1 - \sin x}{1 + \sin x}} + \sqrt{\frac{1 + \sin x}{1 - \sin x}}$ *is equal to*

(a) $2 \sec x$ (b) $-2 \sec x$ (c) $\sec x$ (d) $-\sec x$

7. If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$, then $x^2 + y^2 + z^2$ is independent of

(a) θ, ϕ (b) r, θ (c) r, ϕ (d) r

8. If $\tan x + \sec x = \sqrt{3}$, $0 < x < \pi$, then x is equal to

(a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

9. If $\tan x = -\frac{1}{\sqrt{5}}$ and x lies in the IV quadrant, then the value of $\cos x$ is

(a) $\frac{\sqrt{5}}{\sqrt{6}}$ (b) $\frac{2}{\sqrt{6}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{6}}$

10. If $\frac{3\pi}{4} < \alpha < \pi$, then $\sqrt{2 \cot \alpha + \frac{1}{\sin^2 \alpha}}$ is equal to

(a) $1 - \cot \alpha$ (b) $1 + \cot \alpha$ (c) $-1 + \cot \alpha$ (d) $-1 - \cot \alpha$

11. $\sin^6 A + \cos^6 A + 3 \sin^2 A \cos^2 A =$

(a) 0 (b) 1 (c) 2 (d) 3

12. If $\operatorname{cosec} x - \cot x = \frac{1}{2}$, $0 < x < \frac{\pi}{2}$, then $\cos x$ is equal to

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $-\frac{3}{5}$ (d) $-\frac{5}{3}$

13. If $\operatorname{cosec} x + \cot x = \frac{11}{2}$, then $\tan x =$

(a) $\frac{21}{22}$ (b) $\frac{15}{16}$ (c) $\frac{44}{117}$ (d) $\frac{117}{44}$

14. $\sec^2 x = \frac{4xy}{(x+y)^2}$ is true if and only if

(a) $x + y \neq 0$ (b) $x = y, x \neq 0$ (c) $x = y$ (d) $x \neq 0, y \neq 0$

15. If x is an acute angle and $\tan x = \frac{1}{\sqrt{7}}$, then the value of $\frac{\operatorname{cosec}^2 x - \sec^2 x}{\operatorname{cosec}^2 x + \sec^2 x}$ is

(a) 3/4 (b) 1/2 (c) 2 (d) 5/4

16. The value of $\sin^2 5° + \sin^2 10° + \sin^2 15° + \ldots + \sin^2 85° + \sin^2 90°$ is

(a) 7 (b) 8 (c) 9.5 (d) 10

17. $\sin^2 \frac{\pi}{18} + \sin^2 \frac{\pi}{9} + \sin^2 \frac{7\pi}{18} + \sin^2 \frac{4\pi}{9} =$

(a) 1 (b) 4 (c) 2 (d) 0

18. If $\tan A + \cot A = 4$, then $\tan^4 A + \cot^4 A$ is equal to

(a) 110 (b) 191 (c) 80 (d) 194

19. If $x \sin 45° \cos^2 60° = \frac{\tan^2 60° \operatorname{cosec} 30°}{\sec 45° \cot^{2°} 30°}$, then $x =$

(a) 2 (b) 4 (c) 8 (d) 16

20. If A lies in second quadrant and $3 \tan A + 4 = 0$, then the value of $2 \cot A - 5 \cos A + \sin A$ is equal to

(a) –53/10 (b) 23/10 (c) 37/10 (d) 7/10

[NCERT EXEMPLAR]

21. If $\operatorname{cosec} x + \cot x = \frac{11}{2}$, then $\tan x =$

(a) 21/22 (b) 15/16 (c) 44/117 (d) 117/43

22. If $\tan \theta + \sec \theta = e^x$, then $\cos \theta$ equals

(a) $\frac{e^x + e^{-x}}{2}$ (b) $\frac{2}{e^x + e^{-x}}$ (c) $\frac{e^x - e^{-x}}{2}$ (d) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$

23. If $\sec x + \tan x = k$, $\cos x =$

(a) $\frac{k^2 + 1}{2k}$ (b) $\frac{2k}{k^2 + 1}$ (c) $\frac{k}{k^2 + 1}$ (d) $\frac{k}{k^2 - 1}$

24. If $f(x) = \cos^2 x + \sec^2 x$, then
(a) $f(x) < 1$ (b) $f(x) = 1$ (c) $1 < f(x) < 2$ (d) $f(x) \geq 2$
[NCERT EXEMPLAR]

25. Which of the following is incorrect?
(a) $\sin x = -1/5$ (b) $\cos x = 1$ (c) $\sec x = 1/2$ (d) $\tan x = 20$

26. The value of $\cos 1° \cos 2° \cos 3° \ldots \cos 179°$ is
(a) $1/\sqrt{2}$ (b) 0 (c) 1 (d) -1
[NCERT EXEMPLAR]

27. The value of $\tan 1° \tan 2° \tan 3° \ldots \tan 89°$ is
(a) 0 (b) 1 (c) 1/2 (d) not defined
[NCERT EXEMPLAR]

28. Which of the following is correct?
(a) $\sin 1° > \sin 1$ (b) $\sin 1° < \sin 1$ (c) $\sin 1° = \sin 1$ (d) $\sin 1° = \frac{\pi}{180} \sin 1$
[NCERT EXEMPLAR]

29. If $\tan\theta = -\frac{4}{3}$, then $\sin\theta$ is equal to
(a) $-\frac{4}{5}$ but not $\frac{4}{5}$ (b) $-\frac{4}{5}$ or $\frac{4}{5}$ (c) $\frac{4}{5}$ but not $-\frac{4}{5}$ (d) none of these
[NCERT EXEMPLAR]

30. If $\sin\theta$ and $\cos\theta$ are the roots of the equation $ax^2 - bx + c = 0$, then a, b and c satisfy the relation
(a) $a^2 + b^2 + 2ac = 0$ (b) $a^2 - b^2 + 2ac = 0$ (c) $a^2 + c^2 + 2ab = 0$ (d) $a^2 - b^2 - 2ac = 0$
[NCERT EXEMPLAR]

31. If $\sin\theta + \text{cosec}\,\theta = 2$, then $\sin^2\theta + \text{cosec}^2\theta$ is equal to
(a) 1 (b) 4 (c) 2 (d) none of these

32. Which of the following is incorrect?
(a) $\sin\theta = -\frac{1}{5}$ (b) $\cos\theta = 1$ (c) $\sec\theta = \frac{1}{2}$ (d) $\tan\theta = 20$
[NCERT EXEMPLAR]

33. If for real values of x, $\cos\theta = x + \frac{1}{x}$, then
(a) θ is an acute angle (b) θ is a right angle
(c) θ is an obtuse angle (d) No value of θ is possible

ANSWERS

1. (a)	2. (b)	3. (c)	4. (d)	5. (b)	6. (b)	7. (a)	8. (c)
9. (a)	10. (d)	11. (b)	12. (b)	13. (c)	14. (b)	15. (a)	16. (c)
17. (c)	18. (d)	19. (c)	20. (b)	21. (c)	22. (b)	23. (b)	24. (d)
25. (c)	26. (b)	27. (b)	28. (b)	29. (b)	30. (b)	31. (c)	32. (c)
33. (d)							

SUMMARY

1. Following are some of the fundamental trigonometric identities:

(i) $\sin x = \frac{1}{\operatorname{cosec} x}$ or, $\operatorname{cosec} x = \frac{1}{\sin x}$

(ii) $\cos x = \frac{1}{\sec x}$ or, $\sec x = \frac{1}{\cos x}$ (iii) $\cot x = \frac{1}{\tan x}$ or, $\tan x = \frac{1}{\cot x}$

(iv) $\tan x = \frac{\sin x}{\cos x}$ or, $\cot x = \frac{\cos x}{\sin x}$ (v) $\sin^2 x + \cos^2 x = 1$

(vi) $1 + \tan^2 x = \sec^2 x$ or, $\sec x - \tan x = \frac{1}{\sec x + \tan x}$

(vii) $1 + \cot^2 x = \operatorname{cosec}^2 x$ or, $\operatorname{cosec} x - \cot x = \frac{1}{\operatorname{cosec} x + \cot x}$

2. (i) $\sin(-x) = -\sin x$ or, $\operatorname{cosec}(-x) = -\operatorname{cosec} x$

(ii) $\cos(-x) = \cos x$ or, $\sec(-x) = \sec x$

(iii) $\tan(-x) = -\tan x$ or, $\cot(-x) = -\cot x$

(iv) $\sin\left(\frac{\pi}{2} - x\right) = \cos x$, $\cos\left(\frac{\pi}{2} - x\right) = \sin x$, $\tan\left(\frac{\pi}{2} - x\right) = \cot x$, $\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x$

$\operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$, $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

(v) $\sin\left(\frac{\pi}{2} + x\right) = \cos x$, $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$, $\tan\left(\frac{\pi}{2} + x\right) = -\cot x$, $\cot\left(\frac{\pi}{2} + x\right) = -\tan x$,

$\sec\left(\frac{\pi}{2} + x\right) = -\operatorname{cosec} x$, $\operatorname{cosec}\left(\frac{\pi}{2} + x\right) = \sec x$

(vi) $\sin(\pi - x) = \sin x$, $\cos(\pi - x) = -\cos x$, $\tan(\pi - x) = -\tan x$, $\cot(\pi - x) = -\cot x$

$\sec(\pi - x) = -\sec x$, $\operatorname{cosec}(\pi - x) = \operatorname{cosec} x$

(vii) $\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$, $\cos\left(\frac{3\pi}{2} - x\right) = -\sin x$,

$\tan\left(\frac{3\pi}{2} - x\right) = \cot x$, $\cot\left(\frac{3\pi}{2} - x\right) = \tan x$

$\operatorname{cosec}\left(\frac{3\pi}{2} - x\right) = -\sec x$, $\sec\left(\frac{3\pi}{2} - x\right) = -\operatorname{cosec} x$

(viii) $\sin\left(\frac{3\pi}{2} + x\right) = -\cos x$, $\cos\left(\frac{3\pi}{2} + x\right) = \sin x$,

$\tan\left(\frac{3\pi}{2} + x\right) = -\cot x$, $\cot\left(\frac{3\pi}{2} + x\right) = -\tan x$

$\operatorname{cosec}\left(\frac{3\pi}{2} + x\right) = -\sec x$, $\sec\left(\frac{3\pi}{2} + x\right) = \operatorname{cosec} x$

(ix) $\sin(2\pi - x) = -\sin x$, $\cos(2\pi - x) = \cos x$,

$\tan(2\pi - x) = -\tan x$, $\operatorname{cosec}(2\pi - x) = -\operatorname{cosec} x$

$\sec(2\pi - x) = \sec x$, $\cot(2\pi - x) = -\cot x$

(x) Sine and Cosine functions and their reciprocals i.e. Cosecant and Secant functions are periodic functions with period 2π. Tangent and Cotangent functions are periodic with period π.

(xi) Cosine and secant functions are even functions and all other trigonometric functions are odd functions.

CHAPTER 6

GRAPHS OF TRIGONOMETRIC FUNCTIONS

6.1 INTRODUCTION

In the previous chapters, we have learnt that all trigonometric functions are periodic functions. We have also learnt that sine, cosine, cosecant and secant functions are periodic with period 2π whereas tangent and cotangent functions are periodic with period π. We also know that if a function $f(x)$ is periodic with period T, then $f(ax+b)$ is periodic with period $T/|a|$. Therefore, $\sin(ax+b)$, $\cos(ax+b)$, $\operatorname{cosec}(ax+b)$ and $\sec(ax+b)$ are periodic with period $2\pi/|a|$ and $\tan(ax+b)$ and $\cot(ax+b)$ are periodic with period $\pi/|a|$. For example, $\sin 2x$ and $\cos 3\pi x$ are periodic functions with periods π and $2/3$ respectively.

If the graph of a periodic function with period T is to be drawn in a given interval, then it is sufficient to draw its graph only in an interval of length T. Because, once it is drawn in one such interval, it can-easily be drawn completely by repeating it over the intervals of lengths T. As mentioned in the above paragraph all trigonometric functions are periodic functions. So, we will draw their graphs on the intervals of lengths equal to their periods.

6.2 GRAPH OF SINE FUNCTION

We know that $f(x) = \sin x$ is a periodic function with period 2π. Therefore, it is sufficient to know the graph of $f(x) = \sin x$ in the interval $[0, 2\pi]$. Using the periodicity of the function, we can draw the graph of $f(x) = \sin x$ in other intervals such as $[-2\pi, 0]$, $[2\pi, 4\pi]$ etc.

In order to draw the graph of $f(x) = \sin x$ in the interval $[0, 2\pi]$, we require the values of $\sin x$ at some points in $[0, 2\pi]$. These values are listed in the following table.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$y=\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0
$y=\sin x$	0	0.5	0.71	0.86	1	0.86	0.71	0.5	0	−0.5	−0.71	−0.86	−1	−0.86	−0.71	−0.5	0

On suitable scale we plot the points $(0, 0)$ $(\pi/6, 0.5)$, $(\pi/4, 0.71)$, $(\pi/3, 0.86)$ $(\pi/2, 1)$, $(2\pi/3, 0.86)$, $(3\pi/4, 0.71)$, $(5\pi/6, 0.5$, $(\pi, 0)$, $(7\pi/6, -0.5)$, $(5\pi/4, -0.71)$, $(4\pi/3, -0.86)$ $(3\pi/2, -1)$, $(5\pi/3, -0.86)$, $(7\pi/4, -0.71)$, $(11\pi/6, -0.5)$ and $(2\pi, 0)$ in the xy-plane and join them by a free hand curve to obtain the curve $y = \sin x$ i.e. the graph of $f(x) = \sin x$ in the interval $[0, 2\pi]$ as shown in Fig. 6.1.

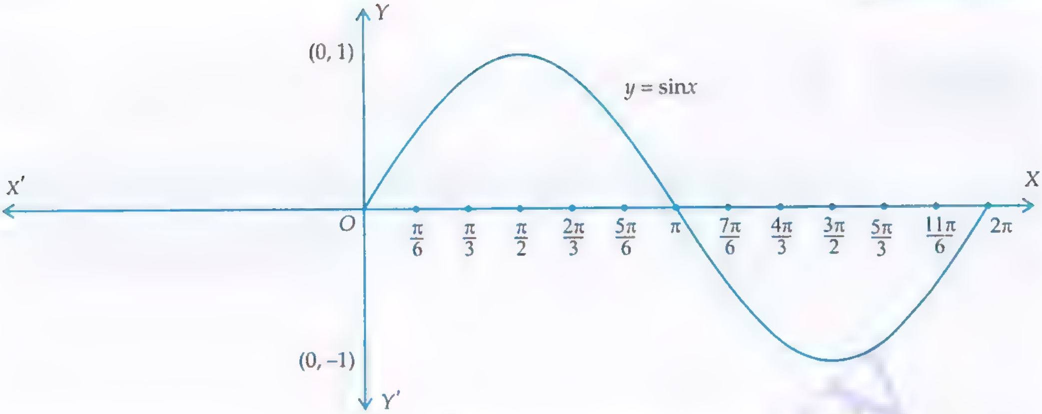


Fig. 6.1 Graph of $f(x) = \sin x$ in $[0, 2\pi]$

As $f(x) = \sin x$ is a periodic function with period 2π. The graph of $f(x) = \sin x$ in the interval $[-2\pi, 0]$ is identical to its graph in $[0, 2\pi]$ as shown in Fig. 6.2.

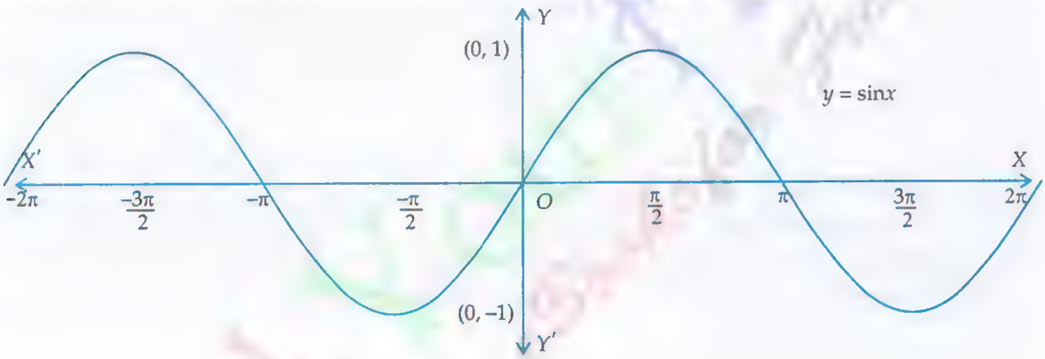


Fig. 6.2 Graph of $f(x) = \sin x$ in $[-2\pi, 2\pi]$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Sketch the graph of the function $f(x) = 3 \sin 2x$.*

SOLUTION We know that $g(x) = \sin x$ is periodic function with period 2π. Therefore, $f(x) = 3 \sin 2x$ is periodic function with period π. So, we will draw the graph of $f(x) = 3 \sin 2x$ in the interval $[0, \pi]$ and to draw (or know) its graph in other intervals such as $[-\pi\ 0]$, $[\pi, 2\pi]$ etc, we may use the periodicity of the function. The values of $f(x) = 3 \sin 2x$ at various points in $[0, \pi]$ are listed in the following table.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{5\pi}{8}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{8}$	$\frac{11\pi}{12}$	π
$2x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$y = 3\sin 2x$	0	$\frac{3}{2}$ $= 15$	$\frac{3}{\sqrt{2}}$ $= 2.1$	$\frac{3\sqrt{3}}{2}$ $= 2.58$	3	$\frac{3\sqrt{3}}{2}$ $= 2.58$	$\frac{3}{\sqrt{2}}$ $= 2.1$	$\frac{3}{2}$ $= 1.5$	0	$-\frac{3}{2}$ $= -1.5$	$\frac{-3}{\sqrt{2}}$ $= -2.1$	$\frac{-3\sqrt{3}}{2}$ $= -2.58$	-3	$\frac{-3\sqrt{3}}{2}$ $= -2.58$	$\frac{-3}{\sqrt{2}}$ $= -2.1$	$\frac{-3}{2}$ $= -1.5$	0

The points $(0, 0)$ $(\pi/12, 1.5)$, $(\pi/8, 2.1)$, $(\pi/6, 2.58)$, $(\pi/4, 3)$, $(\pi/3, 2.58)$, $(3\pi/8, 2.13)$, $(5\pi/12, 1.5)$, $(\pi/2, 0)$,$(7\pi/12, -1.5)$, $(5\pi/8, -2.13)$, $(2\pi/3, -2.58)$ $(3\pi/4, -3)$, $(5\pi/6, -2.58)$, $(7\pi/8, -2.13)$,$(11\pi/12, -1.5)$ and $(\pi, 0)$ are plotted on a suitable scale in the xy-plane and joined by a free hand curve to obtain the graph of the function $f(x) = 3 \sin 2x$ i.e., the curve $y = 3 \sin 2x$ as shown in Fig. 6.3

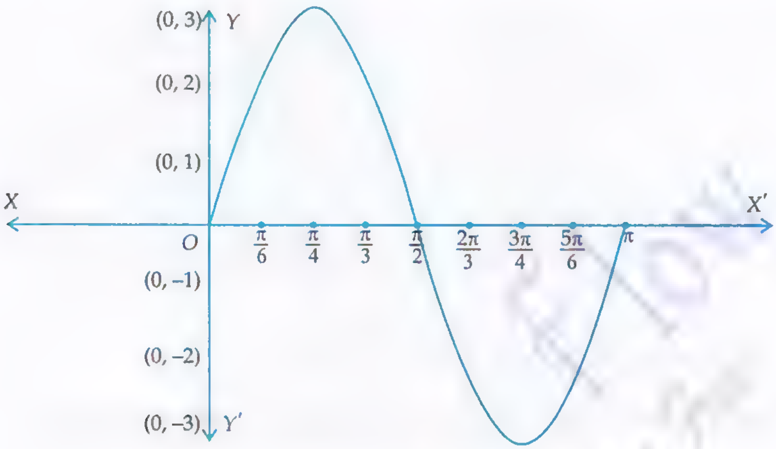


Fig. 6.3 Graph of $f(x) = 3 \sin 2x$ in $[0, \pi]$

EXAMPLE 2 *Sketch the curves $y = \sin x$ and $y = \sin 2x$ on the same scale and same axes.*

SOLUTION We observe that the functions $f(x) = \sin x$ and $g(x) = \sin 2x$ are periodic functions with periods 2π and π respectively. The values of these functions are tabulated below.

Values of $f(x) = \sin x$ in $[0, 2\pi]$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0

Values of $g(x) = \sin 2x$ in $[0, \pi]$

x	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{5\pi}{8}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{8}$	$\frac{11\pi}{8}$	2π
$\sin 2x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0

In order to draw the curves $y = \sin x$ and $y = \sin 2x$, we plot the points given in the above tables and join them by free hand curves as shown in Fig. 6.4.

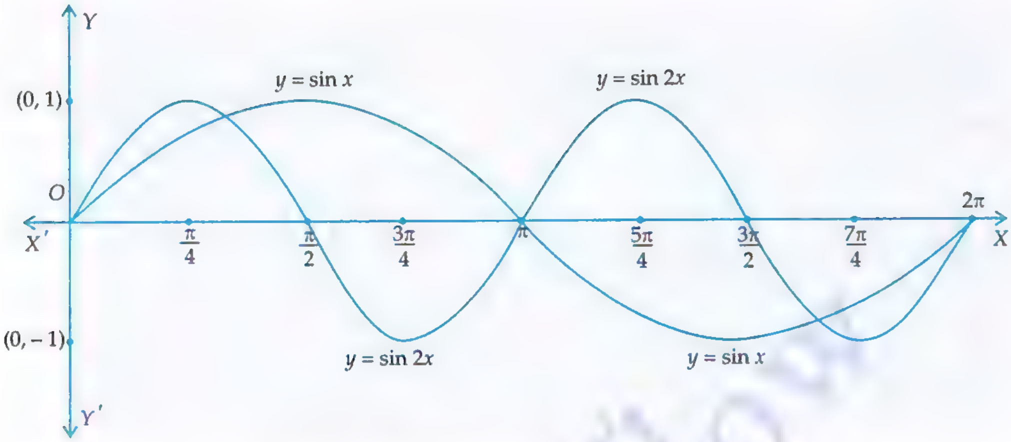


Fig. 6.4 Graph of $y = \sin x$ and $y = \sin 2x$ in $[0, 2\pi]$

EXAMPLE 3 *Sketch the graph of the function* $f(x) = 3\sin\left(2x - \frac{\pi}{4}\right)$.

SOLUTION We know that if $f(x)$ is a periodic function with period T, then $f(ax + b)$ is periodic with period $\frac{T}{|a|}$. As $\sin x$ is periodic with period 2π. Therefore, $f(x) = 3\sin\left(2x - \frac{\pi}{4}\right)$ is periodic with period $\frac{2\pi}{2}$ i.e. π. The values of $f(x)$ for different values of x are listed in the following table.

$2x - \frac{\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
x	$\frac{\pi}{8}$	$\frac{5\pi}{24}$	$\frac{\pi}{4}$	$\frac{7\pi}{24}$	$\frac{3\pi}{8}$	$\frac{11\pi}{24}$	$\frac{\pi}{2}$	$\frac{13\pi}{24}$	$\frac{5\pi}{8}$	$\frac{17\pi}{24}$	$\frac{3\pi}{4}$	$\frac{19\pi}{24}$	$\frac{7\pi}{8}$	$\frac{23\pi}{24}$	π	$\frac{25\pi}{24}$	$\frac{9\pi}{8}$
$3\sin\left(2x - \frac{\pi}{4}\right)$	0	$\frac{3}{2}$	$\frac{3}{\sqrt{2}}$	$\frac{3\sqrt{3}}{2}$	3	$\frac{3\sqrt{3}}{2}$	$\frac{3}{\sqrt{2}}$	$\frac{3}{2}$	0	$-\frac{3}{2}$	$\frac{-3}{\sqrt{2}}$	$\frac{-3\sqrt{3}}{2}$	-3	$\frac{-3\sqrt{3}}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$	$\frac{-3}{2}$	0

In the xy-plane, let us plot the points having x-coordinates given in the second row of the above table and y-coordinate as the corresponding value in the third row. By joining the points so obtained by a free hand curve, we obtain the curve $y = 3\sin\left(2x - \frac{\pi}{4}\right)$ i.e. the graph of the function $f(x) = 3\sin\left(2x - \frac{\pi}{4}\right)$ as shown in Fig. 6.5.

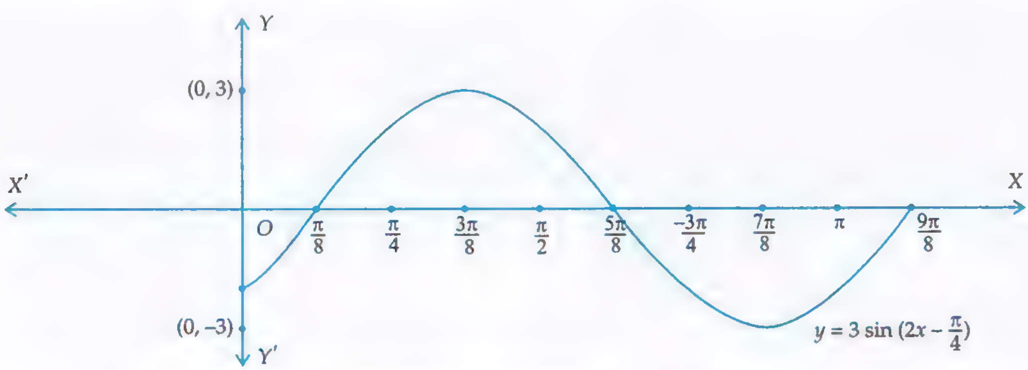


Fig. 6.5 Graph of $f(x) = 3\sin\left(2x - \frac{\pi}{4}\right)$

EXERCISE 6.1

BASIC

1. Sketch the graphs of the following functions:

   (i) $f(x) = 2\sin x,\ 0 \le x \le \pi$

   (ii) $g(x) = 3\sin\left(x - \frac{\pi}{4}\right),\ 0 \le x \le \frac{5\pi}{4}$

   (iii) $h(x) = 2\sin 3x,\ 0 \le x \le \frac{2\pi}{3}$

   (iv) $\phi(x) = 2\sin\left(2x - \frac{\pi}{3}\right),\ 0 \le x \le \frac{7\pi}{5}$

   (v) $\psi(x) = 4\sin 3\left(x - \frac{\pi}{4}\right),\ 0 \le x \le 2\pi$

   (vi) $\theta(x) = \sin\left(\frac{x}{2} - \frac{\pi}{4}\right),\ 0 \le x \le 4\pi$

   (vii) $u(x) = \sin^2 x,\ 0 \le x \le 2\pi$ $\ v(x) = |\sin x|,\ 0 \le x \le 2\pi$

   (viii) $f(x) = 2\sin \pi x,\ 0 \le x \le 2.$

2. Sketch the graphs of the following pairs of functions on the same axes:

   (i) $f(x) = \sin x,\ g(x) = \sin\left(x + \frac{\pi}{4}\right)$

   (ii) $f(x) = \sin x,\ g(x) = \sin 2x$

   (iii) $f(x) = \sin 2x,\ g(x) = 2\sin x$

   (iv) $f(x) = \sin\frac{x}{2},\ g(x) = \sin x$

6.3 GRAPH OF COSINE FUNCTION

In earlier chapters, we have learnt that $f(x) = \cos x$ is a periodic function with period 2π. In order to draw the graph of $f(x) = \cos x$ it is sufficient to know its graph in $[0, 2\pi]$. The values of $f(x) = \cos x$ at various points in $[0, 2\pi]$ are given in the following table.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

On a suitable scale, let us plot the points $(0, 1), (\pi/6, \sqrt{3}/2), (\pi/4, 1/\sqrt{2}), (\pi/3, 1/2), (\pi/2, 0)$, $(2\pi/3, -1/2)$, $(3\pi/4, -1/\sqrt{2})$, $(5\pi/6, -\sqrt{3}/2)$, $(\pi, -1)$, $(7\pi/6, -\sqrt{3}/2)$, $(5\pi/4, -1/\sqrt{2})$, $(4\pi/3, -1/2), (3\pi/2, 0), (5\pi/3, 1/2), (7\pi/4, 1/\sqrt{2}), (11\pi/6, \sqrt{3}/2)$, and $(2\pi, 1)$ in the xy-plane. Now join these points by a free hand curve to obtain the graph of the function $f(x) = \cos x$ i.e. the curve $y = \cos x$ as shown in Fig. 6.6.

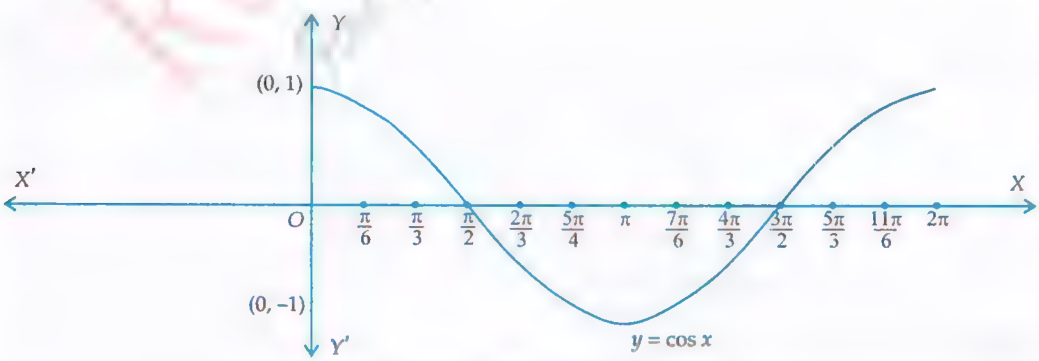


Fig. 6.6 Graph of $f(x) = \cos x, 0 \le x \le 2\pi$

The cosine function i.e. $f(x) = \cos x$ is an even function and the graph of an even function is symmetric about y-axis. So the graph of $f(x) = \cos x$ in $[-2\pi, 2\pi]$ is as shown in Fig. 6.7.

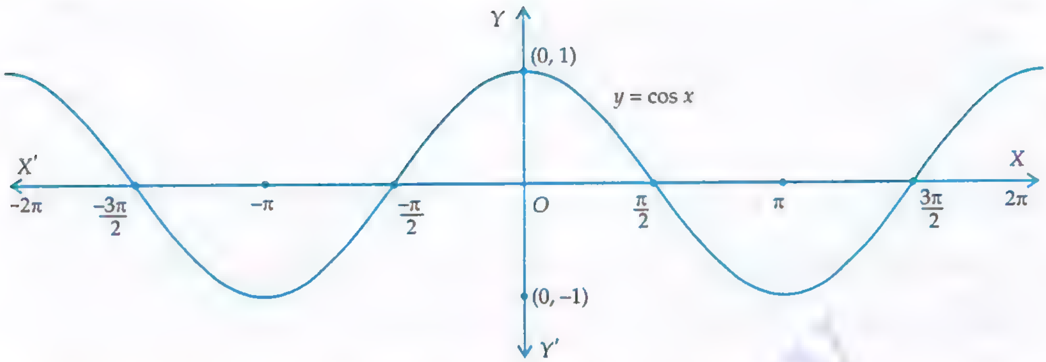


Fig. 6.7 Graph of $f(x) = \cos x, -2\pi \le x \le 2\pi$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Draw the graph of* $f(x) = 3\cos 2x$.

SOLUTION We know that $\cos x$ is a periodic function with period 2π. Therefore, $f(x) = 3\cos 2x$ is periodic with period π. So, it is sufficient to draw the graph of $f(x) = 3\cos 2x$ in the interval $[0, \pi]$. The values of $3\cos 2x$ for different values of x in $[0, \pi]$ are listed below.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{5\pi}{8}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{8}$	$\frac{11\pi}{12}$	π
$3\cos 2x$	3	$\frac{3\sqrt{3}}{2}$	$\frac{3}{\sqrt{2}}$	$\frac{3}{2}$	0	$-\frac{3}{2}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3\sqrt{3}}{2}$	-3	$\frac{-3\sqrt{3}}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$	$-\frac{3}{2}$	0	$\frac{3}{2}$	$\frac{3}{\sqrt{2}}$	$\frac{3\sqrt{3}}{2}$	3

Now, plot the points whose x-coordinates are points in the first row of the above table and the corresponding values in the second row as y-coordinates, By joining these points by a free hand curve, we obtain the graph of $f(x) = 3\cos 2x$ as shown in Fig. 6.8.

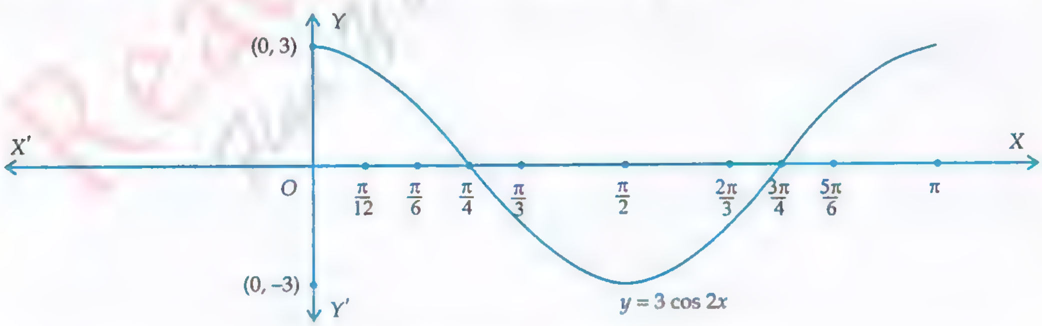


Fig. 6.8 Graph of $f(x) = 3\cos 2x, 0 \le x \le \pi$

EXAMPLE 2 *Sketch the graph of* $f(x) = \cos^2 x$.

SOLUTION We have, $f(x) = \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

We know that $\cos x$ is a periodic function with period 2π. Therefore, $f(x) = \frac{1}{2} + \frac{1}{2}\cos 2x$ is also periodic function with period $\frac{2\pi}{2} = \pi$. So, it is sufficient to draw the graph of $f(x) = \cos^2 x$ in the

interval $[0, \pi]$. In other intervals like $[\pi, 2\pi]$, $[-\pi, 0]$, $[2\pi, 3\pi]$ etc. the graph of $f(x)$ is identical to its in graph in $[0, \pi]$. The values of y (or $f(x)$) at some standard points are listed in the following table.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$y = \cos^2 x$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1

By plotting the points having x-coordinates as points in the first now and y coordinates as the corresponding values in the second row and joining them by a free hand curve, we obtain the graph of the function $f(x) = \cos^2 x$ or the curve $y = \cos^2 x$ as shown in Fig. 6.9.

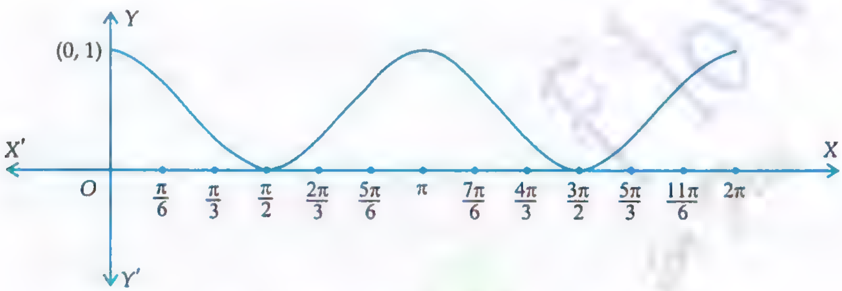


Fig. 6.9 Graph of $f(x) = \cos^2 x, 0 \le x \le 2\pi$

EXAMPLE 3 *Draw the graphs of* $f(x) = \cos 2x$ *and* $g(x) = \cos\left(2x - \frac{\pi}{4}\right)$ *on the same axes and the same scale.*

SOLUTION We know that $\cos x$ is a periodic function with period 2π. Therefore, $f(x) = \cos 2x$ and $g(x) = \cos\left(2x - \frac{\pi}{4}\right)$ are periodic functions with period π. So, to know about their graphs, it is sufficient to draw their graphs in an interval of length π. Let us choose the interval $[0, \pi]$. The values of $f(x) = \cos 2x$ and $g(x) = \cos\left(2x - \frac{\pi}{4}\right)$ at various points in $[0, \pi]$ are listed below.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{5\pi}{8}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{8}$	$\frac{11\pi}{12}$	π
$f(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$\frac{-\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$g(x)$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	1	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	0	$\frac{1-\sqrt{3}}{2\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{-1-\sqrt{3}}{2\sqrt{2}}$	-1	$-\frac{\sqrt{3}+1}{2\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1-\sqrt{3}}{2\sqrt{2}}$	0	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

By plotting the points having their x-coordinates as points in the first row of the above table and y-coordinates as the corresponding values of $f(x)$ in the second row, we obtain the graph of $f(x)$ i.e. the curve $y = \cos 2x$.

Similarly, we draw the graph of $g(x) = \cos\left(2x - \frac{\pi}{4}\right)$ as shown in Fig. 6.10.

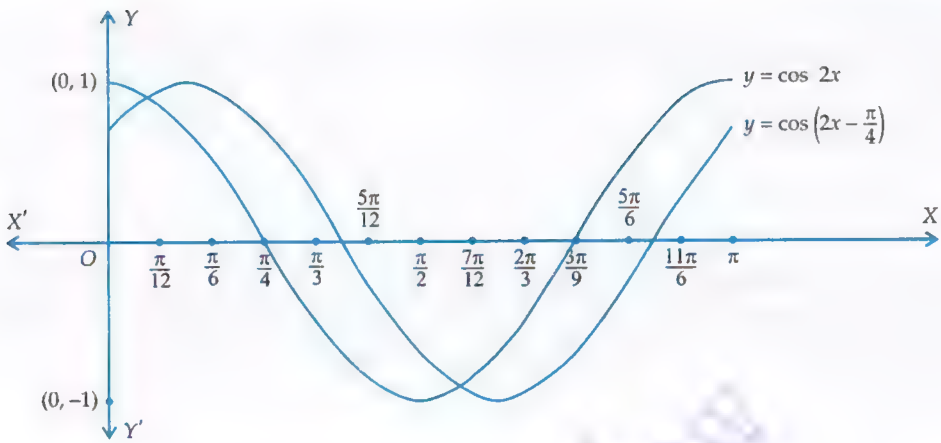


Fig. 6.10 Graphs of $f(x) = \cos 2x$ and $g(x) = \cos\left(2x - \frac{\pi}{4}\right)$

EXERCISE 6.2

BASIC

1. Sketch the graphs of the following trigonometric functions:

 (i) $f(x) = \cos\left(x - \frac{\pi}{4}\right)$ (ii) $g(x) = \cos\left(x + \frac{\pi}{4}\right)$ (iii) $h(x) = \cos^2 2x$

 (iv) $\phi(x) = 2\cos\left(x - \frac{\pi}{6}\right)$ (v) $\psi(x) = \cos 3x$ (vi) $u(x) = \cos^2 \frac{x}{2}$

 (vii) $f(x) = \cos \pi x$ (viii) $g(x) = \cos 2\pi x$

2. Sketch the graphs of the following curves on the same scale and the same axes:

 (i) $y = \cos x$ and $y = \cos\left(x - \frac{\pi}{4}\right)$ (ii) $y = \cos 2x$ and $y = \cos 2\left(x - \frac{\pi}{4}\right)$

 (iii) $y = \cos x$ and $y = \cos \frac{x}{2}$ (iv) $y = \cos^2 x$ and $y = \cos x$

6.4 GRAPH OF TANGENT FUNCTION

We have learnt in the earlier chapters that the tangent function i.e. $f(x) = \tan x$ is a periodic function with period π. So, it is sufficient to know the graph of $f(x) = \tan x$ over an interval of length π, in particular the interval $(-\pi/2, \pi/2)$. The values of $f(x) = \tan x$ at some standard values of x are listed in the following table. As the tangent function is an odd function so the values of $f(x) = \tan x$ at standard points in $(-\pi/2, 0)$ are negative of the corresponding values in $(0, \pi/2)$ and are also listed in the following table.

x	$-\frac{\pi}{2}$	$-\frac{5\pi}{12}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$f(x) = \tan x$	$-\infty$	$-\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	$-\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$	0	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	∞

We also observe that $\tan x$ is an increasing function in $(0, \pi/2)$ and as $x \to \frac{\pi}{2}$ from the left the values of $f(x) = \tan x$ tend to infinity. So, the curve $y = \tan x$ gets closer and closer to the line

$x = \frac{\pi}{2}$ as $x \to \frac{\pi}{2}$ from the left but it never touches the line $x = \frac{\pi}{2}$. The graph of $f(x) = \tan x$ is symmetric in opposite quadrants as the function is an add function. By plotting the points $(-\pi/3, -\sqrt{3})$, $(-\pi/4, -1)$, $(-\pi/6, -1/\sqrt{3})$, $(0, 0)$, $(\pi/6, 1/\sqrt{3})$, $(\pi/4, 1)$, $(\pi/3, \sqrt{3})$ and joining them by a free hand curve, we obtain the sketch of the curve $y = \tan x$ as shown in Fig. 6.11.

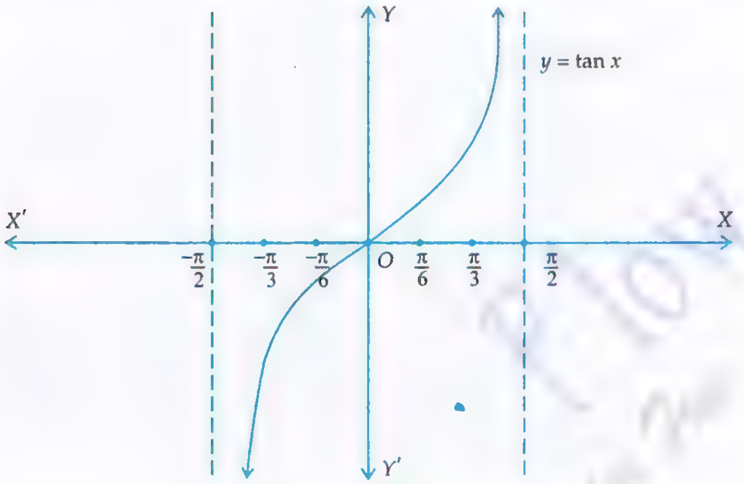


Fig. 6.11 Graph of $f(x) = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

The function $f(x) = \tan x$ is a periodic function with period π. So, the graph of $f(x) = \tan x$ on $(\pi/2, 3\pi/2)$ and $(-3\pi/2, -\pi/2)$ is same as its graph on $(-\pi/2, \pi/2)$ as shown in Fig. 6.12.

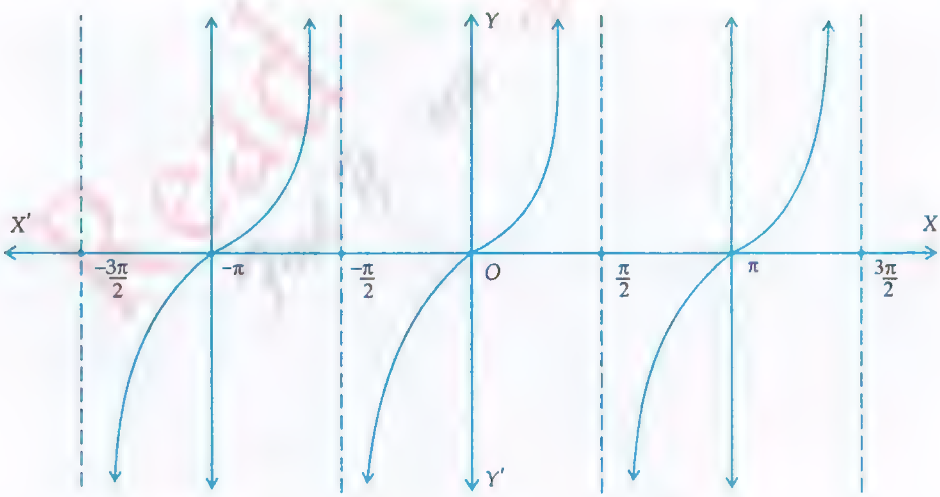


Fig. 6.12 Graph of $f(x) = \tan x, -\frac{3\pi}{2} < x < \frac{3\pi}{2}$

6.5 GRAPH OF COSECANT FUNCTION

In chapter 5, we have learnt that the cosecant function is the reciprocal of the sine function which is periodic with period 2π. So, $f(x) = \text{cosec } x$ is periodic with period 2π. Also, $f(x)$ is defined for

all $x \in R - \{n\pi : n \in Z\}$. In order to know about the graph of $f(x) = \text{cosec } x$, it is sufficient to draw it on an interval of length 2π. Let us choose $[0, 2\pi]$ as interval. The values of $f(x) = \text{cosec } x$ at some standard points in $[0, 2\pi]$ are listed in the following table. We observe that when x is close to zero or π in $(0, \pi)$ the values of $f(x)$ tend to infinity. When $x \to \pi$ or $x \to 2\pi$ in $(\pi, 2\pi)$ the values of $f(x) \to -\infty$.

x	0^+	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π^-	π^+	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$f(x) = \text{cosec } x$	$\to \infty$	2	$\sqrt{2}$ $= 1.41$	$\frac{2}{\sqrt{3}}$ $= 1.5$	1	$\frac{2}{\sqrt{3}}$ $= 1.15$	$\sqrt{2}$ $= 1.41$	2	$\to \infty$	$\to -\infty$	-2	$-\sqrt{2}$ $= -1.41$	-1.15	-1

By plotting points $(\pi/6, 2)$, $(\pi/4, \sqrt{2})$, $(\pi/3, 2/\sqrt{3})$, $(\pi/2, 1)$ $(2\pi/3, 2/\sqrt{3})$, $(3\pi/4, \sqrt{2})$, $(5\pi/6, 2)$, $(7\pi/6, -2)$, $(5\pi/4, -\sqrt{2})$, $(4\pi/3, -2/\sqrt{3})$, $(3\pi/2, -1)$, $(5\pi/3, -2/\sqrt{3})$, $(7\pi/4, -\sqrt{2})$, $(11\pi/6, -2)$ and following these observations, we obtain the graph of the function $f(x) = \text{cosec } x$ i.e. the curve $y = \text{cosec } x$ as shown in Fig. 6.13.

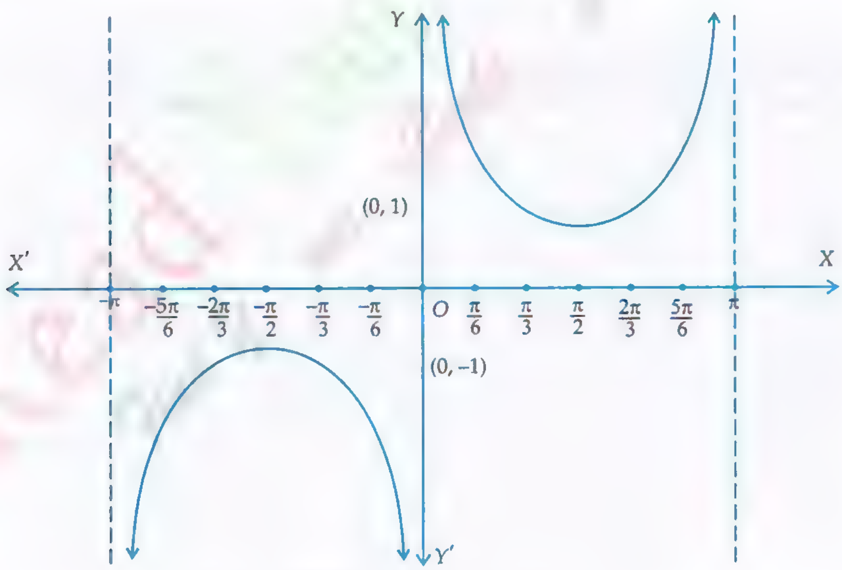


Fig. 6.13 Graph of $f(x) = \text{cosec } x, -\pi < x < \pi, x \neq 0$

The function $f(x) = \text{cosec } x$ is a periodic function with period 2π. So, the graph of $f(x) = \text{cosec } x$ in the interval $[-2\pi, 2\pi]$ is as shown in Fig. 6.14.

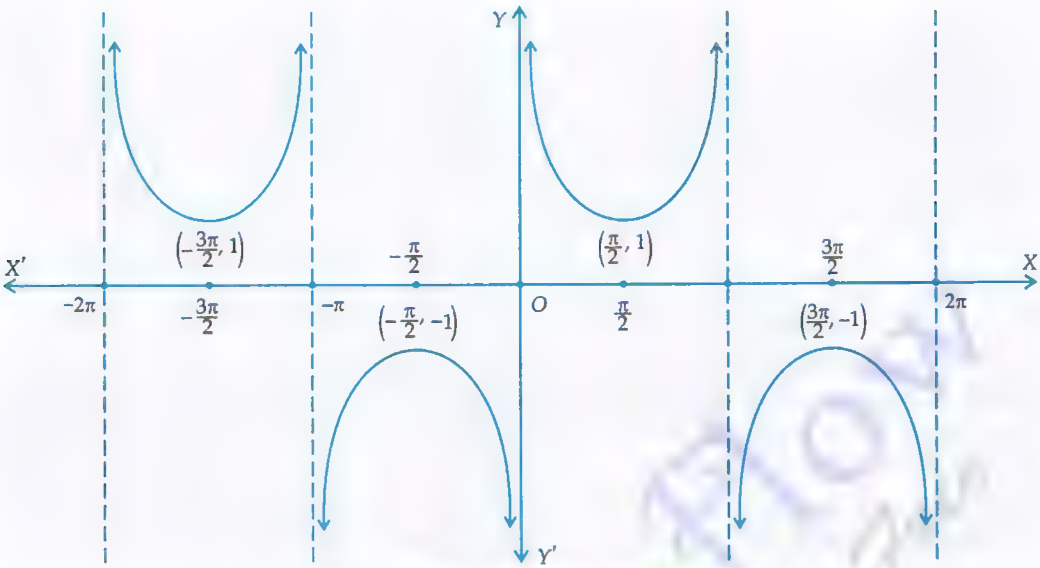


Fig. 6.14 Graph of $y = \operatorname{cosec} x, -2\pi < x < 2\pi$

6.6 GRAPH OF COTANGENT FUNCTIONS

In chapter 5, we have learnt that the cotangent function i.e. $f(x) = \cot x$ is a periodic function with period π. So, it is sufficient to know the graph of $f(x) = \cot x$ over an interval of length π, in particular the interval $(0, \pi)$. The values of $f(x) = \cot x$ at some standard values of x in $(0, \pi)$ are listed in the following table. We also observe that $\cot x$ is decreasing function in $(0, \pi)$ and as $x \to 0^+$, the values of $\cot x \to +\infty$. So, the curve $y = \cot x$ gets closer and closer to the line $x = 0$ i.e. y-axis as $x \to 0^+$. We also observe that $\cot x \to -\infty$ as $x \to \pi^-$ which means that the curve $y = \cot x$ gets closer and closer to the line $x = \pi$ as $x \to \pi$ from left hand side.

By plotting the values of $f(x) = \cot x$ at various points in $(0, \pi)$ and keeping in mind the above observations, we obtain the graph of $f(x) = \cot x$ in $(0, \pi)$ as shown in Fig. 6.15. As $f(x) = \cot x$ is periodic function with period π so the graphs of $f(x) = \cot x$ in $(-2\pi, -\pi)$, $(-\pi, 0)$ and $(\pi, 2\pi)$ are similar to the curve $y = \cot x$ in $(0, \pi)$ as shown in Fig. 6.15.

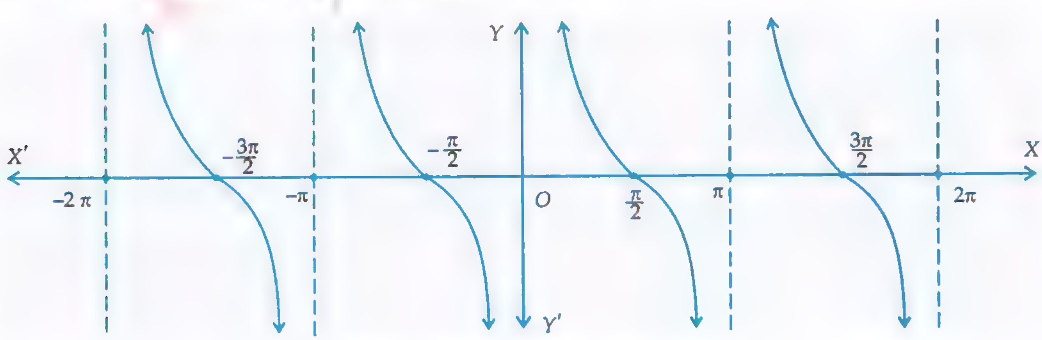


Fig. 6.15 Graph of $y = \cot x, -2\pi < x < 2\pi$

x	0^+	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π^-	π^+	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	$2\pi^-$
$y = \cot x$	$\to\infty$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	$-\infty$	$+\infty$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	$-\infty$

6.7 GRAPH OF SECANT FUNCTION

Similar to the other trigonometric functions the secant function is also a periodic function with period π. In order to known that graph of the secant function i.e. $f(x) = \sec x$, it is sufficient to draw it in an interval of length π, in particular the interval $(-\pi/2, \pi/2)$. We observe that the values of $f(x)$ tend to infinity as $x \to -\pi/2$ from right hand side. So, the graph of $f(x) = \sec x$ comes closer and closer to $x = -\pi/2$ and $x = \pi/2$ but it never touches them. The values of $f(x) = \sec x$ at some standard points in $(-\pi/2, \pi/2)$ are listed in the following table.

x	$-\frac{\pi^+}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi^-}{2}$	$\frac{\pi^+}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$f(x) = \sec x$	$\to\infty$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	$-\infty$	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1

By plotting the points given by the above table and joining them by a free hand curve. We obtain the graph of $f(x) = \sec x$ i.e. the curve $y = \sec x$ as shown in Fig. 6.16.

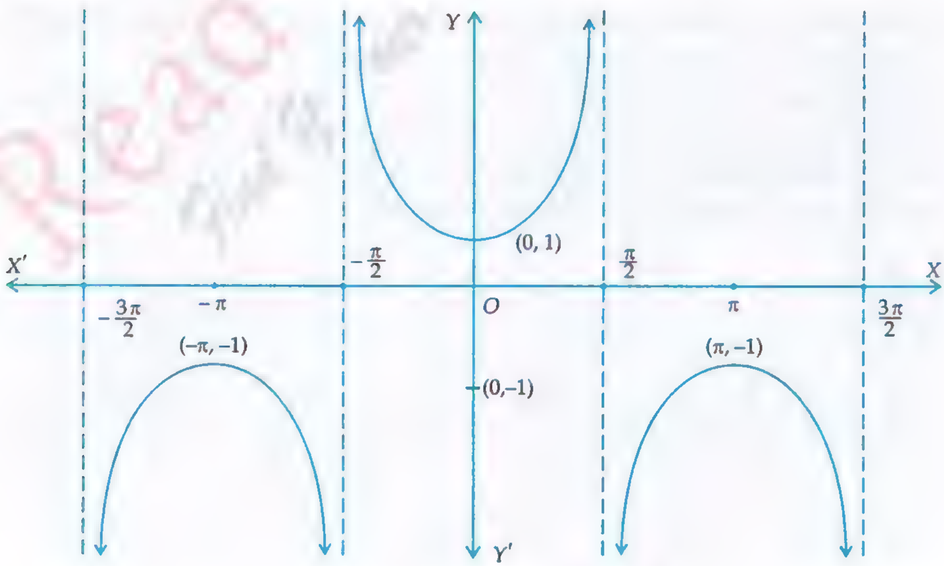


Fig. 6.16 Graph of $y = \sec x, -\frac{3\pi}{2} < x < \frac{3\pi}{2}$

EXERCISE 6.2

BASIC

Sketch the graphs of the following functions:

1. $f(x) = 2 \operatorname{cosec} \pi x$
2. $f(x) = 3 \sec x$
3. $f(x) = \cot 2x$
4. $f(x) = 2 \sec \pi x$
5. $f(x) = \tan^2 x$
6. $f(x) = \cot^2 x$
7. $f(x) = \cot \frac{\pi x}{2}$
8. $f(x) = \sec^2 x$
9. $f(x) = \operatorname{cosec}^2 x$
10. $f(x) = \tan 2x$

CHAPTER 7

VALUES OF TRIGONOMETRIC FUNCTIONS AT SUM OR DIFFERENCE OF ANGLES

7.1 INTRODUCTION

In this chapter, we shall derive formulae which will express the values of trigonometric functions at the sum or difference of two real numbers (or angles) in terms of the values of trigonometric functions at individual numbers (or anlages).

7.2 VALUES OF TRIGONOMETRIC FUNCTIONS AT THE SUM OR DIFFERENCE

7.2.1 COSINE OF THE DIFFERENCE AND SUM OF TWO NUMBERS

THEOREM *For all values of A and B, prove that*

(i) $\cos(A-B) = \cos A \cos B + \sin A \sin B$ (ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

PROOF (i) Let $X'OX$ and YOY' be the coordinate axes. Consider a unit circle with O as the centre.

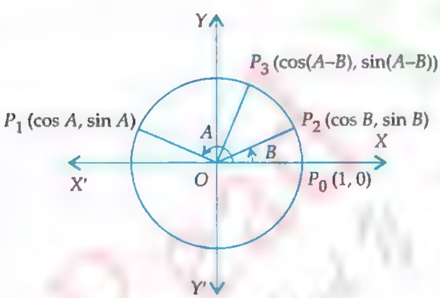


Fig. 7.1

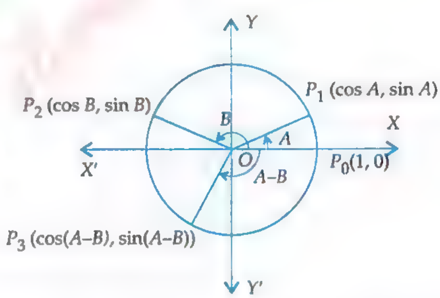


Fig. 7.2

Let P_1, P_2 and P_3 be three points on the circles such that $\angle XOP_1 = A$, $\angle XOP_2 = B$ and $\angle XOP_3 = A - B$ (see Fig. 7.1, 7.2). As we have seen in Section 5.2 that the terminal side of any angle intersects the circle with centre at O and unit radius at a point whose coordinates are respectively the cosine and sine of the angle. Therefore, coordinates of P_1, P_2 and P_3 are $(\cos A, \sin A)$, $(\cos B, \sin B)$ and $(\cos(A-B), \sin(A-B))$ respectively.

We know that equal chords of a circle make equal angles at its centre and chords $P_0 P_3$ and $P_1 P_2$ subtend equal angles at O. Therefore,

$\therefore$ Chord $P_0 P_3$ = Chord $P_1 P_2$

$$\Rightarrow \quad \sqrt{\{\cos(A-B)-1\}^2 + \{\sin(A-B)-0\}^2} = \sqrt{(\cos B - \cos A)^2 + (\sin B - \sin A)^2}$$

$$\Rightarrow \quad \{\cos(A-B)-1\}^2 + \sin^2(A-B) = (\cos B - \cos A)^2 + (\sin B - \sin A)^2$$

$\Rightarrow \quad \cos^2(A-B) - 2\cos(A-B) + 1 + \sin^2(A-B) = \cos^2 B + \cos^2 A - 2\cos A \cos B + \sin^2 B + \sin^2 A - 2\sin A \sin B$

$\Rightarrow \quad 2 - 2\cos(A-B) = 2 - 2\cos A \cos B - 2\sin A \sin B$

$\Rightarrow \quad \cos(A-B) = \cos A \cos B + \sin A \sin B$

Hence, $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(ii) Clearly, $\cos(A+B) = \cos(A-(-B))$

$\Rightarrow \quad \cos(A+B) = \cos A \cos(-B) + \sin A \sin(-B)$ [Using (i)]

$\Rightarrow \quad \cos(A+B) = \cos A \cos B + \sin A \sin B$ [$\because \cos(-B) = \cos B, \sin(-B) = -\sin B$]

Hence, $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Q.E.D.

REMARK *This method of proof of the above formulae is true for all values of A and B whether positive, zero or negative.*

7.2.2 SINE OF THE DIFFERENCE AND SUM OF TWO NUMBERS

THEOREM *For all values of A and B, prove that*

(i) $\sin(A-B) = \sin A \cos B - \cos A \sin B$ (ii) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

PROOF (i) $\sin(A-B) = \cos\left\{\frac{\pi}{2} - (A-B)\right\}$ $\left[\because \cos\left(\frac{\pi}{2} - x\right) = \sin x\right]$

$\Rightarrow \quad \sin(A-B) = \cos\left\{\left(\frac{\pi}{2} - A\right) + B\right\}$

$\Rightarrow \quad \sin(A-B) = \cos\left(\frac{\pi}{2} - A\right)\cos B - \sin\left(\frac{\pi}{2} - A\right)\sin B$ $\left[\because \cos(A+B) = \cos A \cos B - \sin A \sin B\right]$

$\Rightarrow \quad \sin(A-B) = \sin A \cos B - \cos A \sin B$

(ii) $\sin(A+B) = \sin(A-(-B))$

$\Rightarrow \quad \sin(A+B) = \sin A \cos(-B) - \cos A \sin(-B)$ [Using (i)]

$\Rightarrow \quad \sin(A+B) = \sin A \cos B + \cos A \sin B$ [$\because \sin(-B) = -\sin B$]

Q.E.D.

7.2.3 TANGENT OF THE DIFFERENCE AND SUM OF TWO NUMBERS

THEOREM *For those values of A and B for which both sides are defined, prove that*

(i) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ (ii) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

PROOF (i) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$

$\Rightarrow \quad \tan(A+B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

$\Rightarrow \quad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ [On dividing the numerator and denominator by $\cos A \cos B$]

(ii) $\tan(A-B) = \tan(A+(-B))$

$\Rightarrow \quad \tan(A-B) = \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$ [Using (i)]

$$\Rightarrow \quad \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$$

Q.E.D.

Similarly, it can be proved that

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A} \quad \text{and,} \quad \cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

7.3 MORE USEFUL RESULTS

THEOREM *Prove that:*

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(iii) $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$

(iv) $\cos(A+B+C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C - \sin A\sin B\cos C$

(v) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

PROOF (i) $\sin(A+B)\sin(A-B) = (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$

$$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$$

$$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$$

$$= \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B$$

$$= \sin^2 A - \sin^2 B = (1-\cos^2 A) - (1-\cos^2 B) = \cos^2 B - \cos^2 A$$

(ii) $\cos(A+B)\cos(A-B) = (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$$

$$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$$

$$= \cos^2 A - \sin^2 B$$

$$= (1-\sin^2 A) - (1-\cos^2 B) = \cos^2 B - \sin^2 A$$

(iii) $\sin(A+B+C)$

$= \sin((A+B)+C)$

$= \sin(A+B)\cos C + \cos(A+B)\sin C$

$= (\sin A\cos B + \cos A\sin B)\cos C + (\cos A\cos B - \sin A\sin B)\sin C$

$= \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$

(iv) $\cos(A+B+C)$

$= \cos((A+B)+C)$

$= \cos(A+B)\cos C - \sin(A+B)\sin C$

$= (\cos A\cos B - \sin A\sin B)\cos C - (\sin A\cos B + \cos A\sin B)\sin C$

$= \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

(v) $\tan(A+B+C) = \tan\{(A+B)+C\} = \dfrac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$

$$= \frac{\left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) + \tan C}{1 - \left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) \tan C}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I ON FINDING THE VALUES OF $\sin(A \pm B)$, $\cos(A \pm B)$ AND $\tan(A \pm B)$ WHEN VALUES OF ONE OF THE TRIGONOMETRIC FUNCTIONS AT A AND B ARE GIVEN

EXAMPLE 1 *If* $\sin A = \frac{3}{5}$ *and* $\cos B = \frac{9}{41}$, $0 < A < \frac{\pi}{2}$, $0 < B < \frac{\pi}{2}$, *find the values of the following:*

(i) $\sin(A - B)$ (ii) $\sin(A + B)$ (iii) $\cos(A - B)$ (iv) $\cos(A + B)$

SOLUTION We have, $\sin A = \frac{3}{5}$ and, $\cos B = \frac{9}{41}$, where $0 < A, B < \frac{\pi}{2}$.

$\therefore \quad \cos A = \sqrt{1 - \sin^2 A}$ and, $\sin B = \sqrt{1 - \cos^2 B}$

$\Rightarrow \quad \cos A = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$ and, $\sin B = \sqrt{1 - \frac{81}{1681}} = \frac{40}{41}$

(i) $\sin(A - B) = \sin A \cos B - \cos A \sin B = \frac{3}{5} \times \frac{9}{41} - \frac{4}{5} \times \frac{40}{41} = -\frac{133}{205}$

(ii) $\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{3}{5} \times \frac{9}{41} + \frac{4}{5} \times \frac{40}{41} = \frac{187}{205}$

(iii) $\cos(A - B) = \cos A \cos B + \sin A \sin B = \frac{4}{5} \times \frac{9}{41} + \frac{3}{5} \times \frac{40}{41} = \frac{156}{205}$

(iv) $\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{4}{5} \times \frac{9}{41} - \frac{3}{5} \times \frac{40}{41} = -\frac{84}{205}$

EXAMPLE 2 *If* $\sin A = \frac{3}{5}$, $0 < A < \frac{\pi}{2}$ *and* $\cos B = \frac{-12}{13}$, $\pi < B < \frac{3\pi}{2}$, *find the following:*

(i) $\sin(A - B)$ (ii) $\cos(A + B)$ (iii) $\tan(A - B)$

SOLUTION We have, $\sin A = \frac{3}{5}$, where $0 < A < \frac{\pi}{2}$.

$\therefore \quad \cos A = +\sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

In the I quadrant tangent function is positive. Therefore, $\tan A = \frac{\sin A}{\cos A} = \frac{3}{4}$.

It is given that: $\cos B = -\frac{12}{13}$ and $\pi < B < \frac{3\pi}{2}$.

$\therefore \quad \sin B = \pm\sqrt{1 - \cos^2 B}$

$\Rightarrow \quad \sin B = -\sqrt{1 - \cos^2 B} = -\sqrt{1 - \left(\frac{-12}{13}\right)^2} = -\frac{5}{13}$ [$\because$ Sine is negative in the III quadrant]

In the III quadrant tangent function is positive. Therefore, $\tan B = \frac{\sin B}{\cos B} = \frac{5}{12}$.

(i) $\sin(A-B) = \sin A \cos B - \cos A \sin B = \frac{3}{5} \times \frac{-12}{13} - \frac{4}{5} \times \frac{-5}{13} = \frac{-16}{65}$

(ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B = \frac{4}{5} \times \frac{-12}{13} - \frac{3}{5} \times \frac{-5}{13} = \frac{-33}{65}$

(iii) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{3}{4} - \frac{5}{12}}{1 + \frac{3}{4} \times \frac{5}{12}} = \frac{16}{63}$

EXAMPLE 3 *If* $\cos A = \frac{4}{5}$, $\cos B = \frac{12}{13}$, $\frac{3\pi}{2} < A,\ B < 2\pi$, *find the values of the following:*

(i) $\cos(A+B)$ (ii) $\sin(A-B)$

SOLUTION We have, $\cos A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, where $\frac{3\pi}{2} < A, B < 2\pi$. It is given that A and B both lie in the IV quadrant in which $\sin A$ and $\sin B$ are negative. Therefore,

$$\sin A = -\sqrt{1 - \cos^2 A} = -\sqrt{1 - \frac{16}{25}} = -\frac{3}{5} \text{ and,} \sin B = -\sqrt{1 - \cos^2 B} = -\sqrt{1 - \frac{144}{169}} = -\frac{5}{13}$$

(i) $\cos(A+B) = \cos A \cos B - \sin A \sin B = \frac{4}{5} \times \frac{12}{13} - \left(\frac{-3}{5}\right)\left(\frac{-5}{13}\right) = \frac{33}{65}$

(ii) $\sin(A-B) = \sin A \cos B - \cos A \sin B = \frac{-3}{5} \times \frac{12}{13} - \frac{4}{5} \times \frac{-5}{13} = \frac{-16}{65}$

EXAMPLE 4 *If* $\cot \alpha = \frac{1}{2}$, $\sec \beta = -\frac{5}{3}$, *where* $\pi < \alpha < \frac{3\pi}{2}$ *and* $\frac{\pi}{2} < \beta < \pi$. *Find the value of* $\tan(\alpha + \beta)$. *State the quadrant in which* $\alpha + \beta$ *terminates.*

SOLUTION We have, $\cot \alpha = \frac{1}{2} \Rightarrow \tan \alpha = 2$. Since β lies in the second quadrant. Therefore, $\tan \beta$ is negative. Consequently,

$$1 + \tan^2 \beta = \sec^2 \beta \Rightarrow \tan \beta = -\sqrt{\sec^2 \beta - 1} = -\sqrt{\frac{25}{9} - 1} = -\frac{4}{3}$$

$$\therefore \quad \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{2 - \frac{4}{3}}{1 - 2 \times \frac{-4}{3}} = \frac{2}{11}$$

Now, $\pi < \alpha < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi \Rightarrow \frac{3\pi}{2} < \alpha + \beta < \frac{5\pi}{2}$

We know that tangent function is positive in I and III quadrants.

$\therefore \quad \frac{3\pi}{2} < \alpha + \beta < \frac{5\pi}{2}$ and $\tan(\alpha + \beta) = \frac{2}{11} > 0 \Rightarrow \alpha + \beta$ lies in I quadrant.

Type II ON FINDING THE VALUES TRIGONOMETRIC FUNCTIONS AT MULTIPLES OF $\frac{\pi}{12}$

EXAMPLE 5 *Find the values of the following:*

(i) $\sin \frac{5\pi}{12}$ (ii) $\cos \frac{5\pi}{12}$ (iii) $\sin \frac{\pi}{12}$ (iv) $\cos \frac{\pi}{12}$

SOLUTION (i) $\sin \frac{5\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \sin \frac{\pi}{4} \cos \frac{\pi}{6} + \cos \frac{\pi}{4} \sin \frac{\pi}{6} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

(ii) $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(iii) $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(iv) $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

EXAMPLE 6 *Find the values of the following:*

(i) $\tan\frac{\pi}{12}$ [NCERT] (ii) $\tan\frac{5\pi}{12}$ (iii) $\tan\frac{7\pi}{12}$ (iv) $\tan\frac{13\pi}{12}$ [NCERT]

SOLUTION (i) $\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

(ii) $\tan\frac{5\pi}{12} = \tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{6}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

(iii) $\tan\frac{7\pi}{12} = \tan\left(\frac{\pi}{2}+\frac{\pi}{12}\right) = -\cot\frac{\pi}{12} = -\frac{1}{\tan\frac{\pi}{12}} = -\frac{\sqrt{3}+1}{\sqrt{3}-1}$ [Using (i)]

(iv) $\tan\frac{13\pi}{12} = \tan\left(\pi+\frac{\pi}{12}\right) = \tan\frac{\pi}{12} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$ [Using (i)]

EXAMPLE 7 *Prove that:* $\tan\frac{5\pi}{12}+\cot\frac{5\pi}{12}=4.$

SOLUTION From Example 6, we obtain: $\tan\frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \Rightarrow \cot\frac{5\pi}{12} = \frac{1}{\tan\frac{5\pi}{12}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

Substituting the values of $\tan\frac{5\pi}{12}$ and $\cot\frac{5\pi}{12}$, we obtain

$$\text{LHS} = \tan\frac{5\pi}{12}+\cot\frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1}+\frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2+(\sqrt{3}-1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{(4+2\sqrt{3})+(4-2\sqrt{3})}{3-1} = \frac{8}{2} = 4 = \text{RHS}$$

Type III **ON THE APPLICATIONS OF THE FOLLOWING FORMULAE:**

(i) $\sin A\cos B \pm \cos A\sin B = \sin(A\pm B)$ (ii) $\cos A\cos B \pm \sin A\sin B = \cos(A\mp B)$

EXAMPLE 8 *Evaluate the following:*

(i) $\sin\frac{7\pi}{12}\cos\frac{\pi}{4} - \cos\frac{7\pi}{12}\sin\frac{\pi}{4}$ (ii) $\sin\frac{\pi}{4}\cos\frac{\pi}{12} + \cos\frac{\pi}{4}\sin\frac{\pi}{12}$

(iii) $\cos\frac{2\pi}{3}\cos\frac{\pi}{4} - \sin\frac{2\pi}{3}\sin\frac{\pi}{4}$

SOLUTION (i) $\sin\frac{7\pi}{12}\cos\frac{\pi}{4} - \cos\frac{7\pi}{12}\sin\frac{\pi}{4} = \sin\left(\frac{7\pi}{12}-\frac{\pi}{4}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(ii) $\sin\frac{\pi}{4}\cos\frac{\pi}{12}+\cos\frac{\pi}{4}\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{4}+\frac{\pi}{12}\right)=\sin\frac{4\pi}{12}=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

(iii) $\cos\frac{2\pi}{3}\cos\frac{\pi}{4}-\sin\frac{2\pi}{3}\sin\frac{\pi}{4}=\cos\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)=\cos\frac{11\pi}{12}=\cos\frac{11\pi}{12}$

$$=\cos\left(\pi-\frac{\pi}{12}\right)=-\cos\frac{\pi}{12}=-\frac{\sqrt{3}+1}{2\sqrt{2}}$$ [See Ex. 5 (iv)]

EXAMPLE 9 *Prove that:* $\cos\left(\frac{\pi}{4}-A\right)\cos\left(\frac{\pi}{4}-B\right)-\sin\left(\frac{\pi}{4}-A\right)\sin\left(\frac{\pi}{4}-B\right)=\sin(A+B)$ [NCERT]

SOLUTION LHS $=\cos\left(\frac{\pi}{4}-A\right)\cos\left(\frac{\pi}{4}-B\right)-\sin\left(\frac{\pi}{4}-A\right)\sin\left(\frac{\pi}{4}-B\right)$

$$=\cos\left\{\left(\frac{\pi}{4}-A\right)+\left(\frac{\pi}{4}-B\right)\right\}=\cos\left\{\frac{\pi}{2}-(A+B)\right\}=\sin(A+B)=\text{RHS}$$

EXAMPLE 10 *Prove that:* $\sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A=\cos A$ [NCERT]

SOLUTION LHS $=\sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A$

$$=\cos(n+2)A\cos(n+1)A+\sin(n+2)A\sin(n+1)A$$

$$=\cos\{(n+2)A-(n+1)A\}=\cos A=\text{RHS}$$

Type IV ON APPLICATIONS OF THE FORMULAE:

(i) $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$ (ii) $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

(iii) $\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

EXAMPLE 11 *Prove that:*

(i) $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$ [NCERT]

(ii) $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)=-\sqrt{2}\sin x$ [NCERT]

SOLUTION (i) We have, $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$

$$=\left(\cos\frac{\pi}{4}\cos x-\sin\frac{\pi}{4}\sin x\right)+\left(\cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x\right)$$

$$=2\cos\frac{\pi}{4}\cos x=2\times\frac{1}{\sqrt{2}}\times\cos x=\sqrt{2}\cos x$$

(ii) We have, $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)$

$$=\left(\cos\frac{3\pi}{4}\cos x-\sin\frac{3\pi}{4}\sin x\right)-\left(\cos\frac{3\pi}{4}\cos x+\sin\frac{3\pi}{4}\sin x\right)$$

$$=\left(-\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)-\left(-\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)$$

$$=-\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\sin x=-\frac{2}{\sqrt{2}}\sin x=-\sqrt{2}\sin x$$

EXAMPLE 12 *Prove that:* $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$ [NCERT]

SOLUTION $\text{LHS} = \dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\sin x\cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}$

$$= \dfrac{\dfrac{\sin x\cos y + \cos x\sin y}{\cos x\cos y}}{\dfrac{\sin x\cos y - \cos x\sin y}{\cos x\cos y}} \qquad \left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and denominator by } \cos x\cos y\end{array}\right]$$

$$= \dfrac{\tan x + \tan y}{\tan x - \tan y} = \text{RHS}$$

EXAMPLE 13 *Prove that:* $\dfrac{\sin(B-C)}{\cos B\cos C} + \dfrac{\sin(C-A)}{\cos C\cos A} + \dfrac{\sin(A-B)}{\cos A\cos B} = 0$

SOLUTION We have,

$$\text{LHS} = \dfrac{\sin(B-C)}{\cos B\cos C} + \dfrac{\sin(C-A)}{\cos C\cos A} + \dfrac{\sin(A-B)}{\cos A\cos B}$$

$$= \dfrac{\sin B\cos C - \cos B\sin C}{\cos B\cos C} + \dfrac{\sin C\cos A - \cos C\sin A}{\cos C\cos A} + \dfrac{\sin A\cos B - \cos A\sin B}{\cos A\cos B}$$

$$= \dfrac{\sin B\cos C}{\cos B\cos C} - \dfrac{\cos B\sin C}{\cos B\cos C} + \dfrac{\sin C\cos A}{\cos C\cos A} - \dfrac{\cos C\sin A}{\cos C\cos A} + \dfrac{\sin A\cos B}{\cos A\cos B} - \dfrac{\cos A\sin B}{\cos A\cos B}$$

$$= \tan B - \tan C + \tan C - \tan A + \tan A - \tan B = 0 = \text{RHS}$$

EXAMPLE 14 *Prove that:* $\dfrac{\tan\left(\dfrac{\pi}{4}+x\right)}{\tan\left(\dfrac{\pi}{4}-x\right)} = \left(\dfrac{1+\tan x}{1-\tan x}\right)^2$ [NCERT]

SOLUTION $\text{LHS} = \dfrac{\tan\left(\dfrac{\pi}{4}+x\right)}{\tan\left(\dfrac{\pi}{4}-x\right)} = \dfrac{\tan\dfrac{\pi}{4}+\tan x}{1-\tan\dfrac{\pi}{4}\tan x} \times \dfrac{1+\tan\dfrac{\pi}{4}\tan x}{\tan\dfrac{\pi}{4}-\tan x}$

$$= \dfrac{1+\tan x}{1-\tan x} \times \dfrac{1+\tan x}{1-\tan x} = \left(\dfrac{1+\tan x}{1-\tan x}\right)^2 = \text{RHS}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 15 *If* $\tan A - \tan B = x$ *and,* $\cot B - \cot A = y$*, prove that* $\cot(A-B) = \dfrac{1}{x} + \dfrac{1}{y}$.

[NCERT EXEMPLAR]

SOLUTION We have, $\tan A - \tan B = x$ and, $\cot B - \cot A = y$

Now,

$$\cot B - \cot A = y$$

$$\Rightarrow \quad \dfrac{1}{\tan B} - \dfrac{1}{\tan A} = y \Rightarrow \dfrac{\tan A - \tan B}{\tan A\tan B} = y \Rightarrow \dfrac{x}{\tan A\tan B} = y \Rightarrow \tan A\tan B = \dfrac{x}{y}$$

$$\therefore \quad \cot(A-B) = \frac{1}{\tan(A-B)} = \frac{1+\tan A\tan B}{\tan A - \tan B} = \frac{1+\frac{x}{y}}{x} = \frac{x+y}{xy} = \frac{1}{x}+\frac{1}{y}$$

ALITER We have, $\tan A - \tan B = x$ and, $\cot B - \cot A = y$

$$\Rightarrow \quad \frac{1}{\cot A} - \frac{1}{\cot B} = x \text{ and, } \cot B - \cot A = y \Rightarrow \frac{\cot B - \cot A}{\cot A \cot B} = x \text{ and, } \cot B - \cot A = y$$

$$\Rightarrow \quad \frac{y}{\cot A \cot B} = x \text{ and } \cot B - \cot A = y \Rightarrow \cot A \cot B = \frac{y}{x} \text{ and } \cot B - \cot A = y$$

$$\therefore \quad \cot(A-B) = \frac{\cot A\cot B + 1}{\cot B - \cot A} = \frac{\frac{y}{x}+1}{y} = \frac{1}{x}+\frac{1}{y}$$

EXAMPLE 16 *If* $\tan\alpha = \dfrac{1}{\sqrt{x(x^2+x+1)}}$, $\tan\beta = \dfrac{\sqrt{x}}{\sqrt{x^2+x+1}}$ *and* $\tan\gamma = \sqrt{x^{-3}+x^{-2}+x^{-1}}$, *prove that* $\alpha+\beta=\gamma$.

SOLUTION We have,

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{\dfrac{1}{\sqrt{x(x^2+x+1)}} + \dfrac{\sqrt{x}}{\sqrt{x^2+x+1}}}{1-\dfrac{1}{\sqrt{x(x^2+x+1)}} \times \dfrac{\sqrt{x}}{\sqrt{x^2+x+1}}} = \frac{(x+1)\sqrt{x(x^2+x+1)}}{x(x^2+x+1)-x}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{(x+1)\sqrt{x(x^2+x+1)}}{x^2(x+1)} = \frac{\sqrt{x(x^2+x+1)}}{x^2} = \sqrt{\frac{x(x^2+x+1)}{x^4}} = \sqrt{x^{-3}+x^{-2}+x^{-1}}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \tan\gamma$$

$$\Rightarrow \quad \alpha+\beta = \gamma.$$

EXAMPLE 17 *If* α *and* β *are acute angles such that* $\tan\alpha = \dfrac{m}{m+1}$ *and* $\tan\beta = \dfrac{1}{2m+1}$, *prove that* $\alpha+\beta = \dfrac{\pi}{4}$.

SOLUTION We have,

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{\dfrac{m}{m+1}+\dfrac{1}{2m+1}}{1-\dfrac{m}{m+1}\times\dfrac{1}{2m+1}} = \frac{2m^2+m+m+1}{2m^2+3m+1-m} = \frac{2m^2+2m+1}{2m^2+2m+1} = 1 = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \alpha+\beta = \frac{\pi}{4}$$

EXAMPLE 18 *If* $A+B = \dfrac{\pi}{4}$, *prove that:*

(i) $(1+\tan A)(1+\tan B) = 2$ (ii) $(\cot A - 1)(\cot B - 1) = 2$

SOLUTION (i) We have, $A+B = \dfrac{\pi}{4}$

$\therefore \quad \tan(A+B) = \tan\frac{\pi}{4}$

$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$

$\Rightarrow \quad \tan A + \tan B = 1 - \tan A \tan B$

$\Rightarrow \quad \tan A + \tan B + \tan A \tan B = 1 \quad \Rightarrow 1 + \tan A + \tan B + \tan A \tan B = 2$

$\Rightarrow \quad (1 + \tan A) + \tan B(1 + \tan A) = 2 \Rightarrow (1 + \tan A)(1 + \tan B) = 2$

(ii) We have, $A + B = \frac{\pi}{4}$

$\therefore \quad \tan(A+B) = \tan\frac{\pi}{4}$

$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$

$\Rightarrow \quad \tan A + \tan B = 1 - \tan A \tan B$

$\Rightarrow \quad \tan A + \tan B + \tan A \tan B = 1$

$\Rightarrow \quad \frac{\tan A + \tan B + \tan A \tan B}{\tan A \tan B} = \frac{1}{\tan A \tan B}$

$\Rightarrow \quad \cot B + \cot A + 1 = \cot A \cot B$

$\Rightarrow \quad \cot A \cot B - \cot A - \cot B = 1 \quad \Rightarrow \cot A \cot B - \cot A - \cot B + 1 = 2$

$\Rightarrow \quad \cot A(\cot B - 1) - (\cot B - 1) = 2 \quad \Rightarrow (\cot A - 1)(\cot B - 1) = 2$

ALITER We have, $A + B = \frac{\pi}{4}$

$\therefore \quad \cot(A+B) = \cot\frac{\pi}{4}$

$\Rightarrow \quad \frac{\cot A \cot B - 1}{\cot A + \cot B} = 1 \Rightarrow \cot A + \cot B = \cot A \cot B - 1$

$\Rightarrow \quad \cot A \cot B - \cot A - \cot B = 1 \quad \Rightarrow \cot A \cot B - \cot A \cot B + 1 = 1 + 1$

$\Rightarrow \quad \cot A(\cot B - 1) - (\cot B - 1) = 2 \Rightarrow (\cot A - 1)(\cot B - 1) = 2$

EXAMPLE 19 *If* $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$, *show that* $\tan(\alpha - \beta) = (1 - n)\tan\alpha$.

SOLUTION We have,

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\frac{\sin\alpha}{\cos\alpha} - \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}}{1 + \frac{\sin\alpha}{\cos\alpha} \times \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}} \quad \left[\because \tan\beta = \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}\right]$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\sin\alpha - n\sin^3\alpha - n\sin\alpha\cos^2\alpha}{\cos\alpha(1 - n\sin^2\alpha) + n\sin^2\alpha\cos\alpha} \quad \text{[On taking LCM]}$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\sin\alpha - n\sin^3\alpha - n\sin\alpha(1 - \sin^2\alpha)}{\cos\alpha - n\sin^2\alpha\cos\alpha + n\sin^2\alpha\cos\alpha}$$

$$\Rightarrow \quad \tan(\alpha-\beta) = \frac{\sin\alpha - n\sin\alpha}{\cos\alpha} = \frac{\sin\alpha}{\cos\alpha}(1-n) = (1-n)\tan\alpha$$

EXAMPLE 20 *Prove that:*

(i) $\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$

(ii) $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$ [NCERT]

SOLUTION (i) Clearly,

$$3x = 2x + x$$

$$\Rightarrow \tan 3x = \tan(2x+x)$$

$$\Rightarrow \tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$$\Rightarrow \tan 3x(1 - \tan 2x \tan x) = \tan 2x + \tan x$$

$$\Rightarrow \tan 3x - \tan 3x \tan 2x \tan x = \tan 2x + \tan x \Rightarrow \tan 3x - \tan 2x - \tan x = \tan 3x \tan 2x \tan x$$

(ii) Dividing both sides by $\tan x \tan 2x \tan 3x$, we get

$$\frac{\tan 3x \tan 2x \tan x}{\tan 3x \tan 2x \tan x} = \frac{\tan 3x - \tan 2x - \tan x}{\tan 3x \tan 2x \tan x}$$

$$\Rightarrow 1 = \frac{1}{\tan 2x \tan x} - \frac{1}{\tan 3x \tan x} - \frac{1}{\tan 3x \tan 2x} \Rightarrow 1 = \cot x \cot 2x - \cot 3x \cot x - \cot 3x \cot 2x$$

Type V ON THE APPLICATIONS OF THE FORMULAE:

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$ (ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

EXAMPLE 21 *Prove that:* $\dfrac{\tan(x+y)}{\cot(x-y)} = \dfrac{\sin^2 x - \sin^2 y}{\cos^2 x - \sin^2 y}$

SOLUTION $\text{LHS} = \dfrac{\tan(x+y)}{\cot(x-y)} = \dfrac{\sin(x+y)}{\cos(x+y)} \dfrac{\sin(x-y)}{\cos(x-y)} = \dfrac{\sin^2 x - \sin^2 y}{\cos^2 x - \sin^2 y} = \text{RHS}$

EXAMPLE 22 *Prove that :* $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$ [NCERT]

SOLUTION We know that $\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$

$$\therefore \quad \sin^2 6x - \sin^2 4x = \sin(6x+4x)\sin(6x-4x) = \sin 10x \sin 2x$$

EXAMPLE 23 *Prove that :* $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$ [NCERT]

SOLUTION $\text{LHS} = \cos^2 2x - \cos^2 6x$

$$= (1 - \sin^2 2x) - (1 - \sin^2 6x) = \sin^2 6x - \sin^2 2x = \sin(6x+2x)\sin(6x-2x)$$

$$= \sin 8x \sin 4x = \text{RHS}$$

EXAMPLE 24 *Prove that* $\dfrac{\cos^2 33° - \cos^2 57°}{\sin^2 \frac{21°}{2} - \sin^2 \frac{69°}{2}} = -\sqrt{2}$

SOLUTION $\text{LHS} = \dfrac{\cos^2 33° - \cos^2 57°}{\sin^2 \frac{21°}{2} - \sin^2 \frac{69°}{2}} = \dfrac{(1 - \sin^2 33°) - (1 - \sin^2 57°)}{\sin^2 \frac{21°}{2} - \sin^2 \frac{69°}{2}}$

$$= \frac{\sin^2 57° - \sin^2 33°}{\sin^2 \frac{21°}{2} - \sin^2 \frac{69°}{2}}$$

$$= \frac{\sin(57° + 33°) \sin(57° - 33°)}{\sin\left(\frac{21°}{2} + \frac{69°}{2}\right) \sin\left(\frac{21°}{2} - \frac{69°}{2}\right)}$$

$$= \frac{\sin 90° \sin 24°}{\sin 45° \sin(-24°)} = \frac{\sin 24°}{-\frac{1}{\sqrt{2}} \sin 24°} = -\sqrt{2} = \text{RHS}$$

EXAMPLE 25 *Prove that:* $\sin^2\left(\frac{\pi}{8} + \frac{x}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{x}{2}\right) = \frac{1}{\sqrt{2}} \sin x$

SOLUTION Using $\sin^2 A - \sin^2 B = \sin(A+B) \sin(A-B)$, we obtain

$$\text{LHS} = \sin^2\left(\frac{\pi}{8} + \frac{x}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{x}{2}\right)$$

$$= \sin\left\{\left(\frac{\pi}{8} + \frac{x}{2}\right) + \left(\frac{\pi}{8} - \frac{x}{2}\right)\right\} \sin\left\{\left(\frac{\pi}{8} + \frac{x}{2}\right) - \left(\frac{\pi}{8} - \frac{x}{2}\right)\right\} = \sin\frac{\pi}{4} \sin x = \frac{1}{\sqrt{2}} \sin x = \text{RHS}$$

EXAMPLE 26 *Prove that:* $\cos 2\alpha \cos 2\beta + \sin^2(\alpha - \beta) - \sin^2(\alpha + \beta) = \cos 2(\alpha + \beta)$.

SOLUTION Using $\sin^2 A - \sin^2 B = \sin(A+B) \sin(A-B)$, we obtain

$$\text{LHS} = \cos 2\alpha \cos 2\beta + \sin^2(\alpha - \beta) - \sin^2(\alpha + \beta)$$

$$= \cos 2\alpha \cos 2\beta + \sin(\alpha - \beta + \alpha + \beta) \sin(\alpha - \beta - \alpha - \beta)$$

$$= \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta$$

$$= \cos(2\alpha + 2\beta) = \text{RHS}$$

EXAMPLE 27 *Prove that:* $\sin^2 A = \cos^2(A - B) + \cos^2 B - 2\cos(A - B) \cos A \cos B$.

SOLUTION $\text{RHS} = \cos^2(A - B) + \cos^2 B - 2\cos(A - B) \cos A \cos B$

$$= \cos^2 B + \cos^2(A - B) - 2\cos(A - B) \cos A \cos B$$

$$= \cos^2 B + \cos(A - B)\{\cos(A - B) - 2\cos A \cos B\}$$

$$= \cos^2 B + \cos(A - B)\{\cos A \cos B + \sin A \sin B - 2\cos A \cos B\}$$

$$= \cos^2 B + \cos(A - B)\{\sin A \sin B - \cos A \cos B\}$$

$$= \cos^2 B - \cos(A - B)(\cos A \cos B - \sin A \sin B)$$

$$= \cos^2 B - \cos(A - B) \cos(A + B)$$

$$= \cos^2 B - (\cos^2 A - \sin^2 B) = \cos^2 B + \sin^2 B - \cos^2 A$$

$$= 1 - \cos^2 A = \sin^2 A = \text{LHS}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 28 *If* $3 \tan A \tan B = 1$, *prove that* $2\cos(A + B) = \cos(A - B)$.

SOLUTION We have,

$$3 \tan A \tan B = 1 \Rightarrow \frac{3 \sin A \sin B}{\cos A \cos B} = 1 \Rightarrow \frac{\cos A \cos B}{\sin A \sin B} = \frac{3}{1}$$

$$\Rightarrow \quad \frac{\cos A \cos B + \sin A \sin B}{\cos A \cos B - \sin A \sin B} = \frac{3+1}{3-1} \qquad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \quad \frac{\cos(A - B)}{\cos(A + B)} = 2 \Rightarrow 2\cos(A + B) = \cos(A - B)$$

EXAMPLE 29 *If* $\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)=-\frac{3}{2}$, *prove that*

$$\cos\alpha+\cos\beta+\cos\gamma = \sin\alpha+\sin\beta+\sin\gamma = 0$$

SOLUTION We have,

$$\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha) = -\frac{3}{2}$$

$\Rightarrow$ $2\cos\alpha\cos\beta+2\sin\alpha\sin\beta+2\cos\beta\cos\gamma+2\sin\beta\sin\gamma+2\cos\gamma\cos\alpha+2\sin\gamma\sin\alpha=-3$

$\Rightarrow$ $(2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha)+(2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha)+3=0$

$\Rightarrow$ $(2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha)+(2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha+(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)+(\cos^2\gamma+\sin^2\gamma)=0$

$\Rightarrow$ $(\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha)+(\sin^2\alpha+\sin^2\beta+\sin^2\gamma+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha)=0$

$\Rightarrow$ $(\cos\alpha+\cos\beta+\cos\gamma)^2+(\sin\alpha+\sin\beta+\sin\gamma)^2 = 0$

$\Rightarrow$ $\cos\alpha+\cos\beta+\cos\gamma = 0$ and $\sin\alpha+\sin\beta+\sin\gamma = 0$

EXAMPLE 30 *If* $\sin B=3\sin(2A+B)$, *prove that* $2\tan A+\tan(A+B)=0$.

SOLUTION We have,

$$\sin B = 3\sin(2A+B)$$

$\Rightarrow$ $\dfrac{\sin(2A+B)}{\sin B} = \dfrac{1}{3}$

$\Rightarrow$ $\dfrac{\sin\{(A+B)+A\}}{\sin\{(A+B)-A\}} = \dfrac{1}{3}$

$\Rightarrow$ $\dfrac{\sin\{A+B)+A\}+\sin\{(A+B)-A\}}{\sin\{(A+B)+A\}-\sin\{(A+B)-A\}} = \dfrac{1+3}{1-3}$ [Using componendo – dividendo]

$\Rightarrow$ $\dfrac{\{\sin(A+B)\cos A+\cos(A+B)\sin A\}+\{\sin(A+B)\cos A-\cos(A+B)\sin A\}}{\{\sin(A+B)\cos A+\cos(A+B)\sin A\}-\{\sin(A+B)\cos A-\cos(A+B)\sin A\}}=\dfrac{1+3}{1-3}$

$\Rightarrow$ $\dfrac{2\sin(A+B)\cos A}{2\cos(A+B)\sin A} = -2$

$\Rightarrow$ $\tan(A+B)\cot A=-2 \Rightarrow \tan(A+B)=-2\tan A \Rightarrow 2\tan A+\tan(A+B)=0$

EXAMPLE 31 *If* $2\tan\beta+\cot\beta=\tan\alpha$, *prove that* $\cot\beta=2\tan(\alpha-\beta)$.

SOLUTION Clearly,

$$2\tan(\alpha-\beta) = 2\left\{\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\right\}$$

$$=2\left\{\frac{2\tan\beta+\cot\beta-\tan\beta}{1+(2\tan\beta+\cot\beta)\tan\beta}\right\}$$ [Using : $\tan\alpha=2\tan\beta+\cot\beta$]

$$= 2\left\{\frac{\tan\beta+\cot\beta}{1+2\tan^2\beta+1}\right\}$$

$$=\frac{2(\tan\beta+\cot\beta)}{2+2\tan^2\beta}=\frac{2\left\{\tan\beta+\dfrac{1}{\tan\beta}\right\}}{2(1+\tan^2\beta)}=\frac{1}{\tan\beta}=\cot\beta$$

Hence, $\cot\beta = 2\tan(\alpha-\beta)$.

EXAMPLE 32 *If* $\cos(\alpha+\beta)\sin(\gamma+\delta)=\cos(\alpha-\beta)\sin(\gamma-\delta)$, *prove that* $\cot\alpha\cot\beta\cot\gamma=\cot\delta$.

SOLUTION We have,

$$\cos(\alpha+\beta)\sin(\gamma+\delta)=\cos(\alpha-\beta)\sin(\gamma-\delta)$$

$$\Rightarrow \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}=\frac{\sin(\gamma+\delta)}{\sin(\gamma-\delta)}$$

$$\Rightarrow \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{\cos(\alpha-\beta)-\cos(\alpha+\beta)}=\frac{\sin(\gamma+\delta)+\sin(\gamma-\delta)}{\sin(\gamma+\delta)-\sin(\gamma-\delta)} \quad \text{[Using componendo – dividendo]}$$

$$\Rightarrow \frac{2\cos\alpha\cos\beta}{2\sin\alpha\sin\beta}=\frac{2\sin\gamma\cos\delta}{2\cos\gamma\sin\delta}\Rightarrow \cot\alpha\cot\beta=\tan\gamma\cot\delta\Rightarrow\cot\alpha\cot\beta\cot\gamma=\cot\delta$$

EXAMPLE 33 *Prove that:* $\dfrac{\sin(x+\theta)}{\sin(x+\phi)}=\cos(\theta-\phi)+\cot(x+\phi)\sin(\theta-\phi)$.

SOLUTION

$$\text{LHS}=\frac{\sin(x+\theta)}{\sin(x+\phi)}=\frac{\sin\{(x+\phi)+(\theta-\phi)\}}{\sin(x+\phi)}$$

$$=\frac{\sin(x+\phi)\cos(\theta-\phi)+\cos(x+\phi)\sin(\theta-\phi)}{\sin(x+\phi)}$$

$$=\cos(\theta-\phi)+\cot(x+\phi)\sin(\theta-\phi)=\text{RHS}$$

EXAMPLE 34 *If* $\cos(\alpha+\beta)=\dfrac{4}{5}$, $\sin(\alpha-\beta)=\dfrac{5}{13}$ *and* α,β *lie between* 0 *and* $\dfrac{\pi}{4}$, *prove that* $\tan 2\alpha=\dfrac{56}{33}$ [NCERT EXEMPLAR]

SOLUTION It is given that α,β lie between 0 and $\pi/4$. Therefore, $-\pi/4<\alpha-\beta<\pi/4$ and $0<\alpha+\beta<\pi/2$. So, $\cos(\alpha-\beta)$ and $\sin(\alpha+\beta)$ are positive.

Now, $\sin(\alpha+\beta)=\sqrt{1-\cos^2(\alpha+\beta)}\Rightarrow\sin(\alpha+\beta)=\sqrt{1-\dfrac{16}{25}}=\dfrac{3}{5}$

and, $\cos(\alpha-\beta)=\sqrt{1-\sin^2(\alpha-\beta)}\Rightarrow\cos(\alpha-\beta)=\sqrt{1-\dfrac{25}{169}}=\dfrac{12}{13}$

$\therefore\ \tan(\alpha+\beta)=\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\dfrac{3/5}{4/5}=\dfrac{3}{4}$ and, $\tan(\alpha-\beta)=\dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}=\dfrac{5}{12}$

Now,

$$\tan 2\alpha=\tan\{(\alpha+\beta)+(\alpha-\beta)\}=\frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}=\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}}=\frac{56}{33}$$

EXAMPLE 35 *Prove that:* $\tan 70^\circ=\tan 20^\circ+2\tan 50^\circ$.

SOLUTION We have,

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}\Rightarrow \tan x-\tan y=\tan(x-y)(1+\tan x\tan y)$$

Replacing x by 70° and y by 20°, we get

$$\tan 70^\circ-\tan 20^\circ=\tan(70^\circ-20^\circ)(1+\tan 70^\circ\tan 20^\circ)$$
$$=\tan 50^\circ\ (1+\tan 70^\circ\cot 70^\circ)=2\tan 50^\circ$$

EXAMPLE 36 *If* $\tan(\alpha+x)=n\tan(\alpha-x)$, *show that:* $(n+1)\sin 2x=(n-1)\sin 2\alpha$.

SOLUTION We have,

$$\tan(\alpha+x)=n\tan(\alpha-x)$$

$$\Rightarrow \quad \frac{\tan(\alpha + x)}{\tan(\alpha - x)} = \frac{n}{1}$$

$$\Rightarrow \quad \frac{\tan(\alpha + x) + \tan(\alpha - x)}{\tan(\alpha + x) - \tan(\alpha - x)} = \frac{n+1}{n-1} \quad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \quad \frac{\sin(\alpha + x)\cos(\alpha - x) + \cos(\alpha + x)\sin(\alpha - x)}{\sin(\alpha + x)\cos(\alpha - x) - \cos(\alpha + x)\sin(\alpha - x)} = \frac{n+1}{n-1}$$

$$\Rightarrow \quad \frac{\sin(\alpha + x) + (\alpha - x)}{\sin(\alpha + x) - (\alpha - x)} = \frac{n+1}{n-1} \Rightarrow \frac{\sin 2\alpha}{\sin 2x} = \frac{n+1}{n-1} \Rightarrow (n+1)\sin 2x = (n-1)\sin 2\alpha$$

EXAMPLE 37 *Prove that:* $\cot x \cot 2x + \cot 2x \cot 3x + 2 = \cot x(\cot x - \cot 3x)$.

SOLUTION LHS $= \cot x \cot 2x + \cot 2x \cot 3x + 2 = (\cot x \cot 2x + 1) + (\cot 2x \cot 3x + 1)$

$$= \left(\frac{\cos x \cos 2x}{\sin x \sin 2x} + 1\right) + \left(\frac{\cos 2x \cos 3x}{\sin 2x \sin 3x} + 1\right)$$

$$= \left(\frac{\cos 2x \cos x + \sin 2x \sin x}{\sin x \sin 2x}\right) + \left(\frac{\cos 3x \cos 2x + \sin 3x \sin 2x}{\sin 2x \sin 3x}\right)$$

$$= \frac{\cos(2x - x)}{\sin x \sin 2x} + \frac{\cos(3x - 2x)}{\sin 2x \sin 3x} = \frac{\cos x}{\sin x \sin 2x} + \frac{\cos x}{\sin 2x \sin 3x}$$

$$= \cos x\left\{\frac{1}{\sin x \sin 2x} + \frac{1}{\sin 2x \sin 3x}\right\} = \frac{\cos x}{\sin x}\left\{\frac{\sin x}{\sin x \sin 2x} + \frac{\sin x}{\sin 2x \sin 3x}\right\}$$

$$= \cot x\left\{\frac{\sin(2x - x)}{\sin x \sin 2x} + \frac{\sin(3x - 2x)}{\sin 2x \sin 3x}\right\}$$

$$= \cot x\left\{\frac{\sin 2x \cos x - \cos 2x \sin x}{\sin x \sin 2x} + \frac{\sin 3x \cos 2x - \cos 3x \sin 2x}{\sin 2x \sin 3x}\right\}$$

$$= \cot x\{\cot x - \cot 2x + \cot 2x - \cot 3x\} = \cot x(\cot x - \cot 3x) = \text{RHS}$$

EXAMPLE 38 *If* $\tan(\pi \cos x) = \cot(\pi \sin x)$, *prove that* $\cos\left(x - \frac{\pi}{4}\right) = \pm \frac{1}{2\sqrt{2}}$.

SOLUTION We have,

$$\tan(\pi \cos x) = \cot(\pi \sin x)$$

$$\Rightarrow \quad \frac{\sin(\pi \cos x)}{\cos(\pi \cos x)} = \frac{\cos(\pi \sin x)}{\sin(\pi \sin x)}$$

$$\Rightarrow \quad \sin(\pi \cos x)\sin(\pi \sin x) = \cos(\pi \sin x)\cos(\pi \cos x)$$

$$\Rightarrow \quad \cos(\pi \cos x)\cos(\pi \sin x) - \sin(\pi \cos x)\sin(\pi \sin x) = 0$$

$$\Rightarrow \quad \cos(\pi \cos x + \pi \sin x) = 0$$

$$\Rightarrow \quad \pi \cos x + \pi \sin x = \pm \frac{\pi}{2} \quad \left[\because \cos\left(\pm \frac{\pi}{2}\right) = 0\right]$$

$$\Rightarrow \quad \cos x + \sin x = \pm \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \pm \frac{1}{2\sqrt{2}} \quad \text{[Multiplying both sides by } 1/\sqrt{2}\text{]}$$

$$\Rightarrow \quad \cos x \cos \frac{\pi}{4} + \sin x \sin \frac{\pi}{4} = \pm \frac{1}{2\sqrt{2}} \Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \pm \frac{1}{2\sqrt{2}}$$

EXAMPLE 39 *If* $a \tan \alpha + b \tan \beta = (a + b)\tan\left(\frac{\alpha + \beta}{2}\right)$, *where* $\alpha \neq \beta$, *prove that* $a \cos \beta = b \cos \alpha$.

SOLUTION We have,

$$a \tan \alpha + b \tan \beta = (a+b) \tan \left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \quad a\left\{\tan \alpha - \tan \left(\frac{\alpha+\beta}{2}\right)\right\} = b\left\{\tan \left(\frac{\alpha+\beta}{2}\right) - \tan \beta\right\}$$

$$\Rightarrow \quad \frac{a \sin \left(\alpha - \frac{\alpha+\beta}{2}\right)}{\cos \alpha \cos \left(\frac{\alpha+\beta}{2}\right)} = \frac{b \sin \left(\frac{\alpha+\beta}{2} - \beta\right)}{\cos \left(\frac{\alpha+\beta}{2}\right) \cos \beta} \qquad \left[\because \tan A - \tan B = \frac{\sin (A-B)}{\cos A \cos B}\right]$$

$$\Rightarrow \quad \frac{a \sin \left(\frac{\alpha-\beta}{2}\right)}{\cos \alpha} = \frac{b \sin \left(\frac{\alpha-\beta}{2}\right)}{\cos \beta} \Rightarrow a \cos \beta = b \cos \alpha \qquad \left[\because \alpha \neq \beta \therefore \sin \left(\frac{\alpha-\beta}{2}\right) \neq 0\right]$$

EXAMPLE 40 *If* $\sin \alpha + \sin \beta = a$ *and* $\cos \alpha + \cos \beta = b$, *show that*

(i) $\cos (\alpha+\beta) = \dfrac{b^2-a^2}{b^2+a^2}$ (ii) $\sin (\alpha+\beta) = \dfrac{2ab}{a^2+b^2}$

SOLUTION (i) We have,

$$b^2 + a^2 = (\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2$$

$$\Rightarrow b^2 + a^2 = (\cos^2 \alpha + \sin^2 \alpha) + (\cos^2 \beta + \sin^2 \beta) + 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta)$$

$$\Rightarrow b^2 + a^2 = 1 + 1 + 2\cos(\alpha-\beta) = 2 + 2\cos(\alpha-\beta) \qquad \text{...(i)}$$

and, $b^2 - a^2 = (\cos \alpha + \cos \beta)^2 - (\sin \alpha + \sin \beta)^2$

$$\Rightarrow b^2 - a^2 = \cos^2 \alpha + \cos^2 \beta - \sin^2 \alpha - \sin^2 \beta + 2(\cos \alpha \cos \beta - \sin \alpha \sin \beta)$$

$$\Rightarrow b^2 - a^2 = (\cos^2 \alpha - \sin^2 \beta) + (\cos^2 \beta - \sin^2 \alpha) + 2\cos(\alpha+\beta)$$

$$\Rightarrow b^2 - a^2 = \cos(\alpha+\beta)\cos(\alpha-\beta) + \cos(\beta+\alpha)\cos(\beta-\alpha) + 2\cos(\alpha+\beta)$$

$$\Rightarrow b^2 - a^2 = 2\cos(\alpha+\beta)\cos(\alpha-\beta) + 2\cos(\alpha+\beta) \quad [\because \cos(\beta-\alpha) = \cos\{-(\alpha-\beta)\} = \cos(\alpha-\beta)]$$

$$\Rightarrow b^2 - a^2 = \cos(\alpha+\beta)\{2\cos(\alpha-\beta) + 2\}$$

$$\Rightarrow b^2 - a^2 = \cos(\alpha+\beta)(b^2+a^2) \qquad \text{[Using (i)]}$$

Thus, $b^2 - a^2 = (b^2+a^2)\cos(\alpha+\beta) \Rightarrow \cos(\alpha+\beta) = \dfrac{b^2-a^2}{b^2+a^2}$

(ii) $\sin(\alpha+\beta) = \sqrt{1-\cos^2(\alpha+\beta)}$

$$\Rightarrow \sin(\alpha+\beta) = \sqrt{1-\left(\frac{b^2-a^2}{b^2+a^2}\right)^2} = \sqrt{\frac{4a^2b^2}{(a^2+b^2)^2}} = \frac{2ab}{b^2+a^2}$$

EXAMPLE 41 *If* α *and* β *are the solutions of the equation* $a \tan x + b \sec x = c$, *then show that* $\tan(\alpha+\beta) = \dfrac{2ac}{a^2-c^2}$. **[NCERT EXEMPLAR]**

SOLUTION We have,

$$a \tan x + b \sec x = c \qquad \text{...(i)}$$

$\Rightarrow \quad c - a\tan x = b\sec x$

$\Rightarrow \quad (c - a\tan x)^2 = b^2\sec^2 x$

$\Rightarrow \quad c^2 + a^2\tan^2 x - 2ac\tan x = b^2(1+\tan^2 x) \Rightarrow \tan^2 x\,(a^2-b^2) - 2ac\tan x + (c^2-b^2) = 0$ ...(ii)

It is given that α and β are the solutions of equation (i). Therefore, $\tan\alpha$ and $\tan\beta$ are roots of equation (ii).

$$\therefore \quad \tan\alpha + \tan\beta = \frac{2ac}{a^2-b^2} \text{ and, } \tan\alpha\tan\beta = \frac{c^2-b^2}{a^2-b^2}$$

$$\text{Hence, } \tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{\dfrac{2ac}{(a^2-b^2)}}{1-\dfrac{c^2-b^2}{a^2-b^2}} = \frac{2ac}{a^2-c^2}$$

EXAMPLE 42 *If α and β are the solutions of $a\cos x + b\sin x = c$, then show that*

(i) $\cos(\alpha+\beta) = \dfrac{a^2-b^2}{a^2+b^2}$ (ii) $\cos(\alpha-\beta) = \dfrac{2c^2-(a^2+b^2)}{a^2+b^2}$ [NCERT EXEMPLAR]

SOLUTION We have,

$a\cos x + b\sin x = c$ ...(i)

$\Rightarrow \quad a\cos x = c - b\sin x$

$\Rightarrow \quad a^2\cos^2 x = (c - b\sin x)^2$

$\Rightarrow \quad a^2(1-\sin^2 x) = c^2 - 2bc\sin x + b^2\sin^2 x \Rightarrow (a^2+b^2)\sin^2 x - 2bc\sin x + (c^2-a^2) = 0$ ...(ii)

Since α, β are roots of equation (i). Therefore, $\sin\alpha$ and $\sin\beta$ are roots of equation (ii).

$\therefore \quad \sin\alpha\sin\beta = \dfrac{c^2-a^2}{a^2+b^2}$ ...(iii)

Again, $a\cos x + b\sin x = c$

$\Rightarrow \quad b\sin x = c - a\cos x$

$\Rightarrow \quad b^2\sin^2 x = (c - a\cos x)^2$

$\Rightarrow \quad b^2(1-\cos^2 x) = (c - a\cos x)^2 \Rightarrow (a^2+b^2)\cos^2 x - 2ac\cos x + c^2 - b^2 = 0$ ...(iv)

It is given that α, β are the roots of equation (i). So, $\cos\alpha$, $\cos\beta$ are the roots of equation (iv).

$\therefore \quad \cos\alpha\cos\beta = \dfrac{c^2-b^2}{a^2+b^2}$ ...(v)

Using (iii) and (v), we obtain

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{c^2-b^2}{a^2+b^2} - \frac{c^2-a^2}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2}$$

$$\text{and, } \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{c^2-b^2}{a^2+b^2} + \frac{c^2-a^2}{a^2+b^2} = \frac{2c^2-(a^2+b^2)}{a^2+b^2}$$

EXAMPLE 43 *Prove that:* $\dfrac{\sin x}{\cos 3x} + \dfrac{\sin 3x}{\cos 9x} + \dfrac{\sin 9x}{\cos 27x} = \dfrac{1}{2}(\tan 27x - \tan x)$.

SOLUTION We have,

$$\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x}$$

$$= \frac{1}{2}\left\{\frac{2\sin x\cos x}{\cos 3x\cos x} + \frac{2\sin 3x\cos 3x}{\cos 9x\cos 3x} + \frac{2\sin 9x\cos 9x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin(x+x)}{\cos 3x\cos x} + \frac{\sin(3x+3x)}{\cos 9x\cos 3x} + \frac{\sin(9x+9x)}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 2x}{\cos 3x\cos x} + \frac{\sin 6x}{\cos 9x\cos 3x} + \frac{\sin 18x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin(3x-x)}{\cos 3x\cos x} + \frac{\sin(9x-3x)}{\cos 9x\cos 3x} + \frac{\sin(27x-9x)}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 3x\cos x - \cos 3x\sin x}{\cos 3x\cos x} + \frac{\sin 9x\cos 3x - \cos 9x\sin 3x}{\cos 9x\cos 3x} + \frac{\sin 27x\cos 9x - \cos 27x\sin 9x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 3x\cos x}{\cos 3x\cos x} - \frac{\cos 3x\sin x}{\cos 3x\cos x} + \frac{\sin 9x\cos 3x}{\cos 9x\cos 3x} - \frac{\cos 9x\sin 3x}{\cos 9x\cos 3x} + \frac{\sin 27x\cos 9x}{\cos 27x\cos 9x} - \frac{\cos 27x\sin 9x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\{(\tan 3x - \tan x) + (\tan 9x - \tan 3x) + (\tan 27x - \tan 9x)\} = \frac{1}{2}(\tan 27x - \tan x)$$

EXERCISE 7.1

BASIC

1. If $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$, where $0 < A,\ B < \frac{\pi}{2}$, find the values of the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$ (iii) $\sin(A-B)$ (iv) $\cos(A-B)$
2. (a) If $\sin A = \frac{12}{13}$ and $\sin B = \frac{4}{5}$, where $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$, find the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$
   (b) If $\sin A = \frac{3}{5}$, $\cos B = -\frac{12}{13}$, where A and B both lie in second quadrant, find the value of $\sin(A+B)$. [NCERT]
3. If $\cos A = -\frac{24}{25}$ and $\cos B = \frac{3}{5}$, where $\pi < A < \frac{3\pi}{2}$ and $\frac{3\pi}{2} < B < 2\pi$, find the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$
4. If $\tan A = \frac{3}{4}$, $\cos B = \frac{9}{41}$, where $\pi < A < \frac{3\pi}{2}$ and $0 < B < \frac{\pi}{2}$, find $\tan(A+B)$.
5. If $\sin A = \frac{1}{2}$, $\cos B = \frac{12}{13}$, where $\frac{\pi}{2} < A < \pi$ and $\frac{3\pi}{2} < B < 2\pi$, find $\tan(A-B)$.
6. If $\sin A = \frac{1}{2}$, $\cos B = \frac{\sqrt{3}}{2}$,, where $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$, find the following :
   (i) $\tan(A+B)$ (ii) $\tan(A-B)$
7. Evaluate the following:
   (i) $\sin 78° \cos 18° - \cos 78° \sin 18°$ (ii) $\cos 47° \cos 13° - \sin 47° \sin 13°$
   (iii) $\sin 36° \cos 9° + \cos 36° \sin 9°$ (iv) $\cos 80° \cos 20° + \sin 80° \sin 20°$
8. If $\cos A = -\frac{12}{13}$ and $\cot B = \frac{24}{7}$, where A lies in the second quadrant and B in the third quadrant, find the values of the following :
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$ (iii) $\tan(A+B)$

9. Prove that: $\cos\frac{7\pi}{12}+\cos\frac{\pi}{12}=\sin\frac{5\pi}{12}-\sin\frac{\pi}{12}$

10. Prove that: $\frac{\tan A+\tan B}{\tan A-\tan B}=\frac{\sin(A+B)}{\sin(A-B)}$

11. Prove that:

(i) $\frac{\cos 11^\circ+\sin 11^\circ}{\cos 11^\circ-\sin 11^\circ}=\tan 56^\circ$ (ii) $\frac{\cos 9^\circ+\sin 9^\circ}{\cos 9^\circ-\sin 9^\circ}=\tan 54^\circ$ (iii) $\frac{\cos 8^\circ-\sin 8^\circ}{\cos 8^\circ+\sin 8^\circ}=\tan 37^\circ$

12. Prove that:

(i) $\sin\left(\frac{\pi}{3}-x\right)\cos\left(\frac{\pi}{6}+x\right)+\cos\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{6}+x\right)=1$

(ii) $\sin\left(\frac{4\pi}{9}+7\right)\cos\left(\frac{\pi}{9}+7\right)-\cos\left(\frac{4\pi}{9}+7\right)\sin\left(\frac{\pi}{9}+7\right)=\frac{\sqrt{3}}{2}$

(iii) $\sin\left(\frac{3\pi}{8}-5\right)\cos\left(\frac{\pi}{8}+5\right)+\cos\left(\frac{3\pi}{8}-5\right)\sin\left(\frac{\pi}{8}+5\right)=1$

13. Prove that: $\frac{\tan 69^\circ+\tan 66^\circ}{1-\tan 69^\circ\tan 66^\circ}=-1$

BASED ON LOTS

14. (i) If $\tan A=\frac{5}{6}$ and $\tan B=\frac{1}{11}$, prove that $A+B=\frac{\pi}{4}$

(ii) If $\tan A=\frac{m}{m-1}$ and $\tan B=\frac{1}{2m-1}$, then prove that $A-B=\frac{\pi}{4}$

15. Prove that:

(i) $\cos^2\frac{\pi}{4}-\sin^2\frac{\pi}{12}=\frac{\sqrt{3}}{4}$ (ii) $\sin^2(n+1)A-\sin^2 nA=\sin(2n+1)A\sin A$

16. Prove that:

(i) $\frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)}=\tan A$

(ii) $\frac{\sin(A-B)}{\cos A\cos B}+\frac{\sin(B-C)}{\cos B\cos C}+\frac{\sin(C-A)}{\cos C\cos A}=0$

(iii) $\frac{\sin(A-B)}{\sin A\sin B}+\frac{\sin(B-C)}{\sin B\sin C}+\frac{\sin(C-A)}{\sin C\sin A}=0$

(iv) $\sin^2 B=\sin^2 A+\sin^2(A-B)-2\sin A\cos B\sin(A-B)$

(v) $\cos^2 A+\cos^2 B-2\cos A\cos B\cos(A+B)=\sin^2(A+B)$

(vi) $\frac{\tan(A+B)}{\cot(A-B)}=\frac{\tan^2 A-\tan^2 B}{1-\tan^2 A\tan^2 B}$

17. Prove that:

(i) $\tan 8x-\tan 6x-\tan 2x=\tan 8x\tan 6x\tan 2x$

(ii) $\tan\frac{\pi}{12}+\tan\frac{\pi}{6}+\tan\frac{\pi}{12}\tan\frac{\pi}{6}=1$

(iii) $\tan 36^\circ+\tan 9^\circ+\tan 36^\circ\tan 9^\circ=1$

(iv) $\tan 13x-\tan 9x-\tan 4x=\tan 13x\tan 9x\tan 4x$

18. Prove that: $\frac{\tan^2 2x - \tan^2 x}{1 - \tan^2 2x \tan^2 x} = \tan 3x \tan x$

19. (i) If $\frac{\sin (x+y)}{\sin (x-y)} = \frac{a+b}{a-b}$, show that $\frac{\tan x}{\tan y} = \frac{a}{b}$. [NCERT]

(ii) If $\cos (\theta + \phi) = m \cos (\theta - \phi)$, then prove that $\tan \theta = \frac{1-m}{1+m} \cot \phi$. [NCERT EXEMPLAR]

20. If $\tan A = x \tan B$, prove that $\frac{\sin (A-B)}{\sin (A+B)} = \frac{x-1}{x+1}$.

21. If $\tan (A + B) = x$ and $\tan (A - B) = y$, find the values of $\tan 2A$ and $\tan 2B$. [NCERT EXEMPLAR]

22. If $\cos A + \sin B = m$ and $\sin A + \cos B = n$, prove that $2 \sin (A + B) = m^2 + n^2 - 2$.

23. If $\tan A + \tan B = a$ and $\cot A + \cot B = b$, prove that: $\cot (A + B) = \frac{1}{a} - \frac{1}{b}$.

BASED ON HOTS

24. If x lies in the first quadrant and $\cos x = \frac{8}{17}$, then prove that

$$\cos\left(\frac{\pi}{6} + x\right) + \cos\left(\frac{\pi}{4} - x\right) + \cos\left(\frac{2\pi}{3} - x\right) = \frac{23}{17}\left(\frac{\sqrt{3}-1}{2} + \frac{1}{\sqrt{2}}\right)$$ [NCERT EXEMPLAR]

25. If $\tan x + \tan\left(x + \frac{\pi}{3}\right) + \tan\left(x + \frac{2\pi}{3}\right) = 3$, then prove that $\frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x} = 1$.

26. If $\sin (\alpha + \beta) = 1$ and $\sin (\alpha - \beta) = \frac{1}{2}$, where $0 \le \alpha, \beta \le \frac{\pi}{2}$, then find the values of $\tan (\alpha + 2\beta)$ and $\tan (2\alpha + \beta)$.

27. If α, β are two different values of x lying between 0 and 2π which satisfy the equation $6 \cos x + 8 \sin x = 9$, find the value of $\sin (\alpha + \beta)$.

28. If $\sin \alpha + \sin \beta = a$ and $\cos \alpha + \cos \beta = b$, show that

(i) $\sin (\alpha + \beta) = \frac{2ab}{a^2 + b^2}$ (ii) $\cos (\alpha + \beta) = \frac{b^2 - a^2}{b^2 + a^2}$

29. Prove that :

(i) $\frac{1}{\sin (x-a) \sin (x-b)} = \frac{\cot (x-a) - \cot (x-b)}{\sin (a-b)}$

(ii) $\frac{1}{\sin (x-a) \cos (x-b)} = \frac{\cot (x-a) + \tan (x-b)}{\cos (a-b)}$

(iii) $\frac{1}{\cos (x-a) \cos (x-b)} = \frac{\tan (x-b) - \tan (x-a)}{\sin (a-b)}$

30. If $\sin \alpha \sin \beta - \cos \alpha \cos \beta + 1 = 0$, prove that $1 + \cot \alpha \tan \beta = 0$.

31. If $\tan \alpha = x + 1$, $\tan \beta = x - 1$, show that $2 \cot (\alpha - \beta) = x^2$.

32. If angle θ is divided into two parts such that the tangent of one part is λ times the tangent of other, and ϕ is their difference, then show that $\sin \theta = \frac{\lambda+1}{\lambda-1} \sin \phi$. [NCERT EXEMPLAR]

33. If $\tan x = \frac{\sin \alpha - \cos \alpha}{\sin \alpha + \cos \alpha}$, then show that $\sin \alpha + \cos \alpha = \sqrt{2} \cos x$. [NCERT EXEMPLAR]

34. If α and β are two solutions of the equation $a\tan x + b\sec x = c$, then find the values of $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$.

ANSWERS

1. (i) $\frac{56}{65}$ (ii) $\frac{-33}{65}$ (iii) $\frac{-16}{65}$ (iv) $\frac{63}{65}$
2. (a) (i) $\frac{16}{65}$ (ii) $-\frac{63}{65}$ (b) $\frac{-56}{65}$
3. (i) $\frac{3}{5}$ (ii) $-\frac{4}{5}$
4. $-\frac{187}{84}$
5. $\frac{5\sqrt{3}-12}{5+12\sqrt{3}}$
6. (i) 0 (ii) $-\sqrt{3}$
7. (i) $\frac{\sqrt{3}}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{\sqrt{2}}$ (iv) $\frac{1}{2}$
8. (i) $\frac{-36}{325}$ (ii) $\frac{323}{325}$ (iii) $-\frac{36}{323}$
18. $\frac{x+y}{1-xy}, \frac{x-y}{1+xy}$
26. $-\sqrt{3}, -\frac{1}{\sqrt{3}}$
27. $\frac{24}{25}$
34. $\sin(\alpha+\beta) = -\frac{2ac}{a^2+c^2}$, $\cos(\alpha+\beta) = \frac{c^2-a^2}{c^2+a^2}$

HINTS TO SELECTED PROBLEMS

9. $\text{LHS} = \cos\left(\frac{\pi}{2}+\frac{\pi}{12}\right) + \cos\left(\frac{\pi}{2}-\frac{5\pi}{12}\right) = -\sin\frac{\pi}{12} + \sin\frac{5\pi}{12} = \text{RHS}$

11. $$\text{LHS} = \frac{\frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ}}{\frac{\cos 11^\circ - \sin 11^\circ}{\cos 11^\circ}} = \frac{1+\tan 11^\circ}{1-\tan 11^\circ} = \frac{\tan 45^\circ + \tan 11^\circ}{1-\tan 45^\circ \tan 11^\circ} = \tan(45^\circ+11^\circ) = \tan 56^\circ = \text{RHS}$$

19. (i) $\frac{\sin(x+y)}{\sin(x-y)} = \frac{a+b}{a-b}$

$\Rightarrow \frac{\sin(x+y)+\sin(x-y)}{\sin(x+y)-\sin(x-y)} = \frac{(a+b)+(a-b)}{(a+b)-(a-b)}$ [Applying componendo and dividendo]

$\Rightarrow \frac{2\sin x\cos y}{2\cos x\sin y} = \frac{2a}{2b} \Rightarrow \frac{\tan x}{\tan y} = \frac{a}{b}$

(ii) $\cos(\theta+\phi) = m\cos(\theta-\phi)$

$\Rightarrow \frac{\cos(\theta-\phi)}{\cos(\theta+\phi)} = \frac{1}{m}$

$\Rightarrow \frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{\cos(\theta-\phi)-\cos(\theta+\phi)} = \frac{1+m}{1-m}$ [Applying componendo and dividendo]

$\Rightarrow \frac{2\cos\theta\cos\phi}{2\sin\theta\sin\phi} = \frac{1+m}{1-m} \Rightarrow \frac{\cot\phi}{\tan\theta} = \frac{1+m}{1-m} \Rightarrow \tan\theta = \frac{1-m}{1+m}\cot\phi$

29. (i) We have,

$$\frac{1}{\sin(x-a)\sin(x-b)} = \frac{1}{\sin(a-b)}\left\{\frac{\sin(a-b)}{\sin(x-a)\sin(x-b)}\right\}$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin(x-b)\cos(x-a)}{\sin(x-a)\sin(x-b)}-\frac{\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right] = \frac{1}{\sin(a-b)}[\cot(x-a)-\cot(x-b)]$$

Similarly, we can prove other two parts.

32. Let α and β be two parts of angle θ. Then, $\alpha+\beta=\theta$. It is given that $\alpha-\beta=\phi$ and $\tan\alpha=\lambda\tan\beta$.

Now, $\tan\alpha=\lambda\tan\beta$

$$\Rightarrow \frac{\tan\alpha}{\tan\beta}=\frac{\lambda}{1}$$

$$\Rightarrow \frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\frac{\lambda+1}{\lambda-1}$$ [Applying componendo and dividendo]

$$\Rightarrow \frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta}}=\frac{\lambda+1}{\lambda-1} \Rightarrow \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}=\frac{\lambda+1}{\lambda-1}$$

$$\Rightarrow \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{\lambda+1}{\lambda-1} \Rightarrow \frac{\sin\theta}{\sin\phi}=\frac{\lambda+1}{\lambda-1} \Rightarrow \sin\theta=\frac{\lambda+1}{\lambda-1}\sin\phi.$$

33. $\tan x=\dfrac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}=\dfrac{\frac{\sin\alpha-\cos x}{\cos\alpha}}{\frac{\sin\alpha+\cos\alpha}{\cos\alpha}}$ [Dividing numerator and denominator by $\cos\alpha$]

$$\Rightarrow \tan x=\frac{\tan\alpha-1}{\tan\alpha+1} \Rightarrow \tan x=\tan\left(\alpha-\frac{\pi}{4}\right)\Rightarrow x=\alpha-\frac{\pi}{4}\Rightarrow \alpha=x+\frac{\pi}{4}$$

$$\therefore \sin\alpha+\cos\alpha=\sin\left(x+\frac{\pi}{4}\right)+\cos\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}(\sin x+\cos x)+\frac{1}{\sqrt{2}}(\cos x-\sin x)=\sqrt{2}\cos x$$

ALITER $\tan x=\dfrac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}$

$$\Rightarrow \sec x=\sqrt{1+\tan^2 x}=\sqrt{1+\left(\frac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}\right)^2}$$

$$\Rightarrow \frac{1}{\cos x}=\sqrt{\frac{(\sin\alpha+\cos\alpha)^2+(\sin\alpha-\cos\alpha)^2}{(\sin\alpha+\cos\alpha)^2}} \Rightarrow \frac{1}{\cos x}=\sqrt{\frac{2}{(\sin\alpha+\cos\alpha)^2}}$$

$$\Rightarrow \frac{1}{\cos x}=\frac{\sqrt{2}}{\sin\alpha+\cos\alpha} \Rightarrow \sin\alpha+\cos\alpha=\sqrt{2}\cos x$$

7.4 MAXIMUM AND MINIMUM VALUES OF TRIGONOMETRICAL EXPRESSIONS

We have learnt that for those values of x for which trigonometrical functions are defined, we have

$-1 \leq \sin x \leq 1, -1 \leq \cos x \leq 1, -\infty < \tan x < \infty$ $\operatorname{cosec} x \geq 1$ or $\operatorname{cosec} x \leq -1$, $\sec x \geq 1$ or $\sec x \leq -1$ and, $-\infty < \cot x < \infty$

In this section, we will find the maximum and minimum values of trigonometrical expressions of the form $a \cos x + b \sin x$ for varying values of x.

Let $f(x) = a \cos x + b \sin x$ Further, let $a = r \sin \alpha$ and $b = r \cos \alpha$. This assumption is construct a right angled triangle with a and b as two sides and $r = \sqrt{a^2 + b^2}$ as hypotenuse.

Then, $\quad a^2 + b^2 = r^2 \sin^2 \alpha + r^2 \cos^2 \alpha$ and, $\dfrac{a}{b} = \dfrac{r \sin \alpha}{r \cos \alpha}$

$\Rightarrow \quad a^2 + b^2 = r^2 (\sin^2 \alpha + \cos^2 \alpha)$ and, $\dfrac{a}{b} = \tan \alpha \Rightarrow r = \sqrt{a^2 + b^2}$ and, $\tan \alpha = \dfrac{a}{b}$

Substituting the values of a and b in $f(x)$, we obtain

$$f(x) = r \sin \alpha \cos x + r \cos \alpha \sin x = r(\sin \alpha \cos x + \cos \alpha \sin x) = r \sin(\alpha + x)$$

We know that

$-1 \leq \sin(\alpha + x) \leq 1$ for all x

$\Rightarrow \quad -r \leq r \sin(\alpha + x) \leq r$ for all x [Multiplying throughout by r]

$\Rightarrow \quad -\sqrt{a^2 + b^2} \leq f(x) \leq \sqrt{a^2 + b^2}$ for all x

$\Rightarrow \quad -\sqrt{a^2 + b^2} \leq a \cos x + b \sin x \leq \sqrt{a^2 + b^2}$ for all x

Hence, maximum and minimum values of $a \cos x + b \sin x$ are $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$ respectively.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the maximum and minimum values of* $7 \cos x + 24 \sin x$.

SOLUTION We know that the maximum and minimum values of $a \cos x + b \sin x$ are $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$ respectively. Hence, the maximum and minimum values of $7 \cos x + 24 \sin x$ are $\sqrt{7^2 + 24^2} = 25$ and $-\sqrt{7^2 + 24^2} = -25$ respectively.

EXAMPLE 2 *Find the maximum and minimum values of the following expressions:*

(i) $3 \cos x + 5 \sin\left(x - \dfrac{\pi}{6}\right)$ (ii) $4 \sin x - 3 \cos x + 7$

SOLUTION (i) Let $f(x) = 3 \cos x + 5 \sin\left(x - \dfrac{\pi}{6}\right)$. Then,

$$f(x) = 3 \cos x + 5\left\{\sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6}\right\} = 3 \cos x + \frac{5\sqrt{3}}{2} \sin x - \frac{5}{2} \cos x$$

$$\Rightarrow \quad f(x) = \frac{1}{2} \cos x + \frac{5\sqrt{3}}{2} \sin x$$

$\because \quad -\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2} \le \frac{1}{2}\cos x+\frac{5\sqrt{3}}{2}\sin x \le \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2}$ for all x.

$\therefore \quad -\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2} \le f(x) \le \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2}$ for all x

$\Rightarrow \quad -\sqrt{\frac{1}{4}+\frac{75}{4}} \le 3\cos x+5\sin\left(x-\frac{\pi}{6}\right) \le \sqrt{\frac{1}{4}+\frac{75}{4}}$ for all x

$\Rightarrow \quad -\sqrt{19} \le 3\cos x+5\sin\left(x-\frac{\pi}{6}\right) \le \sqrt{19}$ for all x

Hence, $-\sqrt{19}$ and $\sqrt{19}$ are respectively the minimum and the maximum values of $3\cos x+5\sin\left(x-\frac{\pi}{6}\right)$.

(ii) Let $f(x) = 4\sin x - 3\cos x + 7$

We know that

$$-\sqrt{4^2+(-3)^2} \le 4\sin x - 3\cos x \le \sqrt{4^2+(-3)^2} \text{ for all } x$$

$\Rightarrow \quad -5 \le 4\sin x - 3\cos x \le 5$ for all x

$\Rightarrow \quad -5+7 \le 4\sin x - 3\cos x + 7 \le 5+7$ for all x

$\Rightarrow \quad 2 \le f(x) \le 12$ for all x

Hence, minimum and maximum values of $4\sin x - 3\cos x + 7$ are 2 and 12 respectively.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Prove that* $5\cos x + 3\cos\left(x+\frac{\pi}{3}\right)+3$ *lies between* -4 *and* 10.

SOLUTION Let $f(x) = 5\cos x + 3\cos\left(x+\frac{\pi}{3}\right)+3$. Then,

$$f(x) = 5\cos x + 3\left(\cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3}\right)+3 = 5\cos x + \frac{3}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x + 3$$

$\Rightarrow \quad f(x) = \frac{13}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x + 3$ ...(i)

$\because \quad -\sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x \le \sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2}$

$\Rightarrow \quad -7 \le \frac{13}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x \le 7$ for all x

$\Rightarrow \quad -7+3 \le \frac{13}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x + 3 \le 7+3$ for all x

$\Rightarrow \quad -4 \le 5\cos x + 3\cos\left(x+\frac{\pi}{3}\right)+3 \le 10$ for all x [Using (i)]

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Find a and b such that the following inequality holds good for all* x :

$$a \le 3\cos x + 5\sin\left(x-\frac{\pi}{6}\right) \le b.$$

Also, find the greatest and least values of a and b respectively.

SOLUTION Let $f(x) = 3\cos x + 5\sin\left(x - \frac{\pi}{6}\right)$. Then,

$$f(x) = 3\cos x + 5\left(\sin x \cos\frac{\pi}{6} - \cos x \sin\frac{\pi}{6}\right) = 3\cos x + 5\left(\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x\right)$$

$$\Rightarrow \quad f(x) = \frac{1}{2}\cos x + \frac{5\sqrt{3}}{2}\sin x$$

$$\because \quad \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2} \le \frac{1}{2}\cos x + \frac{5\sqrt{3}}{2}\sin x \le \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2} \text{ for all } x$$

$$\Rightarrow \quad -\sqrt{19} \le 3\cos x + 5\sin\left(x - \frac{\pi}{6}\right) \le \sqrt{19} \text{ for all } x$$

Hence, $a \le -\sqrt{19}$ and $b \ge \sqrt{19}$. The greatest value of a is $-\sqrt{19}$ and the least value of b is $\sqrt{19}$.

7.5 TO EXPRESS $a\cos x + b\sin x$ IN THE FORM $r\sin(x \pm \alpha)$ OR $r\cos(x \pm \alpha)$

Sometimes we need to express trigonometrical expressions of the form $a\cos x + b\sin x$ in terms of sine or cosine of single term. We may use the following algorithm to do so.

ALGORITHM

Step I *Multiply and divide* $f(x) = a\cos x + b\sin x$ *by* $\sqrt{a^2 + b^2}$ *to get*

$$f(x) = \sqrt{a^2 + b^2}\left\{\frac{a}{\sqrt{a^2 + b^2}}\cos x + \frac{b}{\sqrt{a^2 + b^2}}\sin x\right\}$$

Step II *In order to express* $f(x)$ *in terms of sine of some term, replace* $\frac{a}{\sqrt{a^2 + b^2}}$ *i.e. coefficient of* $\cos x$ *by* $\sin\alpha$ *and* $\frac{b}{\sqrt{a^2 + b^2}}$ *i.e. coefficient of* $\sin x$ *by* $\cos\alpha$. *This gives the following :*

$$f(x) = \sqrt{a^2 + b^2}\,\{\sin\alpha\cos x + \cos\alpha\sin x\} = \sqrt{a^2 + b^2}\,\sin(x + \alpha)$$

To express $f(x)$ *in terms of cosine of some term, replace coefficient of* $\cos x$ *i.e.* $\frac{a}{\sqrt{a^2 + b^2}}$ *by* $\cos\alpha$ *and coefficient of* $\sin x$ *i.e.* $\frac{b}{\sqrt{a^2 + b^2}}$ *by* $\sin\alpha$. *This gives the following:*

$$f(x) = \sqrt{a^2 + b^2}\,(\cos\alpha\cos x + \sin\alpha\sin x) = \sqrt{a^2 + b^2}\,\cos(x - \alpha).$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Reduce* $\sqrt{3}\sin x + \cos x$ *as a single term consisting (i) sine only (ii) cosine only.*

SOLUTION Let $f(x) = \sqrt{3}\sin x + \cos x$. Then,

$$f(x) = \sqrt{3}\sin x + \cos x$$

$$\Rightarrow f(x) = 2\left\{\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right\} \qquad \left[\text{Multiplying and dividing by } \sqrt{(\sqrt{3})^2 + 1^2} \text{ i.e. by } 2\right]$$

$$\Rightarrow f(x) = 2\left\{\sin x\cos\frac{\pi}{6} + \cos x\sin\frac{\pi}{6}\right\} = 2\sin\left(x + \frac{\pi}{6}\right) \qquad \left[\because \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \text{ and } \frac{1}{2} = \sin\frac{\pi}{6}\right]$$

Again,

$$f(x) = 2\left\{\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right\} = 2\left\{\sin\frac{\pi}{3}\sin x + \cos\frac{\pi}{3}\cos x\right\}$$

$$\Rightarrow f(x) = 2\left\{\cos x\cos\frac{\pi}{3} + \sin x\sin\frac{\pi}{3}\right\} = 2\cos\left(x - \frac{\pi}{3}\right).$$

EXAMPLE 2 *Express* $3\cos x - 4\sin x$ *as sines and cosines of a single expression.*

SOLUTION Let $f(x) = 3\cos x - 4\sin x$. Multiplying and dividing by $\sqrt{3^2 + (-4)^2}$ i.e. by 5, we get

$$f(x) = \sqrt{3^2 + (-4)^2}\left\{\frac{3}{\sqrt{3^2+(-4)^2}}\cos x - \frac{4}{\sqrt{3^2+(-4)^2}}\sin x\right\}$$

$$\Rightarrow f(x) = 5\left(\frac{3}{5}\cos x - \frac{4}{5}\sin x\right)$$

$$\Rightarrow f(x) = 5(\sin\alpha\cos x - \cos\alpha\sin x), \text{ where } \sin\alpha = \frac{3}{5} \text{ and } \cos\alpha = \frac{4}{5}$$

$$\Rightarrow f(x) = 5\sin(\alpha - x), \text{ where } \tan\alpha = \frac{3}{4}$$

Again,

$$f(x) = 5\left(\frac{3}{5}\cos x - \frac{4}{5}\sin x\right)$$

$$\Rightarrow f(x) = 5(\cos\alpha\cos x - \sin\alpha\sin x), \text{ where } \cos\alpha = \frac{3}{5} \text{ and } \sin\alpha = \frac{4}{5}$$

$$\Rightarrow f(x) = 5\cos(\alpha + x), \text{ where } \tan\alpha = \frac{4}{3}$$

EXAMPLE 3 *Find the sign of the expression* $\sin 100° + \cos 100°$.

SOLUTION $\sin 100° + \cos 100°$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin 100° + \frac{1}{\sqrt{2}}\cos 100°\right) = \sqrt{2}(\cos 45°\sin 100° + \sin 45°\cos 100°)$$

$= \sqrt{2}\sin(100° + 45°) = \sqrt{2}\sin 145°$, which is a positive real number. [$\because$ $\sin 145°$ is positive]

EXERCISE 7.2

BASIC

1. Find the maximum and minimum values of each of the following trigonometrical expressions:
   (i) $12\sin x - 5\cos x$
   (ii) $12\cos x + 5\sin x + 4$
   (iii) $5\cos x + 3\sin\left(\frac{\pi}{6} - x\right) + 4$
   (iv) $\sin x - \cos x + 1$
2. Reduce each of the following expressions to the sine and cosine of a single expression:
   (i) $\sqrt{3}\sin x - \cos x$
   (ii) $\cos x - \sin x$
   (iii) $24\cos x + 7\sin x$

BASED ON HOTS

3. Show that $\sin 100° - \sin 10°$ is positive.
4. Prove that $(2\sqrt{3} + 3)\sin x + 2\sqrt{3}\cos x$ lies between $-(2\sqrt{3} + \sqrt{15})$ and $(2\sqrt{3} + \sqrt{15})$.

ANSWERS

1.

	Minimum	Maximum		Minimum	Maximum
(i)	-13	13	(ii)	-9	17
(iii)	-3	11	(iv)	$1-\sqrt{2}$	$1+\sqrt{2}$

2. (i) $2\sin\left(x-\frac{\pi}{6}\right), -2\cos\left(\frac{\pi}{3}+x\right)$ (ii) $\sqrt{2}\sin\left(\frac{\pi}{4}-x\right), \sqrt{2}\cos\left(\frac{\pi}{4}+x\right)$

(iii) $25\sin(\alpha+x)$, where $\tan\alpha=\frac{24}{7}$, $25\cos(x-\alpha)$, where $\tan\alpha=\frac{7}{24}$

HINTS TO SELECTED PROBLEMS

4. $(2\sqrt{3}+3)\sin x+2\sqrt{3}\cos x=(2\sqrt{3}\sin x+\sqrt{3}\cos x)+(\sqrt{3}\cos x+3\sin x)$

Now, $-\sqrt{15}\le 2\sqrt{3}\sin x+\sqrt{3}\cos x\le\sqrt{15}$ and, $-\sqrt{12}\le\sqrt{3}\cos x+3\sin x\le\sqrt{12}$

Adding the two inequalities, we obtain

$\therefore\quad -\sqrt{15}-\sqrt{12}\le 2\sqrt{3}\sin x+\sqrt{3}\cos x+\sqrt{3}\cos x+3\sin x\le\sqrt{15}+\sqrt{12}$

$\Rightarrow\quad -(2\sqrt{3}+\sqrt{15})\le(2\sqrt{3}+3)\sin x+2\sqrt{3}\cos x\le(2\sqrt{3}+\sqrt{15})$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The maximum value of $3\cos x+4\sin x+5$ is ....
2. The minimum value of $4\cos x-3\sin x+7$ is ....
3. If $\sin\theta+\cos\theta=1$, then the value of $\sin 2\theta$ is ....
4. If $A-B=\frac{\pi}{4}$, then $(1+\tan A)(1-\tan B)=$ ....
5. If $A+B=\frac{\pi}{4}$, then $(1+\tan A)(1+\tan B)=$ ....
6. If $\cos(A-B)=\frac{3}{5}$ and $\tan A\tan B=2$, then $\sin A\sin B=$ ....
7. If $\frac{x}{\cos\theta}=\frac{y}{\cos\left(\theta-\frac{2\pi}{3}\right)}=\frac{z}{\cos\left(\theta+\frac{2\pi}{3}\right)}$, then $x+y+z=$ ....
8. The value of $\cot\left(\frac{\pi}{4}+x\right)\cot\left(\frac{\pi}{4}-x\right)$ is ....
9. If $\sin x\cos y=\frac{1}{4}$ and $3\tan x=4\tan y$, then $\sin(x-y)$ is equal to ....
10. If $\cos^2\left(\frac{\pi}{6}+x\right)-\sin^2\left(\frac{\pi}{6}-x\right)=k\cos 2x$ then $k=$ ....
11. If $\tan x=\frac{1}{2}$ and $\tan y=\frac{1}{3}$, then the value of $x+y$ is ....
12. The value of $\tan 5x\tan 3x\tan 2x-\tan 5x+\tan 3x+\tan 2x$ is ....
13. The value of $\cos\frac{\pi}{12}-\sin\frac{\pi}{2}$ is ....

ANSWERS

1. 10 2. 2 3. 0 4. 2 5. 2 6. $\frac{2}{5}$ 7. 0 8. 1
9. $\frac{1}{16}$ 10. $\frac{1}{2}$ 11. $\frac{\pi}{4}$ 12. 0 13. $\frac{1}{\sqrt{2}}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $\alpha+\beta-\gamma=\pi$, and $\sin^2\alpha+\sin^2\beta-\sin^2\gamma=\lambda\sin\alpha\sin\beta\cos\gamma$, then write the value of λ.
2. If $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$, then write the value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.
3. Write the maximum and minimum values of $3\cos x+4\sin x+5$.
4. Write the maximum value of $12\sin x-9\sin^2 x$.
5. If $12\sin x-9\sin^2 x$ attains its maximum value at $x=\alpha$, then write the value of $\sin\alpha$.
6. Write the interval in which the values of $5\cos x+3\cos\left(x+\frac{\pi}{3}\right)+3$ lie.
7. If $\tan(A+B) = p$ and $\tan(A-B) = q$, then write the value of $\tan 2B$.
8. If $\frac{\cos(x-y)}{\cos(x+y)} = \frac{m}{n}$, then write the value of $\tan x\tan y$.
9. If $a = b\cos\frac{2\pi}{3} = c\cos\frac{4\pi}{3}$, then write the value of $ab+bc+ca$.
10. If $A+B = C$, then write the value of $\tan A\tan B\tan C$.
11. If $\sin\alpha-\sin\beta = a$ and $\cos\alpha+\cos\beta = b$, then write the value of $\cos(\alpha+\beta)$.
12. If $\tan\alpha = \frac{1}{1+2^{-x}}$ and $\tan\beta = \frac{1}{1+2^{x+1}}$, then write the value of $\alpha+\beta$ lying in the interval $(0, \pi/2)$.

ANSWERS

1. 2 2. 0 3. Maximum = 10, Minimum = 0 4. 4 5. $\frac{2}{3}$
6. $[-4, 10]$ 7. $\frac{p-q}{1+pq}$ 8. $\frac{m-n}{m+n}$ 9. 0 10. $\tan C-\tan A-\tan B$
11. $\frac{a^2+b^2-2}{2}$ 12. $\frac{\pi}{4}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The value of $\sin^2\frac{5\pi}{12}-\sin^2\frac{\pi}{12}$ is
   (a) 1/2 (b) $\sqrt{3}/2$ (c) 1 (d) 0
2. If $A+B+C=\pi$, then $\sec A(\cos B\cos C-\sin B\sin C)$ is equal to
   (a) 0 (b) -1 (c) 1 (d) none of these

3. $\tan 20° + \tan 40° + \sqrt{3} \tan 20° \tan 40°$ is equal to
(a) $\frac{\sqrt{3}}{4}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\sqrt{3}$ (d) 1

4. If $\tan A = \frac{a}{a+1}$ and $\tan B = \frac{1}{2a+1}$, then the value of $A+B$ is
(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

5. If $3 \sin x + 4 \cos x = 5$, then $4 \sin x - 3 \cos x =$
(a) 0 (b) 5 (c) 1 (d) none of these

6. If in a ΔABC, $\tan A + \tan B + \tan C = 6$, then $\cot A \cot B \cot C =$
(a) 6 (b) 1 (c) 1/6 (d) none of these

7. $\tan 3A - \tan 2A - \tan A$ is equal to
(a) $\tan 3A \tan 2A \tan A$ (b) $-\tan 3A \tan 2A \tan A$
(c) $\tan A \tan 2A - \tan 2A \tan 3A - \tan 3A \tan A$ (d) none of these

8. If $A+B+C=\pi$, then $\frac{\tan A + \tan B + \tan C}{\tan A \tan B \tan C}$ is equal to
(a) $\tan A \tan B \tan C$ (b) 0 (c) 1 (d) none of these

9. If $\cos P = \frac{1}{7}$ and $\cos Q = \frac{13}{14}$, where P and Q both are acute angles. Then, the value of $P-Q$ is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{5\pi}{12}$

10. If $\cot(\alpha+\beta)=0$, then $\sin(\alpha + 2\beta)$ is equal to
(a) $\sin \alpha$ (b) $\cos 2\beta$ (c) $\cos \alpha$ (d) $\sin 2\alpha$

11. $\frac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°}$ is equal to
(a) $\tan 55°$ (b) $\cot 55°$ (c) $-\tan 35°$ (d) $-\cot 35°$

12. The value of $\cos^2\left(\frac{\pi}{6}+x\right) - \sin^2\left(\frac{\pi}{6}-x\right)$ is
(a) $\frac{1}{2}\cos 2x$ (b) 0 (c) $-\frac{1}{2}\cos 2x$ (d) $\frac{1}{2}$

13. If $\tan\theta_1 \tan\theta_2 = k$, then $\frac{\cos(\theta_1-\theta_2)}{\cos(\theta_1+\theta_2)} =$
(a) $\frac{1+k}{1-k}$ (b) $\frac{1-k}{1+k}$ (c) $\frac{k+1}{k-1}$ (d) $\frac{k-1}{k+1}$

14. If $\sin(\pi \cos x) = \cos(\pi \sin x)$, then $\sin 2x =$
(a) $\pm\frac{3}{4}$ (b) $\pm\frac{4}{3}$ (c) $\pm\frac{1}{3}$ (d) none of these

15. If $\tan\theta = \frac{1}{2}$ and $\tan\phi = \frac{1}{3}$, then the value of $\theta+\phi$ is
(a) $\frac{\pi}{6}$ (b) π (c) 0 (d) $\frac{\pi}{4}$
[NCERT EXEMPLAR]

16. The value of $\cos(36° - A)\cos(36° + A) + \cos(54° + A)\cos(54° - A)$ is
(a) $\sin 2A$ (b) $\cos 2A$ (c) $\cos 3A$ (d) $\sin 3A$

17. If $\tan(\pi/4 + x) + \tan(\pi/4 - x) = a$, then $\tan^2(\pi/4 + x) + \tan^2(\pi/4 - x) =$

(a) $a^2 + 1$ (b) $a^2 + 2$ (c) $a^2 - 2$ (d) none of these

18. If $\tan(A - B) = 1$, $\sec(A + B) = \frac{2}{\sqrt{3}}$, then the smallest positive value of B is

(a) $\frac{25\pi}{24}$ (b) $\frac{19\pi}{24}$ (c) $\frac{13\pi}{24}$ (d) $\frac{11\pi}{24}$

19. If $A - B = \pi/4$, then $(1 + \tan A)(1 - \tan B)$ is equal to

(a) 2 (b) 1 (c) 0 (d) 3

20. The maximum value of $\sin^2\left(\frac{2\pi}{3} + x\right) + \sin^2\left(\frac{2\pi}{3} - x\right)$ is

(a) 1/2 (b) 3/2 (c) 1/4 (d) 3/4

21. If $\cos(A - B) = \frac{3}{5}$ and $\tan A \tan B = 2$, then

(a) $\cos A \cos B = \frac{1}{5}$ (b) $\cos A \cos B = -\frac{1}{5}$

(c) $\sin A \sin B = -\frac{1}{5}$ (d) $\sin A \sin B = -\frac{1}{5}$

22. If $\tan 69° + \tan 66° - \tan 69° \tan 66° = 2k$, then $k =$

(a) -1 (b) 1/2 (c) $-1/2$ (d) none of these

23. If $\alpha + \beta = \frac{\pi}{4}$, then the value of $(1 + \tan\alpha)(1 + \tan\beta)$ is

(a) 1 (b) 2 (c) -2 (d) not defined

[NCERT EXEMPLAR]

24. The value of $\sin\left(\frac{\pi}{4} + \theta\right) - \cos\left(\frac{\pi}{4} - \theta\right)$ is

(a) $2\cos\theta$ (b) $2\sin\theta$ (c) 1 (d) 0

[NCERT EXEMPLAR]

25. The minimum value of $3\cos x + 4\sin x + 8$ is

(a) 5 (b) 9 (c) 7 (d) 3

[NCERT EXEMPLAR]

ANSWERS

1. (b)	2. (b)	3. (c)	4. (d)	5. (a)	6. (c)	7. (a)	8. (c)
9. (b)	10. (a)	11. (a)	12. (a)	13. (a)	14. (a)	15. (d)	16. (b)
17. (c)	18. (b)	19. (a)	20. (b)	21. (a)	22. (c)	23. (b)	24. (d)
25. (d)							

SUMMARY

1. (i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(viii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

2. (i) $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

(ii) $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$

(iii) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

3. (i) If $A + B = \pi$, then $\sin A = \sin B$, $\cos A = -\cos B$ and $\tan A = -\tan B$

(ii) If $A + B = 2\pi$, then $\sin A = -\sin B$, $\cos A = \cos B$ and $\tan A = -\tan B$

CHAPTER 8

TRANSFORMATION FORMULAE

8.1 INTRODUCTION

In this chapter, we will establish two sets of transformation formulae: One to transform the products of two sines or two cosines or one sine and one cosine into the sum or difference of two sines or two cosines and the other to convert the sum or difference of two sines or two cosines in the product of two sines or two cosines or one sine and one cosine. These two sets of formulae are of fundamental importance and one should have thorough acquaintance with these formulae.

8.2 FORMULAE TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE

In the previous chapter we have derived the following formulae:

$$\sin A \cos B + \cos A \sin B = \sin(A+B) \quad \text{...(i)}$$

$$\sin A \cos B - \cos A \sin B = \sin(A-B) \quad \text{...(ii)}$$

$$\cos A \cos B - \sin A \sin B = \cos(A+B) \quad \text{...(iii)}$$

$$\cos A \cos B + \sin A \sin B = \cos(A-B) \quad \text{...(iv)}$$

Adding (i) and (ii), we obtain

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

Subtracting (ii) from (i), we get

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$

Adding (iii) and (iv), we get

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

Subtracting (iii) from (iv), we get

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

Thus, we obtain the following formulae :

(a) $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$ (b) $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

(c) $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$ (d) $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

These four formulae convert the product of two sines or two cosines or one sine and one cosine into the sum or difference of two sines or two cosines.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Convert each of the following products into the sum or difference of sines and cosines:*

(i) $2 \sin 5x \cos x$ (ii) $2 \cos 4x \cos 3x$ (iii) $2 \sin 3x \sin x$

(iv) $\sin \frac{5\pi}{12} \cos \frac{\pi}{12}$ (v) $\cos \frac{5\pi}{12} \cos \frac{\pi}{12}$

SOLUTION (i) Using $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$, we obtain

$$2 \sin 5x \cos x = \sin(5x+x) + \sin(5x-x) = \sin 6x + \sin 4x$$

(ii) Using $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$, we obtain

$2 \cos 4x \cos 3x = \cos(4x+3x) + \cos(4x-3x) = \cos 7x + \cos x$

(iii) Using $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$, we obtain

$2 \sin 3x \sin x = \cos(3x-x) - \cos(3x+x) = \cos 2x - \cos 4x$

(iv) $\sin \frac{5\pi}{12} \cos \frac{\pi}{12} = \frac{1}{2}\left(2 \sin \frac{5\pi}{12} \cos \frac{\pi}{12}\right) = \frac{1}{2}\left\{\sin\left(\frac{5\pi}{12}+\frac{\pi}{12}\right) + \sin\left(\frac{5\pi}{12}-\frac{\pi}{12}\right)\right\}$

$= \frac{1}{2}\left(\sin \frac{\pi}{2} + \sin \frac{\pi}{3}\right)$

(v) $\cos \frac{5\pi}{12} \cos \frac{\pi}{12} = \frac{1}{2}\left(2 \cos \frac{5\pi}{12} \cos \frac{\pi}{12}\right) = \frac{1}{2}\left\{\cos\left(\frac{5\pi}{12}+\frac{\pi}{12}\right) + \cos\left(\frac{5\pi}{12}-\frac{\pi}{12}\right)\right\}$

$= \frac{1}{2}\left(\cos \frac{\pi}{2} + \cos \frac{\pi}{3}\right)$

EXAMPLE 2 *Prove that:* $2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0.$ [NCERT]

SOLUTION LHS $= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$

$= \cos\left(\frac{9\pi}{13}+\frac{\pi}{13}\right) + \cos\left(\frac{9\pi}{13}-\frac{\pi}{13}\right) + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$

$= \cos \frac{10\pi}{13} + \cos \frac{8\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$

$= \cos\left(\pi - \frac{3\pi}{13}\right) + \cos\left(\pi - \frac{5\pi}{13}\right) + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$

$= -\cos \frac{3\pi}{13} - \cos \frac{5\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0 = \text{RHS}$ $[\because \cos(\pi - x) = -\cos x]$

EXAMPLE 3 *Prove that:* $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

SOLUTION We have,

LHS $= \cos 20° \cos 40° \cos 60° \cos 80° = \cos 60° (\cos 20° \cos 40°) \cos 80°$

$= \frac{1}{2} \times \frac{1}{2} (2 \cos 20° \cos 40°) \cos 80°$ $\left[\because \cos \frac{\pi}{3} = \frac{1}{2}\right]$

$= \frac{1}{4}\left[\{\cos(40°+20°) + \cos(40°-20°)\} \cos 80°\right]$ $\left[\begin{array}{l}\because 2 \cos A \cos B \\ = \cos(A+B) + \cos(A-B)\end{array}\right]$

$= \frac{1}{4}\left\{(\cos 60° + \cos 20°) \cos 80°\right\} = \frac{1}{4}\left\{\left(\frac{1}{2} + \cos 20°\right) \cos 80°\right\}$

$= \frac{1}{4}\left\{\frac{1}{2} \cos 80° + \cos 80° \cos 20°\right\} = \frac{1}{8}\left\{\cos 80° + 2 \cos 80° \cos 20°\right\}$

$= \frac{1}{8}\left[\cos 80° + \left\{\cos(80°+20°) + \cos(80°-20°)\right\}\right]$ $\left[\begin{array}{l}\because 2 \cos A \cos B \\ = \cos(A+B) + \cos(A-B)\end{array}\right]$

$= \frac{1}{8}\{\cos 80° + \cos 100° + \cos 60°\} = \frac{1}{8}\{\cos 80° + \cos(180°-80°) + \cos 60°\}$

$= \frac{1}{8}\{\cos 80° - \cos 80° + \cos 60°\}$ $\left[\begin{array}{l}\because \cos(180°-x) = -\cos x \\ \therefore \cos(180°-80°) = -\cos 80°\end{array}\right]$

$= \frac{1}{8}\left\{\cos 80° - \cos 80° + \frac{1}{2}\right\} = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16} = \text{RHS}$

EXAMPLE 4 *Prove that:* $\sin 10° \sin 30° \sin 50° \sin 70° = \frac{1}{16}$.

SOLUTION LHS $= \sin 30° (\sin 10° \sin 50°) \sin 70° = \frac{1}{2} (\sin 50° \sin 10°) \sin 70°$

$$= \frac{1}{2} \times \frac{1}{2} (2 \sin 50° \sin 10°) \sin 70° = \frac{1}{4} \left\{ (2 \sin 50° \sin 10°) \sin 70° \right\}$$

$$= \frac{1}{4} \left[\{\cos (50°-10°) - \cos (50°+10°)\} \sin 70° \right] \quad \left[\because \begin{array}{l} 2 \sin A \sin B \\ = \cos (A-B) - \cos (A+B) \end{array}\right]$$

$$= \frac{1}{4} \left\{ (\cos 40° - \cos 60°) \sin 70°] \right\} = \frac{1}{4} \left\{ \sin 70° \cos 40° - \sin 70° \cos 60° \right\}$$

$$= \frac{1}{4} \left\{ \sin 70° \cos 40° - \frac{1}{2} \sin 70° \right\} = \frac{1}{8} \left\{ 2 \sin 70° \cos 40° - \sin 70° \right\}$$

$$= \frac{1}{8} \left\{ \sin (70° + 40°) + \sin (70° - 40°) - \sin 70° \right\} \quad \left[\begin{array}{l} \because 2 \sin A \cos B \\ = \sin (A+B) + \sin (A-B) \end{array}\right]$$

$$= \frac{1}{8} \left\{ \sin 110° + \sin 30° - \sin 70° \right\} = \frac{1}{8} \left\{ \sin (180° - 70°) + \sin 30° - \sin 70° \right\}$$

$$= \frac{1}{8} \left\{ \sin 70° + \frac{1}{2} - \sin 70° \right\} \quad \left[\because \sin (180 - x) = \sin x \therefore \sin (180°-70°) = \sin 70°\right]$$

$$= \frac{1}{8} \times \frac{1}{2} = \frac{1}{16} = \text{RHS}$$

ALITER LHS $= \sin 10° \sin 30° \sin 50° \sin 70°$

$$= \sin (90° - 80°) \sin (90° - 60°) \sin (90° - 40°) \sin (90° - 20°)$$

$$= \cos 80° \cos 60° \cos 40° \cos 20°$$

$$= \cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16} = \text{RHS}$$ [See Ex. 3]

EXAMPLE 5 *Prove that:* $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

SOLUTION LHS $= \sin 20° \sin 40° \sin 60° \sin 80°$

$$= \sin 60° (\sin 20° \sin 40°) \sin 80°$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{2} (2 \sin 20° \sin 40°) \sin 80°$$

$$= \frac{\sqrt{3}}{4} \left[\left\{ \cos (40° - 20°) - \cos (40° + 20°) \right\} \sin 80° \right] \quad \left[\begin{array}{l} \because 2 \sin A \sin B \\ = \cos (A-B) - \cos (A+B) \end{array}\right]$$

$$= \frac{\sqrt{3}}{4} \left\{ (\cos 20° - \cos 60°) \sin 80° \right\} = \frac{\sqrt{3}}{4} \left\{ \left(\cos 20° - \frac{1}{2} \right) \sin 80° \right\}$$

$$= \frac{\sqrt{3}}{8} \left\{ 2 \cos 20° \sin 80° - \sin 80° \right\}$$

$$= \frac{\sqrt{3}}{8} \left\{ \sin (80° + 20°) + \sin (80°-20°) - \sin 80° \right\} \quad \left[\begin{array}{l} \because 2 \sin A \cos B \\ = \sin (A+B) + \sin (A-B) \end{array}\right]$$

$$= \frac{\sqrt{3}}{8} \left\{ \sin 100° + \sin 60° - \sin 80° \right\} = \frac{\sqrt{3}}{8} \left\{ \sin (180° - 80°) + \frac{\sqrt{3}}{2} - \sin 80° \right\}$$

$$= \frac{\sqrt{3}}{8}\left\{\sin 80^\circ + \frac{\sqrt{3}}{2} - \sin 80^\circ\right\} = \frac{\sqrt{3}}{8} \times \frac{\sqrt{3}}{2} = \frac{3}{16} = \text{RHS} \qquad [\because \sin(180^\circ - 80^\circ) = \sin 80^\circ]$$

EXAMPLE 6 *Prove that:* $4 \cos 12^\circ \cos 48^\circ \cos 72^\circ = \cos 36^\circ$

SOLUTION LHS $= 4 \cos 12^\circ \cos 48^\circ \cos 72^\circ = 2(2 \cos 12^\circ \cos 48^\circ) \cos 72^\circ$

$= 2(\cos 60^\circ + \cos 36^\circ) \cos 72^\circ = 2 \cos 60^\circ \cos 72^\circ + 2 \cos 36^\circ \cos 72^\circ$

$= \cos 72^\circ + \cos 108^\circ + \cos 36^\circ = \cos 72^\circ + \cos(180^\circ - 72^\circ) + \cos 36^\circ$

$= \cos 72^\circ - \cos 72^\circ + \cos 36^\circ = \cos 36^\circ = \text{RHS}$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 7 *Prove that:* $\tan 20^\circ \tan 40^\circ \tan 80^\circ = \tan 60^\circ$

SOLUTION LHS $= \tan 20^\circ \tan 40^\circ \tan 80^\circ = \dfrac{\sin 20^\circ \sin 40^\circ \sin 80^\circ}{\cos 20^\circ \cos 40^\circ \cos 80^\circ}$

$$= \frac{(2 \sin 20^\circ \sin 40^\circ) \sin 80^\circ}{(2 \cos 20^\circ \cos 40^\circ) \cos 80^\circ} = \frac{(\cos 20^\circ - \cos 60^\circ) \sin 80^\circ}{(\cos 60^\circ + \cos 20^\circ) \cos 80^\circ}$$

$$= \frac{\sin 80^\circ \cos 20^\circ - (1/2) \sin 80^\circ}{(1/2) \cos 80^\circ + \cos 80^\circ \cos 20^\circ} = \frac{2 \sin 80^\circ \cos 20^\circ - \sin 80^\circ}{\cos 80^\circ + 2 \cos 80^\circ \cos 20^\circ}$$

$$= \frac{\sin 100^\circ + \sin 60^\circ - \sin 80^\circ}{\cos 80^\circ + \cos 100^\circ + \cos 60^\circ} = \frac{\sin(180^\circ - 80^\circ) + \sin 60^\circ - \sin 80^\circ}{\cos 80^\circ + \cos(180^\circ - 80^\circ) + \cos 60^\circ}$$

$$= \frac{\sin 80^\circ + \sin 60^\circ - \sin 80^\circ}{\cos 80^\circ - \cos 80^\circ + \cos 60^\circ} = \frac{\sin 60^\circ}{\cos 60^\circ} = \tan 60^\circ = \text{RHS}$$

EXAMPLE 8 *Prove that:* $\sin A \sin\left(\dfrac{\pi}{3} - A\right) \sin\left(\dfrac{\pi}{3} + A\right) = \dfrac{1}{4} \sin 3A$

SOLUTION LHS $= \sin A \sin\left(\dfrac{\pi}{3} - A\right) \sin\left(\dfrac{\pi}{3} + A\right) = \dfrac{1}{2} \sin A \left\{2 \sin\left(\dfrac{\pi}{3} - A\right) \sin\left(\dfrac{\pi}{3} + A\right)\right\}$

$$= \frac{1}{2} \sin A \left[\cos\left\{\left(\frac{\pi}{3} - A\right) - \left(\frac{\pi}{3} + A\right)\right\} - \cos\left\{\left(\frac{\pi}{3} - A\right) + \left(\frac{\pi}{3} + A\right)\right\}\right]$$

$$= \frac{1}{2} \sin A \left\{\cos(-2A) - \cos\frac{2\pi}{3}\right\} = \frac{1}{2} \sin A \left\{\cos 2A + \frac{1}{2}\right\}$$

$$= \frac{1}{2} \sin A \cos 2A + \frac{1}{4} \sin A = \frac{1}{4}(2 \sin A \cos 2A) + \frac{1}{4} \sin A$$

$$= \frac{1}{4}\left\{\sin(A + 2A) + \sin(A - 2A)\right\} + \frac{1}{4} \sin A$$

$$= \frac{1}{4}\left\{\sin 3A + \sin(-A)\right\} + \frac{1}{4} \sin A = \frac{1}{4} \sin 3A - \frac{1}{4} \sin A + \frac{1}{4} \sin A = \frac{1}{4} \sin 3A = \text{RHS}$$

EXAMPLE 9 *Prove that:* $\cos A \cos\left(\dfrac{\pi}{3} - A\right) \cos\left(\dfrac{\pi}{3} + A\right) = \dfrac{1}{4} \cos 3A.$

SOLUTION LHS $= \cos A \cos\left(\dfrac{\pi}{3} - A\right) \cos\left(\dfrac{\pi}{3} + A\right) = \dfrac{1}{2} \cos A \left\{2 \cos\left(\dfrac{\pi}{3} - A\right) \cos\left(\dfrac{\pi}{3} + A\right)\right\}$

$$= \frac{1}{2} \cos A \left[\cos\left\{\left(\frac{\pi}{3} - A\right) + \left(\frac{\pi}{3} + A\right)\right\} + \cos\left\{\left(\frac{\pi}{3} - A\right) - \left(\frac{\pi}{3} + A\right)\right\}\right]$$

$$= \frac{1}{2} \cos A \left\{\cos\frac{2\pi}{3} + \cos(-2A)\right\} = \frac{1}{2} \cos A \left\{-\frac{1}{2} + \cos 2A\right\}$$

$$= -\frac{1}{4}\cos A + \frac{1}{2}\cos A\cos 2A = -\frac{1}{4}\cos A + \frac{1}{4}(2\cos 2A\cos A)$$

$$= -\frac{1}{4}\cos A + \frac{1}{4}\left\{\cos(2A + A) + \cos(2A - A)\right\}$$

$$= -\frac{1}{4}\cos A + \frac{1}{4}(\cos 3A + \cos A) = \frac{1}{4}\cos 3A = \text{RHS}.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 10 *Prove that:* $4\sin x\sin\left(\frac{\pi}{3} + x\right)\sin\left(\frac{2\pi}{3} + x\right) = \sin 3x.$

SOLUTION LHS $= 4\sin x\sin\left(\frac{\pi}{3} + x\right)\sin\left(\frac{2\pi}{3} + x\right) = 2\sin x\left\{2\sin\left(\frac{2\pi}{3} + x\right)\sin\left(\frac{\pi}{3} + x\right)\right\}$

$$= 2\sin x\left[\cos\left\{\left(\frac{2\pi}{3} + x\right) - \left(\frac{\pi}{3} + x\right)\right\} - \cos\left\{\left(\frac{2\pi}{3} + x\right) + \left(\frac{\pi}{3} + x\right)\right\}\right]$$

$$= 2\sin x\left\{\cos\frac{\pi}{3} - \cos(\pi + 2x)\right\} = 2\sin x\left\{\frac{1}{2} + \cos 2x\right\}$$

$$= \sin x + 2\sin x\cos 2x = \sin x + \{\sin(x + 2x) + \sin(x - 2x)\}$$

$$= \sin x + \sin 3x + \sin(-x) = \sin x + \sin 3x - \sin x = \sin 3x = \text{RHS}$$

EXAMPLE 11 *Show that:* $\tan\left(\frac{\pi}{3} + x\right)\tan\left(\frac{\pi}{3} - x\right) = \frac{2\cos 2x + 1}{2\cos 2x - 1}.$

SOLUTION LHS $= \tan\left(\frac{\pi}{3} + x\right)\tan\left(\frac{\pi}{3} - x\right) = \frac{\sin\left(\frac{\pi}{3} + x\right)\sin\left(\frac{\pi}{3} - x\right)}{\cos\left(\frac{\pi}{3} + x\right)\cos\left(\frac{\pi}{3} - x\right)}$

$$= \frac{2\sin\left(\frac{\pi}{3} + x\right)\sin\left(\frac{\pi}{3} - x\right)}{2\cos\left(\frac{\pi}{3} + x\right)\cos\left(\frac{\pi}{3} - x\right)}$$

$$= \frac{\cos\left\{\left(\frac{\pi}{3} + x\right) - \left(\frac{\pi}{3} - x\right)\right\} - \cos\left\{\left(\frac{\pi}{3} + x\right) + \left(\frac{\pi}{3} - x\right)\right\}}{\cos\left\{\left(\frac{\pi}{3} + x\right) + \left(\frac{\pi}{3} - x\right)\right\} + \cos\left\{\left(\frac{\pi}{3} + x\right) - \left(\frac{\pi}{3} - x\right)\right\}}$$

$$= \frac{\cos 2x - \cos\frac{2\pi}{3}}{\cos\frac{2\pi}{3} + \cos 2x} = \frac{\cos 2x + \frac{1}{2}}{-\frac{1}{2} + \cos 2x} = \frac{2\cos 2x + 1}{2\cos 2x - 1} = \text{RHS}$$

EXAMPLE 12 *If* $\alpha + \beta = 90°$, *find the maximum and minimum values of* $\sin\alpha\sin\beta$.

SOLUTION Let $y = \sin\alpha\sin\beta$. Then,

$$y = \frac{1}{2}(2\sin\alpha\sin\beta) = \frac{1}{2}\{\cos(\alpha - \beta) - \cos(\alpha + \beta)\} = \frac{1}{2}\{\cos(\alpha - \beta) - \cos 90°\} = \frac{1}{2}\cos(\alpha - \beta)$$

We know that

$$-1 \le \cos(\alpha - \beta) \le 1 \Rightarrow \frac{-1}{2} \le \frac{1}{2}\cos(\alpha - \beta) \le \frac{1}{2} \Rightarrow \frac{-1}{2} \le y \le \frac{1}{2} \Rightarrow \frac{-1}{2} \le \sin\alpha\sin\beta \le \frac{1}{2}$$

Hence, $-\frac{1}{2}$ and $\frac{1}{2}$ are respectively the minimum and maximum values of $\sin\alpha \sin\beta$.

EXERCISE 8.1

BASIC

1. Express each of the following as the sum or difference of sines and cosines:
   (i) $2 \sin 3x \cos x$ (ii) $2 \cos 3x \sin 2x$ (iii) $2 \sin 4x \sin 3x$ (iv) $2 \cos 7x \cos 3x$
2. Prove that:
   (i) $2 \sin \frac{5\pi}{12} \sin \frac{\pi}{12} = \frac{1}{2}$ (ii) $2 \cos \frac{5\pi}{12} \cos \frac{\pi}{12} = \frac{1}{2}$ (iii) $2 \sin \frac{5\pi}{12} \cos \frac{\pi}{12} = \frac{\sqrt{3}+2}{2}$
3. Show that:
   (i) $\sin 50° \cos 85° = \frac{1-\sqrt{2} \sin 35°}{2\sqrt{2}}$ (ii) $\sin 25° \cos 115° = \frac{1}{2}(\sin 140° - 1)$
4. Prove that: $4 \cos x \cos\left(\frac{\pi}{3}+x\right)\cos\left(\frac{\pi}{3}-x\right) = \cos 3x$
5. Prove that :
   (i) $\cos 10° \cos 30° \cos 50° \cos 70° = \frac{3}{16}$ (ii) $\cos 40° \cos 80° \cos 160° = -\frac{1}{8}$
   (iii) $\sin 20° \sin 40° \sin 80° = \frac{\sqrt{3}}{8}$ (iv) $\cos 20° \cos 40° \cos 80° = \frac{1}{8}$
   (v) $\tan 20° \tan 40° \tan 60° \tan 80° = 3$ (vi) $\tan 20° \tan 30° \tan 40° \tan 80° = 1$
   (vii) $\sin 10° \sin 50° \sin 60° \sin 70° = \frac{\sqrt{3}}{16}$ (viii) $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

BASED ON LOTS

6. Show that :
   (i) $\sin A \sin (B-C) + \sin B \sin (C-A) + \sin C \sin (A-B) = 0$
   (ii) $\sin (B-C) \cos (A-D) + \sin (C-A) \cos (B-D) + \sin (A-B) \cos (C-D) = 0$

BASED ON HOTS

7. Prove that : $\tan x \tan\left(\frac{\pi}{3}-x\right)\tan\left(\frac{\pi}{3}+x\right) = \tan 3x$.
8. If $\alpha+\beta = \frac{\pi}{2}$, show that the maximum value of $\cos\alpha \cos\beta$ is $\frac{1}{2}$.

ANSWERS

1. (i) $\sin 4x + \sin 2x$ (ii) $\sin 5x - \sin x$ (iii) $\cos x - \cos 7x$ (iv) $\cos 10x + \cos 4x$

8.3 FORMULAE TO TRANSFORM THE SUM OR DIFFERENCE INTO PRODUCT

In the previous section, we have used the following formulae:

$$\sin (A+B) + \sin (A-B) = 2 \sin A \cos B, \quad \sin (A+B) - \sin (A-B) = 2 \cos A \sin B$$
$$\cos (A+B) + \cos (A-B) = 2 \cos A \cos B \text{ and, } \cos (A-B) - \cos (A+B) = 2 \sin A \sin B.$$

Let $A + B = C$ and $A - B = D$. Then, $A = \frac{C+D}{2}$ and $B = \frac{C-D}{2}$.

Substituting the values of A, B, C and D in the above formulae, we get

$$\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \qquad ...(i)$$

$$\sin C - \sin D = 2 \sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right) \quad ...(ii)$$

$$\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \quad ...(iii)$$

$$\cos D - \cos C = 2 \sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

or, $$\cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \quad ...(iv)$$

or, $$\cos C - \cos D = 2 \sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$$

These four formulae are used to convert the sum or difference of two sines or two cosines into the product of sines and cosines.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Express each of the following as a product:*

(i) $\sin 4x + \sin 2x$ (ii) $\sin 6x - \sin 2x$ (iii) $\cos 4x + \cos 8x$ (iv) $\cos 6x - \cos 8x$

SOLUTION (i) $\sin 4x + \sin 2x$

$$= 2 \sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) \quad \left[\because \sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}\right]$$

$$= 2 \sin 3x \cos x$$

(ii) $\sin 6x - \sin 2x$

$$= 2 \sin\left(\frac{6x-2x}{2}\right)\cos\left(\frac{6x+2x}{2}\right) \quad \left[\because \sin C - \sin D = 2 \sin \frac{C-D}{2} \cos \frac{C+D}{2}\right]$$

$$= 2 \sin 2x \cos 4x$$

(iii) $\cos 4x + \cos 8x$

$$= 2 \cos\left(\frac{8x+4x}{2}\right)\cos\left(\frac{8x-4x}{2}\right) \quad \left[\because \cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}\right]$$

$$= 2 \cos 6x \cos 2x$$

(iv) $\cos 6x - \cos 8x$

$$= 2 \sin\left(\frac{6x+8x}{2}\right)\sin\left(\frac{8x-6x}{2}\right) \quad \left[\because \cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}\right]$$

$$= 2 \sin 7x \sin x$$

EXAMPLE 2 *Prove that:* $\cos 18° - \sin 18° = \sqrt{2} \sin 27°$

SOLUTION LHS $= \cos 18° - \sin 18° = \cos 18° - \cos 72°$ $[\because \sin 18° = \sin (90° - 72°) = \cos 72°]$

$$= 2 \sin\left(\frac{18° + 72°}{2}\right)\sin\left(\frac{72° - 18°}{2}\right) = 2 \sin 45° \sin 27° = \sqrt{2} \sin 27° = \text{RHS}$$

EXAMPLE 3 *Prove that:*

(i) $\dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \tan A$ (ii) $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$ [NCERT]

(iii) $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\left(\dfrac{A+B}{2}\right)$ (iv) $\dfrac{\cos 7A+\cos 5A}{\sin 7A-\sin 5A}=\cot A$ [NCERT]

SOLUTION (i) $\text{LHS}=\dfrac{\sin 5A-\sin 3A}{\cos 5A+\cos 3A}=\dfrac{2\sin\left(\dfrac{5A-3A}{2}\right)\cos\left(\dfrac{5A+3A}{2}\right)}{2\cos\left(\dfrac{5A+3A}{2}\right)\cos\left(\dfrac{5A-3A}{2}\right)}=\dfrac{2\sin A\cos 4A}{2\cos 4A\cos A}$

$=\tan A=\text{RHS}$

(ii) $\text{LHS}=\dfrac{\sin 3A+\sin A}{\cos 3A+\cos A}=\dfrac{2\sin\left(\dfrac{3A+A}{2}\right)\cos\left(\dfrac{3A-A}{2}\right)}{2\cos\left(\dfrac{3A+A}{2}\right)\cos\left(\dfrac{3A-A}{2}\right)}=\dfrac{\sin 2A\cos A}{\cos 2A\cos A}=\tan 2A=\text{RHS}$

(iii) $\text{LHS}=\dfrac{\sin A+\sin B}{\cos A+\cos B}=\dfrac{2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)}{2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)}=\tan\left(\dfrac{A+B}{2}\right)=\text{RHS}$

(iv) $\text{LHS}=\dfrac{\cos 7A+\cos 5A}{\sin 7A-\sin 5A}=\dfrac{2\cos\left(\dfrac{7A+5A}{2}\right)\cos\left(\dfrac{7A-5A}{2}\right)}{2\sin\left(\dfrac{7A-5A}{2}\right)\cos\left(\dfrac{7A+5A}{2}\right)}=\dfrac{2\cos 6A\cos A}{2\sin A\cos 6A}=\cot A=\text{RHS}$

EXAMPLE 4 *Prove that:*

(i) $\dfrac{\cos 9x-\cos 5x}{\sin 17x-\sin 3x}=-\dfrac{\sin 2x}{\cos 10x}$ [NCERT] (ii) $\dfrac{\sin 5x+\sin 3x}{\cos 5x+\cos 3x}=\tan 4x$ [NCERT]

(iii) $(\sin 3x+\sin x)\sin x+(\cos 3x-\cos x)\cos x=0$ [NCERT]

SOLUTION (i) $\text{LHS}=\dfrac{\cos 9x-\cos 5x}{\sin 17x-\sin 3x}=\dfrac{-2\sin\left(\dfrac{9x+5x}{2}\right)\sin\left(\dfrac{9x-5x}{2}\right)}{2\sin\left(\dfrac{17x-3x}{2}\right)\cos\left(\dfrac{17x+3x}{2}\right)}=\dfrac{-2\sin 7x\sin 2x}{2\sin 7x\cos 10x}$

$=\dfrac{-\sin 2x}{\cos 10x}=\text{RHS}$

(ii) $\text{LHS}=\dfrac{\sin 5x+\sin 3x}{\cos 5x+\cos 3x}=\dfrac{2\sin\left(\dfrac{5x+3x}{2}\right)\cos\left(\dfrac{5x-3x}{2}\right)}{2\cos\left(\dfrac{5x+3x}{2}\right)\cos\left(\dfrac{5x-3x}{2}\right)}=\dfrac{2\sin 4x\cos x}{2\cos 4x\cos x}=\tan 4x=\text{RHS}$

(iii) $\text{LHS}=(\sin 3x+\sin x)\sin x+(\cos 3x-\cos x)\cos x$

$=\left\{2\sin\left(\dfrac{3x+x}{2}\right)\cos\left(\dfrac{3x-x}{2}\right)\right\}\sin x+\left\{-2\sin\left(\dfrac{3x+x}{2}\right)\sin\left(\dfrac{3x-x}{2}\right)\right\}\cos x$

$=2\sin 2x\cos x\sin x-2\sin 2x\sin x\cos x=0=\text{RHS}$

EXAMPLE 5 *Prove that:* $\cot 4x(\sin 5x+\sin 3x)=\cot x(\sin 5x-\sin 3x)$ [NCERT]

SOLUTION $\text{LHS}=\cot 4x(\sin 5x+\sin 3x)=\cot 4x\times 2\sin\left(\dfrac{5x+3x}{2}\right)\cos\left(\dfrac{5x-3x}{2}\right)$

$=\dfrac{\cos 4x}{\sin 4x}\times 2\sin 4x\cos x=2\cos 4x\cos x$ ...(i)

$$\text{RHS} = \cot x (\sin 5x - \sin 3x) = \cot x \times 2 \sin\left(\frac{5x-3x}{2}\right)\cos\left(\frac{5x+3x}{2}\right)$$

$$= \frac{\cos x}{\sin x} \times 2 \sin x \cos 4x = 2 \cos 4x \cos x \qquad \text{...(ii)}$$

From (i) and (ii), we obtain: LHS = RHS.

EXAMPLE 6 *Prove that:* $\sin x + \sin 3x + \sin 5x + \sin 7x = 4 \cos x \cos 2x \sin 4x$ [NCERT]

SOLUTION $\text{LHS} = \sin x + \sin 3x + \sin 5x + \sin 7x$

$$= (\sin 7x + \sin x) + (\sin 5x + \sin 3x)$$

$$= 2 \sin\left(\frac{7x+x}{2}\right)\cos\left(\frac{7x-x}{2}\right) + 2 \sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)$$

$$= 2 \sin 4x \cos 3x + 2 \sin 4x \cos x = 2 \sin 4x (\cos 3x + \cos x)$$

$$= 2 \sin 4x \times 2 \cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)$$

$$= 2 \sin 4x \times 2 \cos 2x \cos x = 4 \cos x \cos 2x \sin 4x = \text{RHS}$$

EXAMPLE 7 *Prove that:* $1 + \cos 2x + \cos 4x + \cos 6x = 4 \cos x \cos 2x \cos 3x$

SOLUTION $\text{LHS} = 1 + \cos 2x + \cos 4x + \cos 6x = (\cos 0x + \cos 2x) + (\cos 4x + \cos 6x)$

$$= 2 \cos x \cos x + 2 \cos 5x \cos x = 2 \cos x (\cos x + \cos 5x)$$

$$= 2 \cos x (2 \cos 3x \cos 2x) = 4 \cos x \cos 2x \cos 3x = \text{RHS}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 8 *Prove that:*

(i) $(\sin 3A + \sin A) \sin A + (\cos 3A - \cos A) \cos A = 0$

(ii) $\cos 2x \cos \frac{x}{2} - \cos 3x \cos \frac{9x}{2} = \sin 5x \sin \frac{5x}{2}$ [NCERT]

(iii) $\sin \alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right) = 0$

SOLUTION (i) $\text{LHS} = (\sin 3A + \sin A) \sin A + (\cos 3A - \cos A) \cos A$

$$= \left\{2 \sin\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)\right\} \sin A + \left\{-2 \sin\left(\frac{3A+A}{2}\right) \sin\left(\frac{3A-A}{2}\right)\right\} \cos A$$

$$= 2 \sin 2A \cos A \sin A - 2 \sin 2A \sin A \cos A = 0 = \text{RHS}$$

(ii) $\text{LHS} = \cos 2x \cos \frac{x}{2} - \cos 3x \cos \frac{9x}{2} = \frac{1}{2}\left\{2 \cos 2x \cos \frac{x}{2} - 2 \cos 3x \cos \frac{9x}{2}\right\}$

$$= \frac{1}{2}\left[\cos\left\{\left(2x + \frac{x}{2}\right) + \cos\left(2x - \frac{x}{2}\right)\right\} - \left\{\cos\left(3x + \frac{9x}{2}\right) + \cos\left(\frac{9x}{2} - 3x\right)\right\}\right]$$

[Using: $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$]

$$= \frac{1}{2}\left\{\cos \frac{5x}{2} + \cos \frac{3x}{2} - \cos \frac{15x}{2} - \cos \frac{3x}{2}\right\} = \frac{1}{2}\left\{\cos \frac{5x}{2} - \cos \frac{15x}{2}\right\}$$

$$= \frac{1}{2}\left\{2 \sin\left(\frac{\frac{5x}{2} + \frac{15x}{2}}{2}\right) \sin\left(\frac{\frac{15x}{2} - \frac{5x}{2}}{2}\right)\right\} \quad \left[\because \cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}\right]$$

$$= \sin 5x \sin \frac{5x}{2} = \text{RHS}$$

(iii) LHS $= \sin\alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right) = \sin\alpha + \left\{\sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right)\right\}$

$$= \sin\alpha + \left\{2\sin\left(\frac{\alpha + \frac{2\pi}{3} + \alpha + \frac{4\pi}{3}}{2}\right)\cos\left(\frac{\alpha + \frac{4\pi}{3} - \alpha - \frac{2\pi}{3}}{2}\right)\right\}$$

$$= \sin\alpha + 2\sin(\alpha + \pi)\cos\frac{\pi}{3} = \sin\alpha + 2(-\sin\alpha)\left(\frac{1}{2}\right) = \sin\alpha - \sin\alpha = 0 = \text{RHS}$$

EXAMPLE 9 *Prove that:*

(i) $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\left(\frac{\alpha - \beta}{2}\right)$ [NCERT]

(ii) $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\frac{\alpha - \beta}{2}\right)$ [NCERT]

(iii) $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha + \beta + \gamma) = 4\cos\frac{\alpha + \beta}{2}\cos\frac{\beta + \gamma}{2}\cos\frac{\gamma + \alpha}{2}$

SOLUTION (i) LHS $= (\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$

$$= \left\{2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)\right\}^2 + \left\{2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)\right\}^2$$

$$= 4\cos^2\left(\frac{\alpha + \beta}{2}\right)\cos^2\left(\frac{\alpha - \beta}{2}\right) + 4\sin^2\left(\frac{\alpha + \beta}{2}\right)\cos^2\left(\frac{\alpha - \beta}{2}\right)$$

$$= 4\cos^2\left(\frac{\alpha - \beta}{2}\right)\left\{\cos^2\left(\frac{\alpha + \beta}{2}\right) + \sin^2\left(\frac{\alpha + \beta}{2}\right)\right\} = 4\cos^2\left(\frac{\alpha - \beta}{2}\right) = \text{RHS}$$

(ii) LHS $= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

$$= \left\{-2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)\right\}^2 + \left\{2\sin\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right)\right\}^2$$

$$= 4\sin^2\left(\frac{\alpha + \beta}{2}\right)\sin^2\left(\frac{\alpha - \beta}{2}\right) + 4\sin^2\left(\frac{\alpha - \beta}{2}\right)\cos^2\left(\frac{\alpha + \beta}{2}\right)$$

$$= 4\sin^2\left(\frac{\alpha - \beta}{2}\right)\left\{\sin^2\left(\frac{\alpha + \beta}{2}\right) + \cos^2\left(\frac{\alpha + \beta}{2}\right)\right\} = 4\sin^2\left(\frac{\alpha - \beta}{2}\right) = \text{RHS}$$

(iii) LHS $= \cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha + \beta + \gamma) = (\cos\alpha + \cos\beta) + [\cos\gamma + \cos(\alpha + \beta + \gamma)]$

$$= 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) + 2\cos\left(\frac{\alpha + \beta + \gamma + \gamma}{2}\right)\cos\left(\frac{\alpha + \beta + \gamma - \gamma}{2}\right)$$

$$= 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) + 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha + \beta + 2\gamma}{2}\right)$$

$$= 2\cos\left(\frac{\alpha + \beta}{2}\right)\left\{\cos\left(\frac{\alpha - \beta}{2}\right) + \cos\left(\frac{\alpha + \beta + 2\gamma}{2}\right)\right\}$$

$$= 2\cos\left(\frac{\alpha + \beta}{2}\right)\left\{2\cos\left(\frac{\frac{\alpha - \beta}{2} + \frac{\alpha + \beta + 2\gamma}{2}}{2}\right)\cos\left(\frac{\frac{\alpha + \beta + 2\gamma}{2} - \frac{\alpha - \beta}{2}}{2}\right)\right\}$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\right\}$$

$$= 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right) = \text{RHS}$$

EXAMPLE 10 *Prove that:* $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$ [NCERT]

SOLUTION $\text{LHS} = \dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \dfrac{(\cos 4x + \cos 2x) + \cos 3x}{(\sin 4x + \sin 2x) + \sin 3x}$

$$= \frac{2\cos\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) + \cos 3x}{2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) + \sin 3x}$$

$$= \frac{2\cos 3x\cos x + \cos 3x}{2\sin 3x\cos x + \sin 3x} = \frac{\cos 3x(2\cos x + 1)}{\sin 3x(2\cos x + 1)} = \frac{\cos 3x}{\sin 3x} = \cot 3x = \text{RHS}$$

EXAMPLE 11 *Prove that:* $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$

SOLUTION $\text{LHS} = \dfrac{(\sin 7A + \sin A) + (\sin 5A + \sin 3A)}{(\cos 7A + \cos A) + (\cos 5A + \cos 3A)}$

$$= \frac{2\sin\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right) + 2\sin\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}{2\cos\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right) + 2\cos\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}$$

$$= \frac{\sin 4A\cos 3A + \sin 4A\cos A}{\cos 4A\cos 3A + \cos 4A\cos A} = \frac{\sin 4A(\cos 3A + \cos A)}{\cos 4A(\cos 3A + \cos A)} = \tan 4A = \text{RHS}$$

EXAMPLE 12 *Prove that:* $\dfrac{\cos 8A\cos 5A - \cos 12A\cos 9A}{\sin 8A\cos 5A + \cos 12A\sin 9A} = \tan 4A$

SOLUTION $\text{LHS} = \dfrac{2\cos 8A\cos 5A - 2\cos 12A\cos 9A}{2\sin 8A\cos 5A + 2\cos 12A\sin 9A}$

$$= \frac{\{\cos(8A+5A) + \cos(8A-5A)\} - \{\cos(12A+9A) + \cos(12A-9A)\}}{\{\sin(8A+5A) + \sin(8A-5A)\} + \{\sin(9A+12A) + \sin(9A-12A)\}}$$

$$= \frac{\{\cos 13A + \cos 3A\} - \{\cos 21A + \cos 3A\}}{\{\sin 13A + \sin 3A\} + \{\sin 21A + \sin(-3A)\}}$$

$$= \frac{(\cos 13A + \cos 3A) - (\cos 21A + \cos 3A)}{(\sin 13A + \sin 3A) + (\sin 21A - \sin 3A)} = \frac{\cos 13A - \cos 21A}{\sin 13A + \sin 21A}$$

$$= \frac{2\sin\left(\frac{13A+21A}{2}\right)\sin\left(\frac{21A-13A}{2}\right)}{2\sin\left(\frac{3A+21A}{2}\right)\cos\left(\frac{21A-13A}{2}\right)} = \frac{\sin 17A\sin 4A}{\sin 17A\cos 4A} = \tan 4A = \text{RHS}$$

EXAMPLE 13 *Prove that:* $\dfrac{\cos 2A\cos 3A - \cos 2A\cos 7A + \cos A\cos 10A}{\sin 4A\sin 3A - \sin 2A\sin 5A + \sin 4A\sin 7A} = \cot 6A\cot 5A$

SOLUTION LHS $= \dfrac{2\cos 3A\cos 2A - 2\cos 7A\cos 2A + 2\cos 10A\cos A}{2\sin 4A\sin 3A - 2\sin 5A\sin 2A + 2\sin 7A\sin 4A}$

$$= \frac{(\cos 5A + \cos A) - (\cos 9A + \cos 5A) + (\cos 11A + \cos 9A)}{(\cos A - \cos 7A) - (\cos 3A - \cos 7A) + (\cos 3A - \cos 11A)}$$

$$= \frac{\cos A + \cos 11A}{\cos A - \cos 11A} = \frac{2\cos\left(\frac{11A + A}{2}\right)\cos\left(\frac{11A - A}{2}\right)}{2\sin\left(\frac{A + 11A}{2}\right)\sin\left(\frac{11A - A}{2}\right)}$$

$$= \frac{\cos 6A\cos 5A}{\sin 6A\sin 5A} = \cot 6A\cot 5A = \text{RHS}$$

EXAMPLE 14 *Prove that:* $\dfrac{\sin(A - C) + 2\sin A + \sin(A + C)}{\sin(B - C) + 2\sin B + \sin(B + C)} = \dfrac{\sin A}{\sin B}$

SOLUTION LHS $= \dfrac{\sin(A - C) + \sin(A + C) + 2\sin A}{\sin(B - C) + \sin(B + C) + 2\sin B}$

$$= \frac{2\sin\left(\frac{A - C + A + C}{2}\right)\cos\left(\frac{A + C - A + C}{2}\right) + 2\sin A}{2\sin\left(\frac{B + C + B - C}{2}\right)\cos\left(\frac{B + C - B + C}{2}\right) + 2\sin B}$$

$$= \frac{2\sin A\cos C + 2\sin A}{2\sin B\cos C + 2\sin B} = \frac{2\sin A(\cos C + 1)}{2\sin B(\cos C + 1)} = \frac{\sin A}{\sin B} = \text{RHS}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 15 *If* $\sin x = n\sin(x + 2\alpha)$, *prove that* $\tan(x + \alpha) = \dfrac{1 + n}{1 - n}\tan\alpha$.

SOLUTION We have,

$$\sin x = n\sin(x + 2\alpha)$$

$$\Rightarrow \quad \frac{\sin(x + 2\alpha)}{\sin x} = \frac{1}{n} \Rightarrow \frac{\sin(x + 2\alpha) + \sin x}{\sin(x + 2\alpha) - \sin x} = \frac{1 + n}{1 - n} \quad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \quad \frac{2\sin(x + \alpha)\cos\alpha}{2\sin\alpha\cos(x + \alpha)} = \frac{1 + n}{1 - n} \Rightarrow \tan(x + \alpha) = \frac{1 + n}{1 - n}\tan\alpha$$

EXAMPLE 16 *Prove that:*

$$\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n = \begin{cases} 2\cot^n\left(\frac{A - B}{2}\right), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

SOLUTION LHS $= \left(\dfrac{2\cos\frac{A + B}{2}\cos\frac{A - B}{2}}{2\sin\frac{A - B}{2}\cos\frac{A + B}{2}}\right)^n + \left(\dfrac{2\sin\frac{A + B}{2}\cos\frac{A - B}{2}}{-2\sin\frac{A + B}{2}\sin\frac{A - B}{2}}\right)^n$

$$= \left\{\cot\left(\frac{A - B}{2}\right)\right\}^n + \left\{-\cot\left(\frac{A - B}{2}\right)\right\}^n = \cot^n\left(\frac{A - B}{2}\right) + (-1)^n\cot^n\left(\frac{A - B}{2}\right)$$

$$= \cot^n\left(\frac{A - B}{2}\right)^n\left\{1 + (-1)^n\right\} = \begin{cases} 2\cot^n\left(\frac{A - B}{2}\right), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

EXAMPLE 17 *If three angles A, B and C are in A.P., prove that:* $\cot B = \dfrac{\sin A - \sin C}{\cos C - \cos A}$.

SOLUTION RHS $= \dfrac{\sin A - \sin C}{\sin A + \sin C} = \dfrac{2 \sin \frac{A-C}{2} \cos \frac{A+C}{2}}{2 \sin \frac{A+C}{2} \sin \frac{A-C}{2}}$

$$= \cot\left(\frac{A+C}{2}\right) = \cot B = \text{LHS} \qquad [\because A, B, C \text{ are in A.P. } \therefore 2B = A + C]$$

EXAMPLE 18 *If* $\sin \theta + \sin \phi = \sqrt{3}(\cos \phi - \cos \theta)$, *prove that* $\sin 3\theta + \sin 3\phi = 0$

SOLUTION We have, $\sin \theta + \sin \phi = \sqrt{3}(\cos \phi - \cos \theta)$ [NCERT EXEMPLAR]

$$\Rightarrow \quad 2 \sin \frac{\theta+\phi}{2} \cos \frac{\theta-\phi}{2} = 2\sqrt{3} \sin \frac{\theta+\phi}{2} \sin \frac{\theta-\phi}{2}$$

$$\Rightarrow \quad \left\{\cos \frac{\theta-\phi}{2} - \sqrt{3} \sin \frac{\theta-\phi}{2}\right\} \sin\left(\frac{\theta+\phi}{2}\right) = 0$$

$$\Rightarrow \quad \sin\left(\frac{\theta+\phi}{2}\right) = 0 \text{ or, } \cos\left(\frac{\theta-\phi}{2}\right) - \sqrt{3} \sin\left(\frac{\theta-\phi}{2}\right) = 0$$

$$\Rightarrow \quad \sin\left(\frac{\theta+\phi}{2}\right) = 0 \text{ or, } \tan\left(\frac{\theta-\phi}{2}\right) = \frac{1}{\sqrt{3}} = \tan \frac{\pi}{6}$$

$$\Rightarrow \quad \frac{\theta+\phi}{2} = 0 \text{ or, } \frac{\theta-\phi}{2} = \frac{\pi}{6} \Rightarrow \theta = -\phi \text{ or, } \theta - \phi = \frac{\pi}{3}$$

Case I When $\theta = -\phi$: In this case, we obtain

$$\sin 3\theta + \sin 3\phi = \sin 3(-\phi) + \sin 3\phi = -\sin 3\phi + \sin 3\phi = 0$$

Case II When $\theta - \phi = \dfrac{\pi}{3}$: In this case, we obtain

$$\theta - \phi = \frac{\pi}{3} \Rightarrow 3\theta - 3\phi = \pi \Rightarrow 3\theta = \pi + 3\phi$$

$$\therefore \quad \sin 3\theta + \sin 3\phi = \sin(\pi + 3\phi) + \sin 3\phi = -\sin 3\phi + \sin 3\phi = 0$$

EXAMPLE 19 *If* $\dfrac{\sin(x+\alpha)}{\cos(x-\alpha)} = \dfrac{1-m}{1+m}$, *prove that* $\tan\left(\dfrac{\pi}{4} - x\right) \tan\left(\dfrac{\pi}{4} - \alpha\right) = m$.

SOLUTION We have,

$$\frac{\sin(x+\alpha)}{\cos(x-\alpha)} = \frac{1-m}{1+m}$$

$$\Rightarrow \quad \frac{\sin(x+\alpha) + \cos(x-\alpha)}{\sin(x+\alpha) - \cos(x-\alpha)} = \frac{2}{-2m} \qquad \text{[Using componendo-dividendo]}$$

$$\Rightarrow \quad \frac{\sin(x+\alpha) + \sin\left\{\frac{\pi}{2} - (x-\alpha)\right\}}{\sin(x+\alpha) - \sin\left\{\frac{\pi}{2} - (x-\alpha)\right\}} = -\frac{1}{m}$$

$$\Rightarrow \quad \frac{2\sin\left(\dfrac{x+\alpha+\frac{\pi}{2}-x+\alpha}{2}\right)\cos\left(\dfrac{x+\alpha-\frac{\pi}{2}+x-\alpha}{2}\right)}{2\sin\left(\dfrac{x+\alpha-\frac{\pi}{2}+x-\alpha}{2}\right)\cos\left(\dfrac{x+\alpha+\frac{\pi}{2}-x+\alpha}{2}\right)} = -\frac{1}{m}$$

$$\Rightarrow \quad \frac{\sin\left(\frac{\pi}{4}+\alpha\right)\cos\left(-\frac{\pi}{4}+x\right)}{\sin\left(-\frac{\pi}{4}+x\right)\cos\left(\frac{\pi}{4}+\alpha\right)} = -\frac{1}{m} \Rightarrow \frac{\sin\left(\frac{\pi}{4}+\alpha\right)}{\cos\left(\frac{\pi}{4}+\alpha\right)} \cdot \frac{\cos\left(\frac{\pi}{4}-x\right)}{\sin\left(\frac{\pi}{4}-x\right)} = \frac{1}{m}$$

$$\Rightarrow \quad \tan\left(\frac{\pi}{4}+\alpha\right)\cot\left(\frac{\pi}{4}-x\right) = \frac{1}{m} \Rightarrow m = \cot\left(\frac{\pi}{4}+\alpha\right)\tan\left(\frac{\pi}{4}-x\right)$$

$$\Rightarrow \quad m = \tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}+\alpha\right)\right\}\tan\left(\frac{\pi}{4}-x\right) \Rightarrow m = \tan\left(\frac{\pi}{4}-\alpha\right)\tan\left(\frac{\pi}{4}-x\right)$$

EXAMPLE 20 *If* $a\sin x = b\sin\left(x+\frac{2\pi}{3}\right) = c\sin\left(x+\frac{4\pi}{3}\right)$, *prove that* $ab+bc+ca=0$.

SOLUTION We have,

$$a\sin x = b\sin\left(x+\frac{2\pi}{3}\right) = c\sin\left(x+\frac{4\pi}{3}\right) = \lambda \text{ (say)}$$

$$\Rightarrow \quad \frac{\lambda}{a} = \sin x,\ \frac{\lambda}{b} = \sin\left(x+\frac{2\pi}{3}\right) \text{ and } \frac{\lambda}{c} = \sin\left(x+\frac{4\pi}{3}\right)$$

$$\therefore \quad \frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c} = \sin x + \sin\left(x+\frac{2\pi}{3}\right) + \sin\left(x+\frac{4\pi}{3}\right) = \left\{\sin\left(x+\frac{4\pi}{3}\right)+\sin x\right\} + \sin\left(x+\frac{2\pi}{3}\right)$$

$$\Rightarrow \quad \frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c} = 2\sin\left(x+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3} + \sin\left(x+\frac{2\pi}{3}\right) = -\sin\left(x+\frac{2\pi}{3}\right)+\sin\left(x+\frac{2\pi}{3}\right) = 0$$

$$\Rightarrow \quad \lambda\left(\frac{bc+ca+ab}{abc}\right) = 0 \Rightarrow ab+bc+ca = 0$$

EXAMPLE 21 *If* $\sin(y+z-x)$, $\sin(z+x-y)$, $\sin(x+y-z)$ *are in A.P., prove that* $\tan x$, $\tan y$, $\tan z$ *are also in A.P.*

SOLUTION It is given that $\sin(y+z-x)$, $\sin(z+x-y)$ and $\sin(x+y-z)$ are in A.P.

$\therefore \quad \sin(z+x-y)-\sin(y+z-x) = \sin(x+y-z)-\sin(z+x-y)$

$\Rightarrow \quad 2\sin(x-y)\cos z = 2\sin(y-z)\cos x$

$\Rightarrow \quad \sin(x-y)\cos z = \sin(y-z)\cos x$

$\Rightarrow \quad \sin x\cos y\cos z - \cos x\sin y\cos z = \sin y\cos z\cos x - \cos y\sin z\cos x$

$\Rightarrow \quad 2\sin y\cos x\cos z = \sin x\cos y\cos z + \cos x\cos y\sin z$

$\Rightarrow \quad 2\tan y = \tan x + \tan z$ [Dividing throughout by $\cos x\cos y\cos z$]

$\Rightarrow \quad \tan x$, $\tan y$, $\tan z$ are in A.P.

EXAMPLE 22 *If* $\frac{\tan(x+\alpha)}{a} = \frac{\tan(x+\beta)}{b} = \frac{\tan(x+\gamma)}{c}$, *prove that*

$$\frac{a+b}{a-b}\sin^2(\alpha-\beta) + \frac{b+c}{b-c}\sin^2(\beta-\gamma) + \frac{c+a}{c-a}\sin^2(\gamma-\alpha) = 0$$

SOLUTION We have,

$$\frac{\tan(x+\alpha)}{a} = \frac{\tan(x+\beta)}{b}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{\tan(x+\alpha)}{\tan(x+\beta)}$$

$$\Rightarrow \quad \frac{a+b}{a-b} = \frac{\tan(x+\alpha)+\tan(x+\beta)}{\tan(x+\alpha)-\tan(x+\beta)}$$ [Applying Componendo-dividendo]

$$\Rightarrow \quad \frac{a+b}{a-b} = \frac{\sin(2x+\alpha+\beta)}{\sin(\alpha-\beta)}$$ $\left[\because \frac{\tan A+\tan B}{\tan A-\tan B} = \frac{\sin(A+B)}{\sin(A-B)}\right]$

$$\Rightarrow \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta) = \sin(2x+\alpha+\beta)\sin(\alpha-\beta) = \frac{1}{2}\left\{2\sin(2x+\alpha+\beta)\sin(\alpha-\beta)\right\}$$

$$\Rightarrow \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta) = \frac{1}{2}\left\{\cos(2x+2\beta) - \cos(2x+2\alpha)\right\}$$

Similarly, we obtain

$$\frac{b+c}{b-c}\sin^2(\beta-\gamma) = \frac{1}{2}\left\{\cos(2x+2\gamma) - \cos(2x+2\beta)\right\}$$

and, $$\frac{c+a}{c-a}\sin^2(\gamma-\alpha) = \frac{1}{2}\left\{\cos(2x+2\alpha) - \cos(2x+2\gamma)\right\}$$

$$\therefore \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta) + \frac{b+c}{b-c}\sin^2(\beta-\gamma) + \frac{c+a}{c-a}\sin^2(\gamma-\alpha)$$

$$= \frac{1}{2}\left\{\cos(2x+2\beta) - \cos(2x+2\alpha) + \cos(2x+2\gamma) - \cos(2x+2\beta) + \cos(2x+2\alpha) - \cos(2x+2\gamma)\right\}$$

$$= \frac{1}{2} \times 0 = 0$$

EXAMPLE 23 *Prove that :* $\dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x} = 2\cos x$

SOLUTION $\cos 6x + 6\cos 4x + 15\cos 2x + 10$

$= (\cos 6x + \cos 4x) + (5\cos 4x + 5\cos 2x) + (10\cos 2x + 10)$

$= (\cos 6x + \cos 4x) + 5(\cos 4x + \cos 2x) + 10(\cos 2x + \cos 0x)$

$= 2\cos 5x\cos x + 5 \times 2\cos 3x\cos x + 10 \times 2\cos x\cos x = 2\cos x(\cos 5x + 5\cos 3x + 10\cos x)$

$$\therefore \quad \text{LHS} = \frac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x} = \frac{2\cos x(\cos 5x + 5\cos 3x + 10\cos x)}{\cos 5x + 5\cos 3x + 10\cos x}$$

$$= 2\cos x = \text{RHS}$$

EXERCISE 8.2

BASIC

1. Express each of the following as the product of sines and cosines:
   (i) $\sin 12x + \sin 4x$ (ii) $\sin 5x - \sin x$ (iii) $\cos 12x + \cos 8x$
   (iv) $\cos 12x - \cos 4x$ (v) $\sin 2x + \cos 4x$
2. Prove that:
   (i) $\sin 38° + \sin 22° = \sin 82°$ (ii) $\cos 100° + \cos 20° = \cos 40°$
   (iii) $\sin 50° + \sin 10° = \cos 20°$ (iv) $\sin 23° + \sin 37° = \cos 7°$

(v) $\sin 105° + \cos 105° = \cos 45°$ (vi) $\sin 40° + \sin 20° = \cos 10°$

3. Prove that:

(i) $\cos 55° + \cos 65° + \cos 175° = 0$ (ii) $\sin 50° - \sin 70° + \sin 10° = 0$

(iii) $\cos 80° + \cos 40° - \cos 20° = 0$ (iv) $\cos 20° + \cos 100° + \cos 140° = 0$

(v) $\sin \frac{5\pi}{18} - \cos \frac{4\pi}{9} = \sqrt{3} \sin \frac{\pi}{9}$ (vi) $\cos \frac{\pi}{12} - \sin \frac{\pi}{12} = \frac{1}{\sqrt{2}}$

(vii) $\sin 80° - \cos 70° = \cos 50°$ (viii) $\sin 51° + \cos 81° = \cos 21°$

4. Prove that:

(i) $\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2} \sin x$ [NCERT]

(ii) $\cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right) = \sqrt{2} \cos x$ [NCERT]

5. Prove that :

(i) $\sin 65° + \cos 65° = \sqrt{2} \cos 20°$ (ii) $\sin 47° + \cos 77° = \cos 17°$

6. Prove that :

(i) $\cos 3A + \cos 5A + \cos 7A + \cos 15A = 4 \cos 4A \cos 5A \cos 6A$

(ii) $\cos A + \cos 3A + \cos 5A + \cos 7A = 4 \cos A \cos 2A \cos 4A$

(iii) $\sin A + \sin 2A + \sin 4A + \sin 5A = 4 \cos \frac{A}{2} \cos \frac{3A}{2} \sin 3A$

(iv) $\sin 3A + \sin 2A - \sin A = 4 \sin A \cos \frac{A}{2} \cos \frac{3A}{2}$

(v) $\cos 20° \cos 100° + \cos 100° \cos 140° - \cos 140° \cos 200° = -\frac{3}{4}$

(vi) $\sin \frac{x}{2} \sin \frac{7x}{2} + \sin \frac{3x}{2} \sin \frac{11x}{2} = \sin 2x \sin 5x.$

(vii) $\cos x \cos \frac{x}{2} - \cos 3x \cos \frac{9x}{2} = \sin 4x \sin \frac{7x}{2}.$ [NCERT EXEMPLAR]

7. Prove that:

(i) $\frac{\sin A + \sin 3A}{\cos A - \cos 3A} = \cot A$ (ii) $\frac{\sin 9A - \sin 7A}{\cos 7A - \cos 9A} = \cot 8A$

(iii) $\frac{\sin A - \sin B}{\cos A + \cos B} = \tan \frac{A-B}{2}$ (iv) $\frac{\sin A + \sin B}{\sin A - \sin B} = \tan\left(\frac{A+B}{2}\right) \cot\left(\frac{A-B}{2}\right)$

(v) $\frac{\cos A + \cos B}{\cos B - \cos A} = \cot\left(\frac{A+B}{2}\right) \cot\left(\frac{A-B}{2}\right)$

BASED ON LOTS

8. Prove that:

(i) $\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$ (ii) $\frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \frac{\cos 5A}{\cos 3A}$

(iii) $\dfrac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A} = \cot 3A$ [NCERT]

(iv) $\dfrac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A} = \tan A$

(v) $\dfrac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \tan 8A$

(vi) $\dfrac{\sin 3A \cos 4A - \sin A \cos 2A}{\sin 4A \sin A + \cos 6A \cos A} = \tan 2A$

(vii) $\dfrac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A} = \tan 5A$

(viii) $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}$

(ix) $\dfrac{\sin(\theta+\phi) - 2\sin\theta + \sin(\theta-\phi)}{\cos(\theta+\phi) - 2\cos\theta + \cos(\theta-\phi)} = \tan\theta$

(x) $\dfrac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A} = \tan 6A$

(xi) $\dfrac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \cot 6A$

9. Prove that:

(i) $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\beta+\gamma}{2}\right)\sin\left(\dfrac{\gamma+\alpha}{2}\right)$

(ii) $\cos(A+B+C) + \cos(A-B+C) + \cos(A+B-C) + \cos(-A+B+C) = 4\cos A\cos B\cos C$

BASED ON HOTS

10. If $\cos A + \cos B = \dfrac{1}{2}$ and $\sin A + \sin B = \dfrac{1}{4}$, prove that: $\tan\left(\dfrac{A+B}{2}\right) = \dfrac{1}{2}$

11. If $\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$, prove that: $\tan A \tan B = \cot\dfrac{A+B}{2}$

12. If $\sin 2A = \lambda \sin 2B$, prove that : $\dfrac{\tan(A+B)}{\tan(A-B)} = \dfrac{\lambda+1}{\lambda-1}$

13. Prove that:

(i) $\dfrac{\cos(A+B+C) + \cos(-A+B+C) + \cos(A-B+C) + \cos(A+B-C)}{\sin(A+B+C) + \sin(-A+B+C) + \sin(A-B+C) - \sin(A+B-C)} = \cot C$

(ii) $\sin(B-C)\cos(A-D) + \sin(C-A)\cos(B-D) + \sin(A-B)\cos(C-D) = 0$

14. If $\dfrac{\cos(A-B)}{\cos(A+B)} + \dfrac{\cos(C+D)}{\cos(C-D)} = 0$, prove that $\tan A \tan B \tan C \tan D = -1$

15. If $\cos(\alpha+\beta)\sin(\gamma+\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta)$, prove that $\cot\alpha\cot\beta\cot\gamma = \cot\delta$

16. If $y\sin\phi = x\sin(2\theta+\phi)$, prove that $(x+y)\cot(\theta+\phi) = (y-x)\cot\theta$

17. If $\cos(A+B)\sin(C-D) = \cos(A-B)\sin(C+D)$, prove that $\tan A\tan B\tan C + \tan D = 0$

18. If $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$, prove that $xy+yz+zx=0$.

[NCERT EXEMPLAR]

19. If $m\sin\theta = n\sin(\theta+2\alpha)$, prove that $\tan(\theta+\alpha)\cot\alpha = \frac{m+n}{m-n}$. [NCERT EXEMPLAR]

ANSWERS

1. (i) $2\sin 8x\cos 4x$ (ii) $2\sin 2x\cos 3x$ (iii) $2\cos 10x\cos 2x$

(iv) $-2\sin 8x\sin 4x$ (v) $2\cos\left(\frac{\pi}{4}+x\right)\cos\left(\frac{\pi}{4}-3x\right)$

HINTS TO SELECTED PROBLEMS

18. Let $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right) = \lambda$

$\Rightarrow \quad \frac{\lambda}{x} = \cos\theta,\ \frac{\lambda}{y} = \cos\left(\theta+\frac{2\pi}{3}\right)$ and $\frac{\lambda}{z} = \cos\left(\theta+\frac{4\pi}{3}\right)$

$\therefore \quad \frac{\lambda}{x}+\frac{\lambda}{y}+\frac{\lambda}{z} = \cos\theta + \cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)$

$\Rightarrow \quad \lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \cos\theta + \left\{\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)\right\}$

$\Rightarrow \quad \lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \cos\theta + 2\cos(\theta+\pi)\cos\frac{\pi}{3} = \cos\theta - 2\times\frac{1}{2}\cos\theta = 0$

$\Rightarrow \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 0 \Rightarrow \frac{xy+yz+zx}{xyz} = 0 \Rightarrow xy+yz+zx=0.$

19. We have, $m\sin\theta = n\sin(\theta+2\alpha)$

$\Rightarrow \quad \frac{\sin(\theta+2\alpha)}{\sin\theta} = \frac{m}{n}$

$\Rightarrow \quad \frac{\sin(\theta+2\alpha)+\sin\theta}{\sin(\theta+2\alpha)-\sin\theta} = \frac{m+n}{m-n}$ [Applying componendo and dividendo]

$\Rightarrow \quad \frac{2\sin(\theta+\alpha)\cos\alpha}{2\sin\alpha\cos(\theta+\alpha)} = \frac{m+n}{m-n} \Rightarrow \frac{\tan(\theta+\alpha)}{\tan\alpha} = \frac{m+n}{m-n} \Rightarrow \tan(\theta+\alpha)\cot\alpha = \frac{m+n}{m-n}$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The value of $\frac{\sin 70^\circ+\cos 40^\circ}{\cos 70^\circ+\sin 40^\circ}$ is ....
2. If $\tan(A+B)=p$, $\tan(A-B)=q$, then the value of $\tan 2A$ in terms of p and q is ....
3. If $1+\cos 2x+\cos 4x+\cos 6x = k\cos x\cos 2x\cos 3x$, then $k =$ ....
4. The value of $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ$ is ....
5. The value of $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}$ is ....

ANSWERS

1. $\sqrt{3}$ 2. $\frac{p+q}{1-pq}$ 3. 4 4. 0 5. 0

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = \lambda\cos^2\left(\frac{\alpha-\beta}{2}\right)$, write the value of λ.
2. Write the value of $\sin\frac{\pi}{12}\sin\frac{5\pi}{12}$.
3. If $\sin A + \sin B = \alpha$ and $\cos A + \cos B = \beta$, then write the value of $\tan\left(\frac{A+B}{2}\right)$.
4. If $\cos A = m\cos B$, then write the value of $\cot\frac{A+B}{2}\cot\frac{A-B}{2}$.
5. Write the value of the expression $\frac{1-4\sin 10^\circ\sin 70^\circ}{2\sin 10^\circ}$.
6. If $A + B = \frac{\pi}{3}$ and $\cos A + \cos B = 1$, then find the value of $\cos\frac{A-B}{2}$.
7. Write the value of $\sin\frac{\pi}{15}\sin\frac{4\pi}{15}\sin\frac{3\pi}{10}$.
8. If $\sin 2A = \lambda\sin 2B$, then write the value of $\frac{\lambda+1}{\lambda-1}$.
9. Write the value of $\frac{\sin A + \sin 3A}{\cos A + \cos 3A}$.
10. If $\cos(A+B)\sin(C-D) = \cos(A-B)\sin(C+D)$, then write the value $\tan A\tan B\tan C$.

ANSWERS

1. 4 2. $\frac{1}{4}$ 3. $\frac{\alpha}{\beta}$ 4. $\frac{1+m}{1-m}$ 5. 1 6. $\frac{1}{\sqrt{3}}$ 7. $\frac{1}{8}$
8. $\frac{\tan(A+B)}{\tan(A-B)}$ 9. $\tan 2A$ 10. $-\tan D$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. $\cos 40^\circ + \cos 80^\circ + \cos 160^\circ + \cos 240^\circ =$
   (a) 0 (b) 1 (c) 1/2 (d) −1/2
2. $\sin 163^\circ\cos 347^\circ + \sin 73^\circ\sin 167^\circ =$
   (a) 0 (b) 1/2 (c) 1 (d) none of these
3. If $\sin 2\theta + \sin 2\phi = \frac{1}{2}$ and $\cos 2\theta + \cos 2\phi = \frac{3}{2}$, then $\cos^2(\theta-\phi) =$
   (a) 3/8 (b) 5/8 (c) 3/4 (d) 5/4
4. The value of $\cos 52^\circ + \cos 68^\circ + \cos 172^\circ$ is
   (a) 0 (b) 1 (c) 2 (d) 3/2
5. The value of $\sin 78^\circ - \sin 66^\circ - \sin 42^\circ + \sin 6^\circ$ is
   (a) 1/2 (b) −1/2 (c) −1 (d) none of these

6. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha - \cos\beta = b$, then $\tan\frac{\alpha-\beta}{2} =$

(a) $-\frac{a}{b}$ (b) $-\frac{b}{a}$ (c) $\sqrt{a^2+b^2}$ (d) none of these

7. $\cos 35° + \cos 85° + \cos 155° =$

(a) 0 (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\cos 275°$

8. The value of $\sin 50° - \sin 70° + \sin 10°$ is equal to

(a) 1 (b) 0 (c) 1/2 (d) 2

[NCERT EXEMPLAR]

9. $\sin 47° + \sin 61° - \sin 11° - \sin 25°$ is equal to

(a) $\sin 36°$ (b) $\cos 36°$ (c) $\sin 7°$ (d) $\cos 7°$

10. If $\cos A = m\cos B$, then $\cot\frac{A+B}{2}\cot\frac{B-A}{2} =$

(a) $\frac{m-1}{m+1}$ (b) $\frac{m+2}{m-2}$ (c) $\frac{m+1}{m-1}$ (d) none of these

11. If A, B, C are in A.P., then $\frac{\sin A - \sin C}{\cos C - \cos A} =$

(a) $\tan B$ (b) $\cot B$ (c) $\tan 2B$ (d) none of these

12. If $\sin(B+C-A)$, $\sin(C+A-B)$, $\sin(A+B-C)$ are in A.P., then $\cot A$, $\cot B$, $\cot C$ are in

(a) GP (b) HP (c) AP (d) none of these

13. If $\sin x + \sin y = \sqrt{3}(\cos y - \cos x)$, then $\sin 3x + \sin 3y =$

(a) $2\sin 3x$ (b) 0 (c) 1 (d) none of these

14. The value of $\sin\frac{\pi}{18} + \sin\frac{\pi}{9} + \sin\frac{2\pi}{9} + \sin\frac{5\pi}{18}$ is given by

(a) $\sin\frac{7\pi}{18} + \sin\frac{4\pi}{9}$ (b) 1 (c) $\cos\frac{\pi}{6} + \cos\frac{3\pi}{7}$ (d) $\cos\frac{\pi}{9} + \sin\frac{\pi}{9}$

ANSWERS

1. (d) 2. (b) 3. (b) 4. (a) 5. (b) 6. (b) 7. (a) 8. (b)
9. (d) 10. (c) 11. (b) 12. (b) 13. (b) 14. (a)

SUMMARY

1. (i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ (ii) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ (iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

2. (i) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$ (ii) $\sin C - \sin D = 2\sin\frac{C-D}{2}\cos\frac{C+D}{2}$

(iii) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$ (iv) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

CHAPTER 9

VALUES OF TRIGONOMETRIC FUNCTIONS AT MULTIPLES AND SUBMULTIPLES OF AN ANGLE

9.1 INTRODUCTION

In this chapter, we will introduce the formulae expressing the values of trigonometric functions at multiples of x i.e. $2x$, $3x$, $4x$...etc in terms of the values at x. We shall also develop formulae expressing the values of trigonometric functions at x in terms of the values at sub multiples of x i.e. $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}$...etc.

9.2 VALUES OF TRIGONOMETRIC FUNCTIONS AT $2x$ IN TERMS OF VALUES AT x

THEOREM 1 *For the values of angle x for which the two sides are meaningful prove that:*

(i) $\sin 2x = 2 \sin x \cos x$ (ii) $\cos 2x = \cos^2 x - \sin^2 x$

(iii) $\cos 2x = 2\cos^2 x - 1$ or, $1 + \cos 2x = 2\cos^2 x$

(iv) $\cos 2x = 1 - 2\sin^2 x$ or, $1 - \cos 2x = 2\sin^2 x$

(v) $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$ (vi) $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$ (vii) $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

PROOF (i) We know that: $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\therefore \quad \sin 2x = \sin x \cos x + \cos x \sin x$ [Replacing y by x]

$\Rightarrow \quad \sin 2x = 2\sin x \cos x$

(ii) We know that: $\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\therefore \quad \cos 2x = \cos x \cos x - \sin x \sin x$ [Replacing y by x]

$\Rightarrow \quad \cos 2x = \cos^2 x - \sin^2 x$

(iii) We know that:

$$\cos 2x = \cos^2 x - \sin^2 x \Rightarrow \cos 2x = \cos^2 x - (1-\cos^2 x) \Rightarrow \cos 2x = 2\cos^2 x - 1$$

Again, $\cos 2x = 2\cos^2 x - 1 \Rightarrow 1 + \cos 2x = 2\cos^2 x$

(iv) We know that:

$$\cos 2x = \cos^2 x - \sin^2 x \Rightarrow \cos 2x = (1-\sin^2 x) - \sin^2 x \Rightarrow \cos 2x = 1 - 2\sin^2 x$$

Again, $\cos 2x = 1 - 2\sin^2 x \Rightarrow 1 - \cos 2x = 2\sin^2 x$

(v) We know that: $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$\therefore \quad \tan 2x = \dfrac{\tan x + \tan x}{1 - \tan x \tan x}$ [Replacing y by x]

$$\Rightarrow \quad \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

(vi) We know that: $\sin 2x = 2\sin x\cos x$

$$\therefore \quad \sin 2x = \frac{2\sin x\cos x}{1} \Rightarrow \sin 2x = \frac{2\sin x\cos x}{\sin^2 x+\cos^2 x} \qquad [\because \sin^2 x+\cos^2 x = 1]$$

$$\Rightarrow \quad \sin 2x = \frac{\dfrac{2\sin x\cos x}{\cos^2 x}}{\dfrac{\sin^2 x+\cos^2 x}{\cos^2 x}},\ x \neq (2n+1)\frac{\pi}{2}, n \in Z. \qquad \begin{bmatrix}\text{Dividing Numerator and}\\ \text{Denominator by } \cos^2 x\end{bmatrix}$$

$$\Rightarrow \quad \sin 2x = \frac{\dfrac{2\sin x}{\cos x}}{\dfrac{\sin^2 x}{\cos^2 x}+\dfrac{\cos^2 x}{\cos^2 x}} \Rightarrow \sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

(vii) We know that: $\cos 2x = \cos^2 x - \sin^2 x$

$$\therefore \quad \cos 2x = \frac{\cos^2 x-\sin^2 x}{1} = \frac{\cos^2 x-\sin^2 x}{\cos^2 x+\sin^2 x} \qquad [\text{Replacing 1 by } \cos^2 x+\sin^2 x]$$

$$\Rightarrow \quad \cos 2x = \frac{\dfrac{\cos^2 x-\sin^2 x}{\cos^2 x}}{\dfrac{\cos^2 x+\sin^2 x}{\cos^2 x}},\ x \neq (2n+1)\frac{\pi}{2}, n \in Z. \qquad \begin{bmatrix}\text{Dividing Numerator and}\\ \text{Denominator by } \cos^2 x\end{bmatrix}$$

$$\Rightarrow \quad \cos 2x = \frac{\dfrac{\cos^2 x}{\cos^2 x}-\dfrac{\sin^2 x}{\cos^2 x}}{\dfrac{\cos^2 x}{\cos^2 x}+\dfrac{\sin^2 x}{\cos^2 x}} \Rightarrow \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

REMARK *In the above formulae it should be noted that the angle on the RHS is half of the angle on LHS.*

$$\therefore \quad \sin\frac{2\pi}{3} = 2\sin\frac{\pi}{3}\cos\frac{\pi}{3},\ \cos\frac{2\pi}{3} = \cos^2\frac{\pi}{3}-\sin^2\frac{\pi}{3},\ \tan\frac{\pi}{3} = \frac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}} \text{ etc.}$$

9.2.1 VALUES OF TRIGONOMETRIC FUNCTIONS AT x IN TERMS OF THE VALUES AT $\frac{x}{2}$

The relations in section 9.2 are true for all values of the variable x for which the two sides are meaningful. Replacing x by $x/2$ in the above relations, we obtain the following relations:

(i) $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ (ii) $\cos x = \cos^2\frac{x}{2}-\sin^2\frac{x}{2}$

(iii) $\cos x = 2\cos^2\frac{x}{2}-1$ or, $1+\cos x = 2\cos^2\frac{x}{2}$

(iv) $\cos x = 1-2\sin^2\frac{x}{2}$ or, $1-\cos x = 2\sin^2\frac{x}{2}$

(v) $\tan x = \dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$ (vi) $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ (vii) $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

9.2.2 VALUES OF TRIGONOMETRIC FUNCTIONS AT $\frac{x}{2}$ IN TERMS OF $\cos x$

We have,

$$\cos x = 2\cos^2\frac{x}{2} - 1 \Rightarrow 2\cos^2\frac{x}{2} = 1 + \cos x \Rightarrow \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

The sign on the right hand side depends upon the quadrant in which angle $\frac{x}{2}$ lies.

Also,

$$\cos x = 1 - 2\sin^2\frac{x}{2} \Rightarrow 2\sin^2\frac{x}{2} = 1 - \cos x \Rightarrow \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

The sign on the right hand side depends upon the quadrant in which angle $\frac{x}{2}$ lies.

Now, $$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \pm\frac{\sqrt{\frac{1-\cos x}{2}}}{\sqrt{\frac{1+\cos x}{2}}} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

The sign on right hand side depends upon the quadrant in which angle $\frac{x}{2}$ lies.

REMARK *These relations are very useful to find the trigonometric ratios of the angles* $\frac{\pi}{8}, \frac{\pi}{24}, \frac{\pi}{16}$ *etc.*

ILLUSTRATIVE EXAMPLES

BASIC

Type I ON FINDING THE VALUES OF $\sin 2x$, $\cos 2x$, $\tan 2x$ ETC WHEN VALUES OF $\sin x$ OR $\cos x$ OR $\tan x$ ARE GIVEN

EXAMPLE 1 *If* $\sin x = \frac{3}{5}$, *where* $0 < x < \frac{\pi}{2}$, *find the values of* $\sin 2x$, $\cos 2x$, $\tan 2x$ *and* $\sin 4x$.

SOLUTION We have, $\sin x = \frac{3}{5}$, where $0 < x < \frac{\pi}{2}$.

$\therefore \cos^2 x = 1 - \sin^2 x \Rightarrow \cos x = +\sqrt{1-\sin^2 x} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$ $\quad\left[\because \cos x > 0 \text{ for } 0 < x < \frac{\pi}{2}\right]$

$\therefore \tan x = \frac{\sin x}{\cos x} = \frac{3}{4}$

Thus, we obtain

$$\sin 2x = 2\sin x\cos x = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25},\ \cos 2x = 1 - 2\sin^2 x = 1 - 2\times\left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{6}{4}}{1-\frac{9}{16}} = \frac{24}{7} \quad \left[\because \tan x = \frac{3}{4}\right]$$

and, $$\sin 4x = 2\sin 2x\cos 2x = 2\times\frac{24}{25}\times\frac{7}{25} = \frac{336}{625} \quad \left[\because \sin 2x = \frac{24}{25} \text{ and } \cos 2x = \frac{7}{25}\right]$$

EXAMPLE 2 *If* $\tan\alpha = \frac{1}{7}$, $\sin\beta = \frac{1}{\sqrt{10}}$. *Prove that* $\alpha + 2\beta = \frac{\pi}{4}$, *where* $0 < \alpha < \frac{\pi}{2}$ *and* $0 < \beta < \frac{\pi}{2}$.

SOLUTION In order to prove that $\alpha + 2\beta = \frac{\pi}{4}$, it is sufficient to prove that $\tan(\alpha + 2\beta) = \tan\frac{\pi}{4} = 1$.

We have, $\sin\beta = \frac{1}{\sqrt{10}}$, where $0 < \beta < \frac{\pi}{2}$.

$$\therefore \quad \cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{1}{10}} = \frac{3}{\sqrt{10}} \text{ and, } \tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}}} = \frac{1}{3}$$

In order to find the value of $\tan(\alpha + 2\beta)$, we require the values of $\tan\alpha$ and $\tan 2\beta$. The value of $\tan\alpha$ is given. So, let us find $\tan 2\beta$.

$$\text{Now,} \quad \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} \Rightarrow \tan 2\beta = \frac{2\times\frac{1}{3}}{1-\frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{2\times 9}{3\times 8} = \frac{3}{4}$$

Thus, we have $\tan\alpha = \frac{1}{7}$ and $\tan 2\beta = \frac{3}{4}$

$$\therefore \quad \tan(\alpha+2\beta) = \frac{\tan\alpha + \tan 2\beta}{1-\tan\alpha\tan 2\beta} = \frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\times\frac{3}{4}} = \frac{4+21}{28-3} = 1 \Rightarrow \alpha + 2\beta = \frac{\pi}{4}$$

Type II ON PROVING RESULTS AND IDENTITIES BASED UPON THE FOLLOWING FORMULAE:

$\sin 2x = 2\sin x\cos x$, $1+\cos 2x = 2\cos^2 x$, $1-\cos 2x = 2\sin^2 x$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$, $1+\cos x = 2\cos^2\frac{x}{2}$, $1-\cos x = 2\sin^2\frac{x}{2}$

EXAMPLE 3 *Prove that :*

(i) $\frac{\sin 2x}{1+\cos 2x} = \tan x$ (ii) $\frac{\sin 2x}{1-\cos 2x} = \cot x$

(iii) $\frac{1+\sin 2x+\cos 2x}{1+\sin 2x-\cos 2x} = \cot x$ (iv) $\frac{1+\sin x-\cos x}{1+\sin x+\cos x} = \tan\frac{x}{2}$

(v) $\frac{\cos 2x}{1+\sin 2x} = \tan\left(\frac{\pi}{4}-x\right)$ (vi) $\frac{\cos x}{1+\sin x} = \tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$

SOLUTION (i) We have, $\sin 2x = 2\sin x\cos x$ and $1+\cos 2x = 2\cos^2 x$.

$$\therefore \quad \text{LHS} = \frac{\sin 2x}{1+\cos 2x} = \frac{2\sin x\cos x}{2\cos^2 x} = \tan x = \text{RHS}$$

(ii)
$$\text{LHS} = \frac{\sin 2x}{1-\cos 2x} = \frac{2\sin x\cos x}{2\sin^2 x} = \cot x = \text{RHS}$$

(iii)
$$\text{LHS} = \frac{1+\sin 2x+\cos 2x}{1+\sin 2x-\cos 2x} = \frac{(1+\cos 2x)+\sin 2x}{(1-\cos 2x)+\sin 2x}$$

$$= \frac{2\cos^2 x + 2\sin x\cos x}{2\sin^2 x + 2\sin x\cos x} = \frac{2\cos x(\cos x+\sin x)}{2\sin x(\cos x+\sin x)} = \frac{\cos x}{\sin x} = \cot x = \text{RHS}$$

(iv)
$$\text{LHS} = \frac{1+\sin x-\cos x}{1+\sin x+\cos x} = \frac{(1-\cos x)+\sin x}{(1+\cos x)+\sin x}$$

$$= \frac{2\sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{2\sin\frac{x}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)} = \tan\frac{x}{2} = \text{RHS}$$

(v) $$\text{LHS} = \frac{\cos 2x}{1 + \sin 2x} = \frac{\sin\left(\frac{\pi}{2} - 2x\right)}{1 + \cos\left(\frac{\pi}{2} - 2x\right)} \qquad \left[\because \cos x = \sin\left(\frac{\pi}{2} - x\right), \sin x = \cos\left(\frac{\pi}{2} - x\right)\right]$$

$$= \frac{2\sin\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - x\right)}{2\cos^2\left(\frac{\pi}{4} - x\right)} \qquad \left[\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \text{ and } 1 + \cos x = 2\cos^2\frac{x}{2}\right]$$

$$= \tan\left(\frac{\pi}{4} - x\right) = \text{RHS}$$

(vi) $$\text{LHS} = \frac{\cos x}{1 + \sin x} = \frac{\sin\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)} = \frac{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)} = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = \text{RHS}$$

EXAMPLE 4 *Show that:* $\sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos 8x}}} = 2\cos x,\ 0 < x < \frac{\pi}{8}$

SOLUTION $$\text{LHS} = \sqrt{2 + \sqrt{2 + \sqrt{2(1 + \cos 8x)}}}$$

$$= \sqrt{2 + \sqrt{2 + \sqrt{2(2\cos^2 4x)}}} \qquad \left[\because 1 + \cos 8x = 2\cos^2\frac{8x}{2} = 2\cos^2 4x\right]$$

$$= \sqrt{2 + \sqrt{2 + \sqrt{(4\cos^2 4x)}}} = \sqrt{2 + \sqrt{2 + 2\cos 4x}} = \sqrt{2 + \sqrt{2(1 + \cos 4x)}}$$

$$= \sqrt{2 + \sqrt{2(2\cos^2 2x)}} \qquad [\because 1 + \cos 4x = 2\cos^2 2x]$$

$$= \sqrt{2 + 2\cos 2x} = \sqrt{2(1 + \cos 2x)} = \sqrt{2(2\cos^2 x)} = 2\cos x = \text{RHS}$$

EXAMPLE 5 *Prove that:* $\cos 4x = 1 - 8\sin^2 x\cos^2 x$. [NCERT]

SOLUTION $\text{LHS} = \cos 4x = \cos 2(2x) = 1 - 2\sin^2 2x = 1 - 2(\sin 2x)^2 = 1 - 2(2\sin x\cos x)^2$

$= 1 - 8\sin^2 x\cos^2 x = \text{RHS}$

EXAMPLE 6 *Prove that:* $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2\left(\frac{x+y}{2}\right)$.

SOLUTION $$\text{LHS} = (\cos x + \cos y)^2 + (\sin x - \sin y)^2$$

$$= (\cos^2 x + \cos^2 y + 2\cos x\cos y) + (\sin^2 x + \sin^2 y - 2\sin x\sin y)$$

$$= (\cos^2 x + \sin^2 y) + (\cos^2 x + \sin^2 y) + 2(\cos x\cos y - \sin x\sin y)$$

$$= 1 + 1 + 2\cos(x + y) = 2 + 2\cos(x + y) = 2\{1 + \cos(x + y)\}$$

$$= 2 \times 2\cos^2\left(\frac{x+y}{2}\right) = 4\cos^2\left(\frac{x+y}{2}\right) = \text{RHS}. \qquad \left[\because 1 + \cos x = 2\cos^2\frac{x}{2}\right]$$

EXAMPLE 7 *Prove that:* $\dfrac{\sec 8x - 1}{\sec 4x - 1} = \dfrac{\tan 8x}{\tan 2x}$ [NCERT EXEMPLAR]

SOLUTION $\text{LHS} = \dfrac{\sec 8x - 1}{\sec 4x - 1} = \dfrac{\dfrac{1}{\cos 8x} - 1}{\dfrac{1}{\cos 4x} - 1} = \dfrac{1 - \cos 8x}{\cos 8x} \times \dfrac{\cos 4x}{1 - \cos 4x}$

$$= \frac{2 \sin^2 4x}{\cos 8x} \times \frac{\cos 4x}{2 \sin^2 2x} \qquad \left[\because 1 - \cos 8x = 2\sin^2 4x \text{ and, } 1 - \cos 4x = 2 \sin^2 2x\right]$$

$$= \frac{(2 \sin 4x \cos 4x)}{\cos 8x} \times \frac{\sin 4x}{2 \sin^2 2x} = \left(\frac{2 \sin 4x \cos 4x}{\cos 8x}\right) \times \left(\frac{2 \sin 2x \cos 2x}{2 \sin^2 2x}\right)$$

$$= \left(\frac{\sin 2(4x)}{\cos 8x}\right) \times \left(\frac{\cos 2x}{\sin 2x}\right) = \left(\frac{\sin 8x}{\cos 8x}\right) \times \left(\frac{\cos 2x}{\sin 2x}\right) = \tan 8x \cot 2x = \frac{\tan 8x}{\tan 2x} = \text{RHS}$$

EXAMPLE 8 *Prove that:* $\left(1 + \cos \dfrac{\pi}{8}\right)\left(1 + \cos \dfrac{3\pi}{8}\right)\left(1 + \cos \dfrac{5\pi}{8}\right)\left(1 + \cos \dfrac{7\pi}{8}\right) = \dfrac{1}{8}$.

[NCERT EXEMPLAR]

SOLUTION We observe that

$$\cos \frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos \frac{\pi}{8} \quad \text{and,} \quad \cos \frac{5\pi}{8} = \cos\left(\pi - \frac{3\pi}{8}\right) = -\cos \frac{3\pi}{8}$$

$$\therefore \quad \text{LHS} = \left(1 + \cos \frac{\pi}{8}\right)\left(1 + \cos \frac{3\pi}{8}\right)\left(1 + \cos \frac{5\pi}{8}\right)\left(1 + \cos \frac{7\pi}{8}\right)$$

$$= \left(1 + \cos \frac{\pi}{8}\right)\left(1 + \cos \frac{3\pi}{8}\right)\left(1 - \cos \frac{3\pi}{8}\right)\left(1 - \cos \frac{\pi}{8}\right)$$

$$= \left\{\left(1 + \cos \frac{\pi}{8}\right)\left(1 - \cos \frac{\pi}{8}\right)\right\}\left\{\left(1 + \cos \frac{3\pi}{8}\right)\left(1 - \cos \frac{3\pi}{8}\right)\right\}$$

$$= \left(1 - \cos^2 \frac{\pi}{8}\right)\left(1 - \cos^2 \frac{3\pi}{8}\right) = \sin^2 \frac{\pi}{8} \sin^2 \frac{3\pi}{8}$$

$$= \frac{1}{4}\left(2 \sin^2 \frac{\pi}{8}\right)\left(2 \sin^2 \frac{3\pi}{8}\right) = \frac{1}{4}\left\{\left(1 - \cos \frac{\pi}{4}\right)\left(1 - \cos \frac{3\pi}{4}\right)\right\} \qquad [\because 2 \sin^2 x = 1 - \cos 2x]$$

$$= \frac{1}{4}\left\{\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{2}}\right)\right\} = \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8} = \text{RHS}$$

EXAMPLE 9 *Prove that:*

(i) $\cos^4 \dfrac{\pi}{8} + \cos^4 \dfrac{3\pi}{8} + \cos^4 \dfrac{5\pi}{8} + \cos^4 \dfrac{7\pi}{8} = \dfrac{3}{2}$ (ii) $\sin^4 \dfrac{\pi}{8} + \sin^4 \dfrac{3\pi}{8} + \sin^4 \dfrac{5\pi}{8} + \sin^4 \dfrac{7\pi}{8} = \dfrac{3}{2}$

[NCERT EXEMPLAR]

SOLUTION (i) We know that: $\cos(\pi - x) = -\cos x$ and $\dfrac{7\pi}{8} = \pi - \dfrac{\pi}{8}$, $\dfrac{5\pi}{8} = \pi - \dfrac{3\pi}{8}$.

$$\therefore \quad \cos \frac{7\pi}{8} = -\cos \frac{\pi}{8} \text{ and } \cos \frac{5\pi}{8} = -\cos \frac{3\pi}{8} \Rightarrow \cos^4 \frac{7\pi}{8} = \cos^4 \frac{\pi}{8} \text{ and } \cos^4 \frac{5\pi}{8} = \cos^4 \frac{3\pi}{8}$$

Now, $\text{LHS} = \cos^4 \dfrac{\pi}{8} + \cos^4 \dfrac{3\pi}{8} + \cos^4 \dfrac{5\pi}{8} + \cos^4 \dfrac{7\pi}{8} = \cos^4 \dfrac{\pi}{8} + \cos^4 \dfrac{3\pi}{8} + \cos^4 \dfrac{3\pi}{8} + \cos^4 \dfrac{\pi}{8}$

$$= 2 \cos^4 \frac{\pi}{8} + 2 \cos^4 \frac{3\pi}{8} = 2\left\{\left(\cos^2 \frac{\pi}{8}\right)^2 + \left(\cos^2 \frac{3\pi}{8}\right)^2\right\}$$

$$= 2\left[\left\{\frac{1+\cos\frac{\pi}{4}}{2}\right\}^2 + \left\{\frac{1+\cos\frac{3\pi}{4}}{2}\right\}^2\right] \qquad \left[\because \frac{1+\cos 2x}{2} = \cos^2 x\right]$$

$$= \frac{2}{4}\left\{\left(1+\cos\frac{\pi}{4}\right)^2 + \left(1+\cos\frac{3\pi}{4}\right)^2\right\} = \frac{2}{4}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right\}$$

$$= \frac{1}{2}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right\} = \frac{1}{2}\left\{\left(1+\frac{1}{2}+\sqrt{2}\right)+\left(1+\frac{1}{2}-\sqrt{2}\right)\right\} = \frac{3}{2} = \text{RHS}$$

(ii) We know that $\sin(\pi - x) = \sin x$

$$\therefore \quad \sin^4\frac{7\pi}{8} = \sin^4\left(\pi - \frac{\pi}{8}\right) = \sin^4\frac{\pi}{8} \quad \text{and,} \quad \sin^4\frac{5\pi}{8} = \sin^4\left(\pi - \frac{3\pi}{8}\right) = \sin^4\frac{3\pi}{8}$$

Now, $\text{LHS} = \sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8} = \sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{\pi}{8}$

$$= 2\left\{\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8}\right\} = 2\left\{\left(\sin^2\frac{\pi}{8}\right)^2 + \left(\sin^2\frac{3\pi}{8}\right)^2\right\}$$

$$= 2\left[\left\{\frac{1-\cos\frac{\pi}{4}}{2}\right\}^2 + \left\{\frac{1-\cos\frac{3\pi}{4}}{2}\right\}^2\right] \qquad \left[\because \frac{1-\cos 2x}{2} = \sin^2 x\right]$$

$$= \frac{2}{4}\left\{\left(1-\cos\frac{\pi}{4}\right)^2 + \left(1-\cos\frac{3\pi}{4}\right)^2\right\}$$

$$= \frac{1}{2}\left\{\left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2\right\} = \frac{1}{2}\left\{\left(1+\frac{1}{2}-\sqrt{2}\right)+\left(1+\frac{1}{2}+\sqrt{2}\right)\right\} = \frac{3}{2} = \text{RHS}$$

EXAMPLE 10 *Prove that:* $\tan 4x = \dfrac{4\tan x(1-\tan^2 x)}{1-6\tan^2 x + \tan^4 x}$. **[NCERT]**

SOLUTION $\text{LHS} = \tan 4x = \tan\left(2(2x)\right) = \dfrac{2\tan 2x}{1-\tan^2 2x}$

$$= \frac{2\left(\frac{2\tan x}{1-\tan^2 x}\right)}{1-\left(\frac{2\tan x}{1-\tan^2 x}\right)^2} = \frac{4\tan x(1-\tan^2 x)}{(1-\tan^2 x)^2 - 4\tan^2 x} = \frac{4\tan x(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x} = \text{RHS}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 11 *Prove that:*

(i) $\cos^2 x + \cos^2\left(x+\frac{2\pi}{3}\right) + \cos^2\left(x-\frac{2\pi}{3}\right) = \frac{3}{2}$ (ii) $\cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right) + \cos^2\left(x-\frac{\pi}{3}\right) = \frac{3}{2}$

SOLUTION (i) $\text{LHS} = \cos^2 x + \cos^2\left(x+\frac{2\pi}{3}\right) + \cos^2\left(x-\frac{2\pi}{3}\right)$

$$= \frac{1}{2}\left\{2\cos^2 x + 2\cos^2\left(x + \frac{2\pi}{3}\right) + 2\cos^2\left(x - \frac{2\pi}{3}\right)\right\}$$

$$= \frac{1}{2}\left[1 + \cos 2x + \left\{1 + \cos 2\left(x + \frac{2\pi}{3}\right)\right\} + \left\{1 + \cos 2\left(x - \frac{2\pi}{3}\right)\right\}\right]$$

$$= \frac{1}{2}\left[1 + \cos 2x + 1 + \cos\left(2x + \frac{4\pi}{3}\right) + 1 + \cos\left(2x - \frac{4\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3 + \cos 2x + \left\{\cos\left(2x + \frac{4\pi}{3}\right) + \cos\left(2x - \frac{4\pi}{3}\right)\right\}\right]$$

$$= \frac{1}{2}\left[3 + \cos 2x + 2\cos 2x \cos\frac{4\pi}{3}\right] \qquad [\because \cos(x+y) + \cos(x-y) = 2\cos x \cos y]$$

$$= \frac{1}{2}\left[3 + \cos 2x + 2(\cos 2x)\left(-\frac{1}{2}\right)\right] = \frac{1}{2}(3 + \cos 2x - \cos 2x) = \frac{3}{2} = \text{RHS}$$

(ii) $\text{LHS} = \cos^2 x + \cos^2\left(x + \frac{\pi}{3}\right) + \cos^2\left(x - \frac{\pi}{3}\right)$

$$= \frac{1}{2}\left\{2\cos^2 x + 2\cos^2\left(x + \frac{\pi}{3}\right) + 2\cos^2\left(x - \frac{\pi}{3}\right)\right\}$$

$$= \frac{1}{2}\left\{(1 + \cos 2x) + 1 + \cos\left(2x + \frac{2\pi}{3}\right) + 1 + \cos\left(2x - \frac{2\pi}{3}\right)\right\}$$

$$= \frac{1}{2}\left[3 + \cos 2x + \left\{\cos\left(2x + \frac{2\pi}{3}\right) + \cos\left(2x - \frac{2\pi}{3}\right)\right\}\right]$$

$$= \frac{1}{2}\left\{3 + \cos 2x + 2\cos 2x \cos\frac{2\pi}{3}\right\} \qquad [\because \cos(x+y) + \cos(x-y) = 2\cos x \cos y]$$

$$= \frac{1}{2}\left\{3 + \cos 2x + 2(\cos 2x) \times -\frac{1}{2}\right\} = \frac{1}{2}\{3 + \cos 2x - \cos 2x\} = \frac{3}{2} = \text{RHS}$$

EXAMPLE 12 *Prove that:*

(i) $\dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$ [NCERT] (ii) $\dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$ [NCERT]

(iii) $\dfrac{\tan 5x + \tan 3x}{\tan 5x - \tan 3x} = 4\cos 2x \cos 4x$

(iv) $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$ [NCERT]

SOLUTION (i) $\text{LHS} = \dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \dfrac{(\sin 5x + \sin x) - 2\sin 3x}{\cos 5x - \cos x}$

$$= \frac{2\sin\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right) - 2\sin 3x}{-2\sin\left(\frac{5x+x}{2}\right)\sin\left(\frac{5x-x}{2}\right)} = \frac{2\sin 3x\cos 2x - 2\sin 3x}{-2\sin 3x \sin 2x}$$

$$= -\frac{2\sin 3x(1 - \cos 2x)}{-2\sin 3x \sin 2x} = \frac{1 - \cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin\cos x} = \tan x = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = \frac{2\sin\left(\frac{x-3x}{2}\right)\cos\left(\frac{x+3x}{2}\right)}{-(\cos^2 x - \sin^2 x)} = \frac{2\sin(-x)\cos 2x}{-\cos 2x} = \frac{-2\sin x\cos 2x}{-\cos 2x}$$

$$= 2\sin x = \text{RHS}$$

(iii) We have,

$$\text{L.H.S} = \frac{\tan 5x + \tan 3x}{\tan 5x - \tan 3x} = \frac{\frac{\sin 5x}{\cos 5x} + \frac{\sin 3x}{\cos 3x}}{\frac{\sin 5x}{\cos 5x} - \frac{\sin 3x}{\cos 3x}} = \frac{\sin 5x\cos 3x + \cos 5x\sin 3x}{\sin 5x\cos 3x - \cos 5x\sin 3x} = \frac{\sin(5x+3x)}{\sin(5x-3x)}$$

$$= \frac{\sin 8x}{\sin 2x} = \frac{2\sin 4x\cos 4x}{\sin 2x} = \frac{2\times(2\sin 2x\cos 2x)\cos 4x}{\sin 2x} = 4\cos 2x\cos 4x = \text{RHS}$$

(iv) $\text{LHS} = \sin 2x + 2\sin 4x + \sin 6x = (\sin 6x + \sin 2x) + 2\sin 4x$

$$= 2\sin\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right) + 2\sin 4x = 2\sin 4x\cos 2x + 2\sin 4x$$

$$= 2\sin 4x(\cos 2x + 1) = 2\sin 4x \times 2\cos^2 x = 4\cos^2 x\sin 4x = \text{RHS}$$

EXAMPLE 13 *Show that :* $2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta) = \cos 2\alpha$

[NCERT EXEMPLAR]

SOLUTION $\text{LHS} = 2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta)$

$$= 2\sin^2\beta + 2\cos(\alpha+\beta)(2\sin\alpha\sin\beta) + \cos 2(\alpha+\beta)$$

$$= 2\sin^2\beta + 2\cos(\alpha+\beta)\{\cos(\alpha-\beta) - \cos(\alpha+\beta)\} + \cos 2(\alpha+\beta)$$

$$= 2\sin^2\beta + 2\cos(\alpha+\beta)\cos(\alpha-\beta) - 2\cos^2(\alpha+\beta) + \cos 2(\alpha+\beta)$$

$$= 2\sin^2\beta + 2(\cos^2\alpha - \sin^2\beta) - 2\cos^2(\alpha+\beta) + 2\cos^2(\alpha+\beta) - 1$$

$$= 2\sin^2\beta + 2\cos^2\alpha - 2\sin^2\beta - 2\cos^2(\alpha+\beta) + 2\cos^2(\alpha+\beta) - 1 = 2\cos^2\alpha - 1$$

$$= \cos 2\alpha = \text{RHS}$$

EXAMPLE 14 *Show that :* $\sqrt{3}\,\text{cosec}\,20^\circ - \sec 20^\circ = 4.$ [NCERT EXEMPLAR]

SOLUTION $\text{LHS} = \sqrt{3}\,\text{cosec}\,20^\circ - \sec 20^\circ$

$$= \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ} = \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ\cos 20^\circ}$$

$$= \frac{2\left\{\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right\}}{\sin 20^\circ\cos 20^\circ} = \frac{2(\sin 60^\circ\cos 20^\circ - \cos 60^\circ\sin 20^\circ)}{\sin 20^\circ\cos 20^\circ}$$

$$= \frac{2\sin(60^\circ - 20^\circ)}{\sin 20^\circ\cos 20^\circ} = \frac{2\sin 40^\circ}{\sin 20^\circ\cos 20^\circ} = \frac{4\sin 40^\circ}{2\sin 20^\circ\cos 20^\circ} = \frac{4\sin 40^\circ}{\sin 40^\circ} = 4 = \text{RHS}$$

Type III ON FINDING THE VALUES OF $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ AND $\tan\frac{x}{2}$ WHEN VALUES OF $\sin x$ OR $\cos x$ OR $\tan x$ ARE GIVEN

EXAMPLE 15 If $0 \le x \le 2\pi$, find $\sin\frac{x}{2}$, $\cos\frac{x}{2}$, and $\tan\frac{x}{2}$, when:

(i) $\tan x = -\frac{4}{3}$, x lies in quadrant II (ii) $\cos x = -\frac{1}{3}$, x lies in quadrant III

(iii) $\sin x = -\frac{1}{2}$, x lies in quadrant IV.

SOLUTION (i) It is given that x lies in the second quadrant in which $\cos x$ is negative.

$$\therefore \quad \cos x = -\frac{1}{\sqrt{1+\tan^2 x}} = -\frac{1}{\sqrt{1+\frac{16}{9}}} = -\frac{3}{5}$$

Again x lies in the second quadrant.

i.e. $\frac{\pi}{2} < x < \pi \Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2} \Rightarrow \frac{x}{2}$ lies in first quadrant $\Rightarrow \sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are positive

$$\therefore \quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \Rightarrow \cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} = \sqrt{\frac{1-3/5}{2}} = \frac{1}{\sqrt{5}}$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \Rightarrow \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1+3/5}{2}} = \frac{2}{\sqrt{5}}$$

$$\text{and,} \quad \tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} \Rightarrow \tan\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\frac{1+3/5}{1-3/5}} = 2$$

(ii) It is given that x lies in the third quadrant.

i.e. $\pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \Rightarrow \frac{x}{2}$ lies in IInd quadrant $\Rightarrow \cos\frac{x}{2} < 0$, $\sin\frac{x}{2} > 0$ and $\tan\frac{x}{2} < 0$

$$\therefore \quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\Rightarrow \quad \cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}} = -\sqrt{\frac{1-1/3}{2}} = -\frac{1}{\sqrt{3}} \qquad \left[\because \cos\frac{x}{2} \text{ is} - \text{ve and } \cos x = -\frac{1}{3}\right]$$

$$\text{and,} \quad \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\Rightarrow \quad \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1+1/3}{2}} = \sqrt{\frac{2}{3}} \qquad \left[\because \sin\frac{x}{2} > 0 \text{ and } \cos x = \frac{1}{3}\right]$$

$$\text{and,} \quad \tan\frac{x}{2} = \frac{\sin x/2}{\cos x/2} = \sqrt{\frac{2}{3}} \times -\sqrt{3} = -\sqrt{2}$$

(iii) It is given that x lies in the fourth quadrant in which $\cos x$ is positive.

$$\therefore \quad \sin x = -\frac{1}{2} \Rightarrow \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$$

Again, x lies in the fourth quadrant

i.e. $\frac{3\pi}{2} < x < 2\pi \Rightarrow \frac{3\pi}{4} < \frac{x}{2} < \pi \Rightarrow \frac{x}{2}$ lies in IInd quadrant $\Rightarrow \cos\frac{x}{2} < 0$, $\sin\frac{x}{2} > 0$ and $\tan\frac{x}{2} < 0$

$$\therefore \quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\Rightarrow \quad \cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}} = -\sqrt{\frac{1+\sqrt{3}/2}{2}} = -\frac{\sqrt{2+\sqrt{3}}}{2} \qquad \left[\because \cos\frac{x}{2} < 0 \text{ and } \cos x = \frac{\sqrt{3}}{2}\right]$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\Rightarrow \quad \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1-\sqrt{3}/2}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2} \qquad \left[\because \sin\frac{x}{2} > 0 \text{ and } \cos x = \frac{\sqrt{3}}{2}\right]$$

and, $\tan\frac{x}{2} = \frac{\sin(x/2)}{\cos(x/2)} = \frac{\sqrt{2-\sqrt{3}}}{2} \times \frac{-2}{\sqrt{2+\sqrt{3}}} = -\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$

EXAMPLE 16 *If* $\tan x = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2}$, *find the values of* $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$. [NCERT]

SOLUTION It is given that $\tan x = \frac{3}{4}$ and, $\pi < x < \frac{3\pi}{2}$.

$\therefore \quad \cos x = -\frac{1}{\sqrt{1+\tan^2 x}} = -\frac{1}{\sqrt{1+\frac{9}{16}}} = -\frac{4}{5}$ $\left[\because \pi < x < \frac{3\pi}{2} \therefore \cos x \text{ is negative}\right]$

Again, $\pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \Rightarrow \cos x < 0$ and $\sin x > 0$

$\therefore \quad \cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}} = -\sqrt{\frac{1-\frac{4}{5}}{2}} = -\frac{1}{\sqrt{10}}$ and, $\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1+\frac{4}{5}}{2}} = \frac{3}{\sqrt{10}}$

$\therefore \quad \tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{3/\sqrt{10}}{-1/\sqrt{10}} = -3$

Hence, $\cos\frac{x}{2} = -\frac{1}{\sqrt{10}}$, $\sin\frac{x}{2} = \frac{3}{\sqrt{10}}$ and $\tan\frac{x}{2} = -3$

Type IV ON FINDING THE VALUES TRIGONOMETRICAL FUNCTIONS FOR $\frac{\pi}{24}, \frac{\pi}{16}, \frac{\pi}{8}$

Formulae: $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$, $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$, $\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$

EXAMPLE 17 *Find the values of*

(i) $\cos\frac{\pi}{8}$ (ii) $\sin\frac{\pi}{8}$ (iii) $\tan\frac{\pi}{8}$ [NCERT EXEMPLAR] (iv) $\sin\frac{\pi}{24}$ (v) $\cos\frac{\pi}{24}$

SOLUTION (i) We know that $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$. Putting $x = \frac{\pi}{4}$, we get

$$\cos\frac{\pi}{8} = \sqrt{\frac{1+\cos\pi/4}{2}} = \sqrt{\frac{1+1/\sqrt{2}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} \qquad \left[\because \cos\frac{\pi}{8} \text{ is positive}\right]$$

(ii) We have, $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$. Putting $x = \frac{\pi}{4}$, we get

$$\sin\frac{\pi}{8} = \sqrt{\frac{1-\cos\pi/4}{2}} = \sqrt{\frac{1-1/\sqrt{2}}{2}} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \qquad \left[\because \sin\frac{\pi}{8} \text{ is positive}\right]$$

(iii) We have, $\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$. Putting $x = \frac{\pi}{4}$, we get

$$\tan\frac{\pi}{8} = \sqrt{\frac{1-\cos\pi/4}{1+\cos\pi/4}} = \sqrt{\frac{1-1/\sqrt{2}}{1+1/\sqrt{2}}} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \sqrt{\frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)}} = \sqrt{2}-1$$

(iv) We observe that

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Putting $x = \frac{\pi}{12}$ in $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$, we get

$$\sin\frac{\pi}{24} = \sqrt{\frac{1-\cos\frac{\pi}{12}}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}{2}} = \sqrt{\frac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}}} = \sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{8}} = \frac{\sqrt{4-\sqrt{6}-\sqrt{2}}}{2\sqrt{2}}$$

(v) Putting $x = \frac{\pi}{12}$ in $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$, we get

$$\cos\frac{\pi}{24} = \sqrt{\frac{1+\cos\frac{\pi}{12}}{2}} \qquad \left[\because \cos\frac{\pi}{24} = \cos 7\frac{1}{2}^\circ \text{ is positive}\right]$$

$$\Rightarrow \quad \cos\frac{\pi}{24} = \sqrt{\frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{2}} = \sqrt{\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}}} = \sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{8}} = \frac{\sqrt{4+\sqrt{6}+\sqrt{2}}}{2\sqrt{2}}$$

EXAMPLE 18 *Prove that:*

(i) $\cot\frac{\pi}{24} = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$ (ii) $\tan\frac{\pi}{16} = \sqrt{4+2\sqrt{2}} - (\sqrt{2}+1)$

(iii) $\tan 142\frac{1}{2}^\circ = 2+\sqrt{2}-\sqrt{3}-\sqrt{6}$

SOLUTION (i) LHS $= \cot\frac{\pi}{24}$

$$= \frac{\cos\frac{\pi}{24}}{\sin\frac{\pi}{24}} = \frac{2\cos\frac{\pi}{24}\cos\frac{\pi}{24}}{2\sin\frac{\pi}{24}\cos\frac{\pi}{24}} = \frac{2\cos^2\frac{\pi}{24}}{2\sin\frac{\pi}{24}\cos\frac{\pi}{24}} = \frac{1+\cos\frac{\pi}{12}}{\sin\frac{\pi}{12}}$$

$$= \frac{1+\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)} = \frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1} = \frac{(2\sqrt{2}+\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{2\sqrt{6}+2\sqrt{2}+3+\sqrt{3}+\sqrt{3}+1}{3-1} = \frac{2\sqrt{6}+2\sqrt{2}+2\sqrt{3}+4}{2}$$

$$= \sqrt{2}+\sqrt{3}+2+\sqrt{6} = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6} = \text{RHS}$$

(ii) $$\text{LHS} = \tan\frac{\pi}{16} = \frac{\sin\frac{\pi}{16}}{\cos\frac{\pi}{16}} = \frac{\sin\frac{\pi}{16}}{\cos\frac{\pi}{16}} \times \frac{2\sin\frac{\pi}{16}}{2\sin\frac{\pi}{16}} = \frac{2\sin^2\frac{\pi}{16}}{2\sin\frac{\pi}{16}\cos\frac{\pi}{16}} = \frac{1-\cos\frac{\pi}{8}}{\sin\frac{\pi}{8}}$$

$$= \frac{1-\sqrt{\frac{1+\cos\frac{\pi}{4}}{2}}}{\sqrt{\frac{1-\cos\frac{\pi}{4}}{2}}} = \frac{\sqrt{2}-\sqrt{1+\cos\frac{\pi}{4}}}{\sqrt{1-\cos\frac{\pi}{4}}} \qquad \left[\because \cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} \text{ and } \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}\right]$$

$$= \frac{\sqrt{2}-\sqrt{1+\frac{1}{\sqrt{2}}}}{\sqrt{1-\frac{1}{\sqrt{2}}}} = \frac{\sqrt{2}-\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}}}}{\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}}}} = \frac{\sqrt{2\sqrt{2}}-\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}-1}}$$

$$= \frac{\sqrt{2\sqrt{2}}-\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}-1}} \times \frac{\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}+1}} = \frac{\sqrt{2\sqrt{2}}\sqrt{\sqrt{2}+1}-\sqrt{(\sqrt{2}+1)^2}}{\sqrt{(\sqrt{2}+1)(\sqrt{2}-1)}}$$

$$= \frac{\sqrt{2\sqrt{2}(\sqrt{2}+1)}-(\sqrt{2}+1)}{2-1} = \sqrt{4+2\sqrt{2}}-(\sqrt{2}+1) = \text{RHS}$$

(iii) $\text{LHS} = \tan 142\frac{1}{2}^\circ = \tan\left(180^\circ - 37\frac{1}{2}^\circ\right) = \tan 37\frac{1}{2}^\circ = -\tan\frac{5\pi}{24}$

$$= -\frac{\sin\frac{5\pi}{24}}{\cos\frac{5\pi}{24}} = -\frac{2\sin^2\frac{5\pi}{24}}{2\sin\frac{5\pi}{24}\cos\frac{5\pi}{24}} = -\frac{1-\cos\frac{5\pi}{12}}{\sin\frac{5\pi}{12}} = -\frac{1-\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)}$$

$$= -\frac{1-\left(\cos\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{4}\sin\frac{\pi}{6}\right)}{\sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6}} = \left(\frac{1-\frac{\sqrt{3}-1}{2\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}\right) = -\left(\frac{2\sqrt{2}-\sqrt{3}+1}{\sqrt{3}+1}\right)$$

$$= -\frac{(2\sqrt{2}-\sqrt{3}+1)}{(\sqrt{3}+1)} \times \frac{(\sqrt{3}-1)}{\sqrt{3}-1} = -\left\{\frac{(2\sqrt{2}-\sqrt{3}+1)(\sqrt{3}-1)}{3-1}\right\}$$

$$= -\left\{\frac{2\sqrt{2}(\sqrt{3}-1)-(\sqrt{3}-1)^2}{2}\right\} = -\left\{\frac{2\sqrt{2}(\sqrt{3}-1)-(3+1-2\sqrt{3})}{2}\right\}$$

$$= -\left\{\sqrt{2}(\sqrt{3}-1) - (2-\sqrt{3})\right\} = 2+\sqrt{2}-\sqrt{3}-\sqrt{6} = \text{RHS}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 19 Prove that: $\cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A = \dfrac{\sin 2^n A}{2^n \sin A}$

SOLUTION $\text{LHS} = \cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A$

$$= \frac{1}{2\sin A}\left\{(2\sin A\cos A)\cos 2A\cos 2^2 A\cos 2^3 A \ldots \cos 2^{n-1} A\right\}$$

$$= \frac{1}{2\sin A}\left\{\sin 2A\cos 2A\cos 2^2 A\cos 2^3 A \ldots \cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^2\sin A}\left\{(2\sin 2A\cos 2A)\cos 2^2 A\cos 2^3 A \ldots \cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^2\sin A}\left\{\sin 2(2A)\cos 2^2 A\cos 2^3 A \ldots \cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^3\sin A}\left\{(2\sin 2^2 A\cos 2^2 A)\cos 2^3 A \ldots \cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^3 \sin A}\left\{\sin (2 \times 2^2 A) \cos 2^3 A \ldots \cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^3 \sin A}\left\{\sin 2^3 A \cos 2^3 A \cos 2^4 A \ldots \cos 2^{n-1} A\right\}$$

$$= \ldots = \frac{1}{2^{n-1} \sin A}\left\{\sin 2^{n-1} A \cos 2^{n-1} A\right\} = \frac{1}{2^n \sin A}\left\{2 \sin 2^{n-1} A \cos 2^{n-1} A\right\}$$

$$= \frac{1}{2^n \sin A} \sin (2 \times 2^{n-1} A) = \frac{1}{2^n \sin A} \sin 2^n A = \text{RHS}$$

<u>NOTE</u> *The result proved in the above example may be used as a standard formula.*

Type V PROBLEMS BASED ON THE FORMULA

$$\cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$$

EXAMPLE 20 *Prove that:* $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

SOLUTION LHS $= \cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{2}(\cos 20° \cos 40° \cos 80°)$

$$= \frac{1}{2}\left(\cos A \cos 2A \cos 2^2 A\right), \text{ where } A = 20°$$

$$= \frac{1}{2}\left(\frac{\sin 2^3 A}{2^3 \sin A}\right) = \frac{1}{2^4} \frac{\sin 8A}{\sin A}$$

$$= \frac{1}{2^4} \frac{\sin 160°}{\sin 20°} = \frac{1}{2^4} \frac{\sin (180° - 20°)}{\sin 20°} = \frac{1}{2^4} \frac{\sin 20°}{\sin 20°} = \frac{1}{2^4} = \frac{1}{16} = \text{RHS}$$

EXAMPLE 21 *Prove that:* $\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} = -\frac{1}{8}$.

SOLUTION LHS $= \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} = \cos A \cos 2A \cos 2^2 A$, where $A = \frac{\pi}{7}$

$$= \frac{\sin 2^3 A}{2^3 \sin A} = \frac{\sin \frac{8\pi}{7}}{8 \sin \frac{\pi}{7}} = \frac{\sin\left(\pi + \frac{\pi}{7}\right)}{8 \sin \frac{\pi}{7}} = -\frac{\sin \frac{\pi}{7}}{8 \sin \frac{\pi}{7}} = -\frac{1}{8} = \text{RHS}$$

EXAMPLE 22 *If* $\theta = \frac{\pi}{2^n + 1}$, *prove that:* $2^n \cos \theta \cos 2\theta \cos 2^2 \theta \ldots \cos 2^{n-1} \theta = 1$.

SOLUTION We have, $\theta = \frac{\pi}{2^n + 1} \Rightarrow 2^n \theta + \theta = \pi \Rightarrow 2^n \theta = \pi - \theta$

$$\therefore\ 2^n \cos \theta \cos 2\theta \cos 2^2 \theta \ldots \cos 2^{n-1} \theta = 2^n \left\{\frac{\sin 2^n \theta}{2^n \sin \theta}\right\} = \frac{\sin 2^n \theta}{\sin \theta} = \frac{\sin (\pi - \theta)}{\sin \theta} = 1 \quad [\because\ 2^n \theta = \pi - \theta]$$

EXAMPLE 23 *Prove that:* $\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7} = \frac{1}{8}$

SOLUTION LHS $= \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7} = \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos\left(\pi - \frac{4\pi}{7}\right)$

$$= -\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} = -\left(-\frac{1}{8}\right) = \frac{1}{8} = \text{RHS}$$

[See Example 21]

EXAMPLE 24 *Prove that:* $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15}=\frac{1}{16}$

SOLUTION LHS $=\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\left(\pi-\frac{\pi}{15}\right)$

$$=\left(\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\right)\left(-\cos\frac{\pi}{15}\right) \qquad [\because \cos(\pi-\theta)=-\cos\theta]$$

$$=-\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}$$

$$=-\cos A\cos 2A\cos 2^2A\cos 2^3A, \quad \text{where} \quad A=\frac{\pi}{15}$$

$$=-\frac{\sin 2^4 A}{2^4\sin A}=-\frac{\sin 16A}{2^4\sin A}=-\frac{\sin\frac{16\pi}{15}}{16\sin\frac{\pi}{15}}=-\frac{\sin\left(\pi+\frac{\pi}{15}\right)}{16\sin\frac{\pi}{15}}=\frac{\sin\frac{\pi}{15}}{16\sin\frac{\pi}{15}}=\frac{1}{16}=\text{RHS}$$

EXAMPLE 25 *Prove that:* $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}=\frac{1}{64}$

SOLUTION We know that $A+B=\pi \Rightarrow \sin A=\sin(\pi-B)=\sin B$

$\therefore \quad \frac{\pi}{14}+\frac{13\pi}{14}=\pi,\ \frac{3\pi}{14}+\frac{11\pi}{14}=\pi,\ \frac{5\pi}{14}+\frac{9\pi}{14}=\pi$

$\Rightarrow \quad \sin\frac{\pi}{14}=\sin\frac{13\pi}{14},\ \sin\frac{3\pi}{14}=\sin\frac{11\pi}{14},\ \sin\frac{5\pi}{14}=\sin\frac{9\pi}{14}$

Now, LHS $=\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$

$$=\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{3\pi}{14}\sin\frac{\pi}{14}$$

$$=\left\{\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right\}^2\times\sin\frac{7\pi}{14}=\left\{\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right\}^2\times 1$$

$$=\left\{\cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{14}\right)\right\}^2=\left\{\cos\frac{6\pi}{14}\cos\frac{4\pi}{14}\cos\frac{2\pi}{14}\right\}^2$$

$$=\left\{\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right\}^2=\left(\frac{1}{8}\right)^2=\frac{1}{64}$$

[See Example 23]

EXAMPLE 26 *Find the value of* $\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$.

SOLUTION $\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}=\cos\left(\frac{\pi}{2}-\frac{\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{7\pi}{18}\right)$

$$=\cos\frac{4\pi}{9}\cos\frac{2\pi}{9}\cos\frac{\pi}{9}=\frac{\sin\left(2^3\times\frac{\pi}{9}\right)}{2^3\sin\frac{\pi}{9}}=\frac{\sin\frac{8\pi}{9}}{8\sin\frac{\pi}{9}}=\frac{\sin\left(\pi-\frac{\pi}{9}\right)}{8\sin\frac{\pi}{9}}=\frac{\sin\frac{\pi}{9}}{8\sin\frac{\pi}{9}}=\frac{1}{8}$$

EXAMPLE 27 *Prove that:* $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{128}$

SOLUTION LHS $=\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$

$$=\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\}\cos\frac{5\pi}{15}$$

$$= \frac{1}{2}\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\} \qquad \left[\because \cos\frac{5\pi}{15} = \cos\frac{\pi}{3} = \frac{1}{2}\right]$$

$$= -\frac{1}{2}\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\}\left[\because \cos\frac{7\pi}{15} = \cos\left(\pi - \frac{8\pi}{15}\right) = -\cos\frac{8\pi}{15}\right]$$

$$= -\frac{1}{2}\left\{\frac{\sin\frac{2^4\pi}{15}}{2^4\sin\frac{\pi}{15}}\right\} \times \left\{\frac{\sin\left(2^2 \times \frac{3\pi}{15}\right)}{2^2\sin\frac{3\pi}{15}}\right\} = -\frac{1}{2}\left\{\frac{\sin\frac{16\pi}{15}}{16\sin\frac{\pi}{15}}\right\} \times \left\{\frac{\sin\frac{12\pi}{15}}{4\sin\frac{3\pi}{15}}\right\}$$

$$= -\frac{1}{2}\left\{\frac{\sin\left(\pi + \frac{\pi}{15}\right)}{16\sin\frac{\pi}{15}}\right\} \times \left\{\frac{\sin\left(\pi - \frac{3\pi}{15}\right)}{4\sin\frac{3\pi}{15}}\right\} = -\frac{1}{2}\left\{\frac{-\sin\frac{\pi}{15}}{16\sin\frac{\pi}{15}}\right\} \times \left\{\frac{\sin\frac{3\pi}{15}}{4\sin\frac{3\pi}{15}}\right\}$$

$$= -\frac{1}{2} \times -\frac{1}{16} \times \frac{1}{4} = \frac{1}{128} = \text{RHS}$$

EXAMPLE 28 *Prove that:* $(1 + \sec 2x)(1 + \sec 4x)(1 + \sec 8x)\ldots(1 + \sec 2^n x) = \tan 2^n x \cot x, n \in N.$

SOLUTION LHS $= (1 + \sec 2x)(1 + \sec 4x)(1 + \sec 8x)\ldots(1 + \sec 2^n x)$

$$= \frac{(1 + \cos 2x)(1 + \cos 4x)(1 + \cos 8x)\ldots(1 + \cos 2^n x)}{\cos 2x \cos 4x \cos 8x \ldots \cos 2^n x}$$

$$= \frac{(2\cos^2 x)(2\cos^2 2x)(2\cos^2 2^2 x)\ldots(2\cos^2 2^{n-1} x)}{\cos 2x \cos 2^2 x \cos 2^3 x \ldots \cos 2^n x}$$

$$= \frac{2^n \cos x}{\cos 2^n x}\left\{\cos x \cos 2x \cos 2^2 x \ldots \cos 2^{n-1} x\right\} = \frac{2^n \cos x}{\cos 2^n x}\left\{\frac{\sin 2^n x}{2^n \sin x}\right\}$$

$$= \tan 2^n x \cot x = \text{RHS}$$

Type VI MISCELLANEOUS PROBLEMS BASED UPON FOLLOWING FORMULAE

$$\sin 2x = 2\sin x\cos x,\ \cos 2x = \cos^2 x - \sin^2 x,\ \cos 2x = 2\cos^2 x - 1,$$

$$\cos 2x = 1 - 2\sin^2 x,\ \tan 2x = \frac{2\tan x}{1 - \tan^2 x},\ \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},\ \sin 2x = \frac{2\tan x}{1 + \tan^2 x}$$

EXAMPLE 29 *If* $\tan^2 x = 2\tan^2\phi + 1$, *prove that* $\cos 2x + \sin^2\phi = 0.$

SOLUTION Using $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$, we obtain

$$\therefore \quad \cos 2x + \sin^2\phi = \frac{1 - \tan^2 x}{1 + \tan^2 x} + \sin^2\phi = \frac{1 - (2\tan^2\phi + 1)}{1 + (2\tan^2\phi + 1)} + \sin^2\phi \ [\because \tan^2 x = 2\tan^2\phi + 1]$$

$$= \frac{-2\tan^2\phi}{2 + 2\tan^2\phi} + \sin^2\phi = \frac{-\tan^2\phi}{\sec^2\phi} + \sin^2\phi = -\sin^2\phi + \sin^2\phi = 0$$

EXAMPLE 30 *Prove that:*

(i) $\dfrac{1 + \cos 4x}{\cot x - \tan x} = \dfrac{1}{2}\sin 4x$ (ii) $\dfrac{\cos 5x + \cos 4x}{1 - 2\cos 3x} = -\cos 2x - \cos x$

(iii) $\dfrac{\cos 7x - \cos 8x}{1 + 2\cos 5x} = \cos 2x - \cos 3x$

SOLUTION (i) We have,

$$\text{LHS} = \frac{1+\cos 4x}{\cot x - \tan x} = \frac{2\cos^2 2x \times \cos x \sin x}{\cos^2 x - \sin^2 x} = \frac{2\cos^2 2x \times 2\sin x \cos x}{2\cos 2x}$$

$$= \cos 2x \sin 2x = \frac{1}{2}(2\sin 2x \cos 2x) = \frac{1}{2}\sin 4x = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} = \frac{\sin 3x(\cos 5x + \cos 4x)}{\sin 3x(1 - 2\cos 3x)} \quad [\text{Multiplying and dividing by } \sin 3x]$$

$$= \frac{\left\{2\sin\frac{3x}{2}\cos\frac{3x}{2}\right\}\left\{2\cos\frac{9x}{2}\cos\frac{x}{2}\right\}}{\sin 3x - 2\sin 3x\cos 3x} = \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{\sin 3x - \sin 6x}$$

$$= \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{2\sin\left(\frac{3x-6x}{2}\right)\cos\left(\frac{3x+6x}{2}\right)} = \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{2\sin\left(-\frac{3x}{2}\right)\cos\frac{9x}{2}}$$

$$= \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{-2\sin\frac{3x}{2}\cos\frac{9x}{2}} = -2\cos\frac{3x}{2}\cos\frac{x}{2} = -(\cos 2x + \cos x) = \text{RHS}$$

(iii) We have,

$$\text{LHS} = \frac{\cos 7x - \cos 8x}{1 + 2\cos 5x}$$

$$= \frac{2\sin\frac{5x}{2}(\cos 7x - \cos 8x)}{2\sin\frac{5x}{2}(1 + 2\cos 5x)} \quad \left[\text{Multiplying numerator and denominator by } 2\sin\frac{5x}{2}\right]$$

$$= \frac{2\sin\frac{5x}{2}\cos 7x - 2\sin\frac{5x}{2}\cos 8x}{2\sin\frac{5x}{2} + 4\sin\frac{5x}{2}\cos 5x} = \frac{2\sin\frac{5x}{2}\cos 7x - 2\sin\frac{5x}{2}\cos 8x}{2\left\{\sin\frac{5x}{2} + 2\sin\frac{5x}{2}\cos 5x\right\}}$$

$$= \frac{\left(\sin\frac{19x}{2} - \sin\frac{9x}{2}\right) - \left(\sin\frac{21x}{2} - \sin\frac{11x}{2}\right)}{2\left\{\sin\frac{5x}{2} + \sin\frac{15x}{2} - \sin\frac{5x}{2}\right\}}$$

$$= \frac{\left(\sin\frac{19x}{2} + \sin\frac{11x}{2}\right) - \left(\sin\frac{9x}{2} + \sin\frac{21x}{2}\right)}{2\sin\frac{15x}{2}}$$

$$= \frac{2\sin\left(\frac{\frac{19x}{2}+\frac{11x}{2}}{2}\right)\cos\left(\frac{\frac{19x}{2}-\frac{11x}{2}}{2}\right) - 2\sin\left(\frac{\frac{9x}{2}+\frac{21x}{2}}{2}\right)\cos\left(\frac{\frac{9x}{2}-\frac{21x}{2}}{2}\right)}{2\sin\frac{15x}{2}}$$

$$= \frac{2 \sin \frac{15x}{2} \cos 2x - 2 \sin \frac{15x}{2} \cos(-3x)}{2 \sin \frac{15x}{2}} = \cos 2x - \cos 3x = \text{RHS}$$

EXAMPLE 31 *If* $\tan \alpha = \frac{p}{q}$, *where* $\alpha = 6\beta$, *α being an acute angle, prove that :*

$$\frac{1}{2}\left\{p \operatorname{cosec} 2\beta - q \sec 2\beta\right\} = \sqrt{p^2 + q^2}.$$

SOLUTION We have, $\tan \alpha = \frac{p}{q}$. Therefore, $\sin \alpha = \frac{p}{\sqrt{p^2 + q^2}}$ and, $\cos \alpha = \frac{q}{\sqrt{p^2 + q^2}}$

Now,

$$\text{LHS} = \frac{1}{2}\left\{p \operatorname{cosec} 2\beta - q \sec 2\beta\right\} = \frac{\sqrt{p^2 + q^2}}{2}\left\{\frac{p}{\sqrt{p^2 + q^2}} \operatorname{cosec} 2\beta - \frac{q}{\sqrt{p^2 + q^2}} \sec 2\beta\right\}$$

$$= \frac{\sqrt{p^2 + q^2}}{2}\left\{\sin \alpha \operatorname{cosec} 2\beta - \cos \alpha \sec 2\beta\right\} = \frac{\sqrt{p^2 + q^2}}{2}\left\{\frac{\sin \alpha}{\sin 2\beta} - \frac{\cos \alpha}{\cos 2\beta}\right\}$$

$$= \frac{\sqrt{p^2 + q^2}}{2}\left\{\frac{\sin \alpha \cos 2\beta - \cos \alpha \sin 2\beta}{\sin 2\beta \cos 2\beta}\right\} = \sqrt{p^2 + q^2}\left\{\frac{\sin(\alpha - 2\beta)}{2 \sin 2\beta \cos 2\beta}\right\}$$

$$= \sqrt{p^2 + q^2}\left\{\frac{\sin(6\beta - 2\beta)}{\sin 4\beta}\right\} = \sqrt{p^2 + q^2} = \text{RHS}. \qquad [\because \alpha = 6\beta]$$

EXAMPLE 32 *Prove that:* $\tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha = \cot \alpha$

SOLUTION We find that

$$\cot x - \tan x = \frac{1}{\tan x} - \tan x = \frac{1 - \tan^2 x}{\tan x} = 2\left\{\frac{1 - \tan^2 x}{2 \tan x}\right\} = \frac{2}{\tan 2x} = 2 \cot 2x \quad ...(i)$$

$\therefore$ LHS $= \tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha$

$= \cot \alpha - \cot \alpha + \tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha$

$= \cot \alpha - \{\cot \alpha - \tan \alpha - 2 \tan 2\alpha - 4 \tan 4\alpha - 8 \cot 8\alpha\}$

$= \cot \alpha - \{(\cot \alpha - \tan \alpha) - 2 \tan 2\alpha - 4 \tan 4\alpha - 8 \cot 8\alpha\}$

$= \cot \alpha - \{2 \cot 2\alpha - 2 \tan 2\alpha - 4 \tan 4\alpha - 8 \cot 8\alpha\}$

$= \cot \alpha - \{2(\cot 2\alpha - \tan 2\alpha) - 4 \tan 4\alpha - 8 \cot 8\alpha\}$ [Using (i)]

$= \cot \alpha - \{2 \times 2 \cot 2(2\alpha) - 4 \tan 4\alpha - 8 \cot 8\alpha\}$ [Using (i)]

$= \cot \alpha - \{4(\cot 4\alpha - \tan 4\alpha) - 8 \cot 8\alpha\}$ [Using (i)]

$= \cot \alpha - \{4 \times 2 \cot 2(4\alpha) - 8 \cot 8\alpha\} = \cot \alpha - (8 \cot 8\alpha - 8 \cot 8\alpha) = \cot \alpha = \text{RHS}$

EXAMPLE 33 *If* $\tan \beta = \frac{\tan \alpha + \tan \gamma}{1 + \tan \alpha \tan \gamma}$, *prove that* $\sin 2\beta = \frac{\sin 2\alpha + \sin 2\gamma}{1 + \sin 2\alpha \sin 2\gamma}$

SOLUTION It is given that $\tan \beta = \frac{\tan \alpha + \tan \gamma}{1 + \tan \alpha \tan \gamma}$

$$\Rightarrow \quad \tan\beta = \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\gamma}{\cos\gamma}}{1 + \frac{\sin\alpha}{\cos\alpha} \times \frac{\sin\gamma}{\cos\gamma}} = \frac{\sin\alpha\cos\gamma + \cos\alpha\sin\gamma}{\cos\alpha\cos\gamma + \sin\alpha\sin\gamma} = \frac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)} \quad \text{...(i)}$$

$$\therefore \quad \sin 2\beta = \frac{2\tan\beta}{1+\tan^2\beta} = \frac{\frac{2\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}}{1 + \frac{\sin^2(\alpha+\gamma)}{\cos^2(\alpha-\gamma)}} \quad \text{[Using (i)]}$$

$$= \frac{2\sin(\alpha+\gamma)\cos(\alpha-\gamma)}{\cos^2(\alpha-\gamma)+\sin^2(\alpha+\gamma)} = \frac{2(\sin 2\alpha + \sin 2\gamma)}{2\cos^2(\alpha-\gamma) + 2\sin^2(\alpha+\gamma)}$$

$$= \frac{2(\sin 2\alpha + \sin 2\gamma)}{1+\cos(2\alpha-2\gamma)+1-\cos(2\alpha+2\gamma)} = \frac{2(\sin 2\alpha + \sin 2\gamma)}{2+\cos(2\alpha-2\gamma)-\cos(2\alpha+2\gamma)}$$

$$= \frac{2(\sin 2\alpha + \sin 2\gamma)}{2+2\sin 2\alpha\sin 2\gamma} = \frac{\sin 2\alpha + \sin 2\gamma}{1+\sin 2\alpha \sin 2\gamma}$$

EXAMPLE 34 *If* $\sin(\theta+\alpha) = a$ *and* $\sin(\theta+\beta) = b$, *prove that* $\cos 2(\alpha-\beta) - 4ab\cos(\alpha-\beta) = 1 - 2a^2 - 2b^2$.

[NCERT EXEMPLAR]

SOLUTION We have, $\sin(\theta+\alpha) = a$ and $\sin(\theta+\beta) = b$

$$\therefore \quad \cos(\theta+\alpha) = \sqrt{1-\sin^2(\theta+\alpha)} = \sqrt{1-a^2} \text{ and, } \cos(\theta+\beta) = \sqrt{1-\sin^2(\theta+\beta)} = \sqrt{1-b^2}$$

Now,

$$\cos(\alpha-\beta) = \cos\{(\theta+\alpha)-(\theta+\beta)\} = \cos(\theta+\alpha)\cos(\theta+\beta) + \sin(\theta+\alpha)\sin(\theta+\beta)$$

$$= \sqrt{1-a^2}\sqrt{1-b^2} + ab = ab + \sqrt{1-a^2-b^2+a^2b^2}$$

$$\therefore \quad \cos 2(\alpha-\beta) - 4ab\cos(\alpha-\beta) = 2\cos^2(\alpha-\beta) - 1 - 4ab\cos(\alpha-\beta)$$

$$= 2\left\{ab + \sqrt{1-a^2-b^2+a^2b^2}\right\}^2 - 1 - 4ab\left\{ab + \sqrt{1-a^2-b^2+a^2b^2}\right\}$$

$$= 2\left\{a^2b^2 + 2ab\sqrt{1-a^2-b^2+a^2b^2} + 1 - a^2 - b^2 + a^2b^2\right\} - 1 - 4a^2b^2 - 4ab\sqrt{1-a^2-b^2+a^2b^2}$$

$$= 2 - 2a^2 - 2b^2 - 1 = 1 - 2a^2 - 2b^2.$$

EXAMPLE 35 *If* $\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \tan^3\left(\frac{\pi}{4} + \frac{\iota}{2}\right)$, *prove that* $\sin x = \dfrac{3\sin\alpha + \sin^3\alpha}{1 + 3\sin^2\alpha}$.

SOLUTION We have,

$$\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \tan^3\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$$\Rightarrow \quad \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} = \left\{\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right\}^3 \Rightarrow \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}} = \left\{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\right\}^3$$

$$\Rightarrow \quad \left\{\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right\}^2 = \left[\left\{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\right\}^2\right]^3 \quad \text{[On squaring both sides]}$$

$$\Rightarrow \quad \frac{1 + 2\sin\frac{x}{2}\cos\frac{x}{2}}{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}} = \left\{\frac{1 + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1 - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right\}^3 \Rightarrow \frac{1+\sin x}{1-\sin x} = \left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)^3$$

$$\Rightarrow \quad \frac{(1+\sin x)-(1-\sin x)}{(1+\sin x)+(1-\sin x)} = \frac{(1+\sin\alpha)^3-(1-\sin\alpha)^3}{(1+\sin\alpha)^3+(1-\sin\alpha)^3} \quad \left[\text{Applying componendo-dividendo}\right]$$

$$\Rightarrow \quad \frac{2\sin x}{2} = \frac{6\sin\alpha + 2\sin^3\alpha}{2+6\sin^2\alpha} \Rightarrow \sin x = \frac{3\sin\alpha + \sin^3\alpha}{1+3\sin^2\alpha}$$

EXAMPLE 36 *If* $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2}$, *prove that* $\cos\phi = \frac{\cos\theta - e}{1 - e\cos\theta}$.

SOLUTION We have,

$$\cos\phi = \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}$$

$$\Rightarrow \quad \cos\phi = \frac{1 - \frac{1+e}{1-e}\tan^2\frac{\theta}{2}}{1 + \frac{1+e}{1-e}\tan^2\frac{\theta}{2}} \quad \left[\because \tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2} \Rightarrow \tan\frac{\phi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\theta}{2}\right]$$

$$\Rightarrow \quad \cos\phi = \frac{(1-e)-(1+e)\tan^2\frac{\theta}{2}}{(1-e)+(1+e)\tan^2\frac{\theta}{2}} = \frac{(1-\tan^2\theta/2) - e(1+\tan^2\theta/2)}{(1+\tan^2\theta/2) - e(1-\tan^2\theta/2)}$$

$$\Rightarrow \quad \cos\phi = \frac{\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2} - e}{1 - e\,\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}} \quad \left[\text{Dividing numerator and denominator by } 1+\tan^2\frac{\theta}{2}\right]$$

$$\Rightarrow \quad \cos\phi = \frac{\cos\theta - e}{1 - e\cos\theta} \quad \left[\because \cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right]$$

EXAMPLE 37 *Prove that:*

$$\frac{2\cos 2^n x + 1}{2\cos x + 1} = (2\cos x - 1)(2\cos 2x - 1)(2\cos 2^2 x - 1)\ldots(2\cos 2^{n-1}x - 1)$$

SOLUTION RHS $= (2\cos x - 1)(2\cos 2x - 1)(2\cos 2^2 x - 1)\ldots(2\cos 2^{n-1}x - 1)$

$$= \frac{1}{(2\cos x + 1)}\left\{(2\cos x + 1)(2\cos x - 1)(2\cos 2x - 1)(2\cos 2^2 x - 1)\ldots(2\cos 2^{n-1}x - 1\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{(4\cos^2 x - 1)(2\cos 2x - 1)(2\cos 2^2 x - 1)\ldots(2\cos 2^{n-1}x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{\{2(1+\cos 2x) - 1\}(2\cos 2x - 1)(2\cos 2^2 x - 1)\ldots(2\cos 2^{n-1}x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{(2\cos 2x + 1)(2\cos 2x - 1)(2\cos 2^2 x - 1)\dots(2\cos 2^{n-1} x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{(4\cos^2 2x - 1)(2\cos 2^2 x - 1)(2\cos 2^3 x - 1)\dots(2\cos 2^{n-1} x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{\left\{2(\cos 4x + 1) - 1\right\}(2\cos 2^2 x - 1)(2\cos 2^3 x - 1)\dots(2\cos 2^{n-1} x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{(2\cos 2^2 x + 1)(2\cos 2^2 x - 1)(2\cos 2^3 x - 1)\dots\dots(2\cos 2^{n-1} x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{(4\cos^2 2^2 x - 1)(2\cos 2^3 x - 1)\dots\dots(2\cos 2^{n-1} x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{\left\{2(1 + \cos 2^3 x) - 1\right\}(2\cos 2^3 x - 1)\dots(2\cos 2^{n-1} x - 1)\right\}$$

$$= \frac{1}{(2\cos x + 1)}\left\{(2\cos 2^3 x + 1)(2\cos 2^3 x - 1)\dots(2\cos 2^{n-1} x - 1)\right\}$$

....

$$= \frac{1}{(2\cos x + 1)}(2\cos 2^{n-1} x + 1)(2\cos 2^{n-1} x - 1)$$

$$= \frac{1}{(2\cos x + 1)}(4\cos^2 2^{n-1} x - 1) = \frac{1}{(2\cos x + 1)}\left\{2(\cos 2 . 2^{n-1} x + 1) - 1\right\}$$

$$= \frac{1}{(2\cos x + 1)}(2\cos 2^n x + 2 - 1) = \frac{2\cos 2^n x + 1}{2\cos x + 1} = \text{LHS}$$

Type VII ON CONDITIONAL IDENTITIES

EXAMPLE 38 *If* $\tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\phi}{2}$, *prove that* $\cos\theta = \frac{a\cos\phi + b}{a + b\cos\phi}$.

SOLUTION Putting $\tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\phi}{2}$ in $\cos\theta = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$, we obtain

$$\Rightarrow \quad \cos\theta = \frac{1 - \frac{a-b}{a+b}\tan^2\frac{\phi}{2}}{1 + \frac{a-b}{a+b}\tan^2\frac{\phi}{2}} = \frac{(a+b) - (a-b)\tan^2\frac{\phi}{2}}{(a+b) + (a-b)\tan^2\frac{\phi}{2}}$$

$$\Rightarrow \quad \cos\theta = \frac{a\left(1 - \tan^2\frac{\phi}{2}\right) + b\left(1 + \tan^2\frac{\phi}{2}\right)}{a\left(1 + \tan^2\frac{\phi}{2}\right) + b\left(1 - \tan^2\frac{\phi}{2}\right)}$$

$$\Rightarrow \quad \cos\theta = \frac{a\left(\dfrac{1-\tan^2\dfrac{\phi}{2}}{1+\tan^2\dfrac{\phi}{2}}\right)+b}{a+b\left(\dfrac{1-\tan^2\dfrac{\phi}{2}}{1+\tan^2\dfrac{\phi}{2}}\right)} \quad \left[\text{Dividing numerator and denominator by } 1+\tan^2\frac{\phi}{2}\right]$$

$$\Rightarrow \quad \cos\theta = \frac{a\cos\phi+b}{a+b\cos\phi}$$

EXAMPLE 39 *If* $\cos\theta = \cos\alpha\cos\beta$, *prove that* $\tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2} = \tan^2\frac{\beta}{2}$.

SOLUTION We have,

$$\cos\theta = \cos\alpha\cos\beta$$

$$\Rightarrow \quad \cos\beta = \frac{\cos\theta}{\cos\alpha}$$

$$\Rightarrow \quad \frac{1-\tan^2\dfrac{\beta}{2}}{1+\tan^2\dfrac{\beta}{2}} = \frac{\cos\theta}{\cos\alpha}$$

$$\Rightarrow \quad \frac{\left(1-\tan^2\dfrac{\beta}{2}\right)+\left(1+\tan^2\dfrac{\beta}{2}\right)}{\left(1-\tan^2\dfrac{\beta}{2}\right)-\left(1+\tan^2\dfrac{\beta}{2}\right)} = \frac{\cos\theta+\cos\alpha}{\cos\theta-\cos\alpha} \quad [\text{Applying componendo - dividendo}]$$

$$\Rightarrow \quad \frac{2}{-2\tan^2\dfrac{\beta}{2}} = \frac{2\cos\dfrac{\theta+\alpha}{2}\cos\dfrac{\theta-\alpha}{2}}{-2\sin\dfrac{\theta+\alpha}{2}\sin\dfrac{\theta-\alpha}{2}}$$

$$\Rightarrow \quad \frac{1}{\tan^2\dfrac{\beta}{2}} = \frac{1}{\tan\dfrac{\theta+\alpha}{2}\tan\dfrac{\theta-\alpha}{2}}$$

$$\Rightarrow \quad \tan^2\frac{\beta}{2} = \tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2}$$

EXAMPLE 40 *If* $\cos\theta = \dfrac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$, *prove that* $\tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$.

SOLUTION We have,

$$\cos\theta = \frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$$

$$\Rightarrow \quad \frac{1-\tan^2\dfrac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}} = \frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$$

$$\Rightarrow \quad \frac{\left(1-\tan^2\dfrac{\theta}{2}\right)+\left(1+\tan^2\dfrac{\theta}{2}\right)}{\left(1-\tan^2\dfrac{\theta}{2}\right)-\left(1+\tan^2\dfrac{\theta}{2}\right)} = \frac{(\cos\alpha-\cos\beta)+(1-\cos\alpha\cos\beta)}{(\cos\alpha-\cos\beta)-(1-\cos\alpha\cos\beta)}$$

$$\Rightarrow \quad \frac{2}{-2\tan^2\dfrac{\theta}{2}} = \frac{1+\cos\alpha-\cos\beta-\cos\alpha\cos\beta}{-\{1-\cos\alpha+\cos\beta-\cos\alpha\cos\beta\}}$$

$$\Rightarrow \quad \frac{1}{\tan^2 \frac{\theta}{2}} = \frac{(1+\cos\alpha)(1-\cos\beta)}{(1-\cos\alpha)(1+\cos\beta)}$$

$$\Rightarrow \quad \frac{1}{\tan^2 \frac{\theta}{2}} = \frac{2\cos^2 \frac{\alpha}{2} \times 2\sin^2 \frac{\beta}{2}}{2\sin^2 \frac{\alpha}{2} \times 2\cos^2 \frac{\beta}{2}}$$

$$\Rightarrow \quad \tan^2 \frac{\theta}{2} = \tan^2 \frac{\alpha}{2} \cot^2 \frac{\beta}{2} \Rightarrow \tan \frac{\theta}{2} = \pm \tan \frac{\alpha}{2} \cot \frac{\beta}{2}$$

EXAMPLE 41 *If* $\cos\theta = \dfrac{\cos\alpha \cos\beta}{1-\sin\alpha \sin\beta}$, *prove that one value of* $\tan \dfrac{\theta}{2} = \dfrac{\tan \frac{\alpha}{2} - \tan \frac{\beta}{2}}{1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}$.

SOLUTION We have, $\cos\theta = \dfrac{\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta}$

Now, $\tan^2 \dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta}$

$$\Rightarrow \quad \tan^2 \frac{\theta}{2} = \frac{1 - \frac{\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta}}{1 + \frac{\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta}} = \frac{1-\sin\alpha\sin\beta - \cos\alpha\cos\beta}{1-\sin\alpha\sin\beta+\cos\alpha\cos\beta}$$

$$\Rightarrow \quad \tan^2 \frac{\theta}{2} = \frac{1-(\cos\alpha\cos\beta+\sin\alpha\sin\beta)}{1+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)} = \frac{1-\cos(\alpha-\beta)}{1+\cos(\alpha+\beta)} = \frac{2\sin^2\left(\frac{\alpha-\beta}{2}\right)}{2\cos^2\left(\frac{\alpha+\beta}{2}\right)}$$

$$\Rightarrow \quad \tan \frac{\theta}{2} = \pm \frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)} = \pm \frac{\sin\frac{\alpha}{2}\cos\frac{\beta}{2} - \cos\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$\Rightarrow \quad \tan \frac{\theta}{2} = \pm \frac{\tan\frac{\alpha}{2} - \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}} \quad \left[\text{Dividing numerator and denominator by } \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\right]$$

EXAMPLE 42 *If* $\sin\alpha + \sin\beta = a$ *and* $\cos\alpha + \cos\beta = b$, *prove that*

(i) $\cos(\alpha-\beta) = \dfrac{a^2+b^2-2}{2}$ (ii) $\tan \dfrac{\alpha-\beta}{2} = \pm\sqrt{\dfrac{4-a^2-b^2}{a^2+b^2}}$

SOLUTION (i) We have,

$$\sin\alpha + \sin\beta = a \text{ and } \cos\alpha + \cos\beta = b$$

$$\Rightarrow \quad (\sin\alpha+\sin\beta)^2 + (\cos\alpha+\cos\beta)^2 = a^2+b^2$$

$$\Rightarrow \quad (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta+\sin^2\beta) + 2(\cos\alpha\cos\beta+\sin\alpha\sin\beta) = a^2+b^2$$

$$\Rightarrow \quad 2+2\cos(\alpha-\beta) = a^2+b^2 \Rightarrow \cos(\alpha-\beta) = \frac{a^2+b^2-2}{2}$$

(ii) Now, $\tan^2 \dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta}$

$$\Rightarrow \quad \tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1-\cos(\alpha-\beta)}{1+\cos(\alpha-\beta)} \qquad [\text{Replacing } \theta \text{ by } (\alpha-\beta)]$$

$$\Rightarrow \quad \tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1-\dfrac{a^2+b^2-2}{2}}{1+\dfrac{a^2+b^2-2}{2}} \qquad \left[\because \cos(\alpha-\beta) = \frac{a^2+b^2-2}{2}\right]$$

$$\Rightarrow \quad \tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{4-a^2-b^2}{a^2+b^2} \Rightarrow \tan\left(\frac{\alpha-\beta}{2}\right) = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

EXAMPLE 43 *If α and β are distinct roots of $a\cos x + b\sin x = c$, prove that* $\sin(\alpha+\beta) = \dfrac{2ab}{a^2+b^2}$.

SOLUTION It is given that α and β are distinct roots of $a\cos x + b\sin x = c$

$$\therefore \quad a\cos\alpha + b\sin\alpha = c \text{ and } a\cos\beta + b\sin\beta = c$$

$$\Rightarrow \quad (a\cos\alpha + b\sin\alpha) - (a\cos\beta + b\sin\beta) = c - c$$

$$\Rightarrow \quad a(\cos\alpha - \cos\beta) + (b\sin\alpha - \sin\beta) = 0$$

$$\Rightarrow \quad -2a\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} + 2b\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} = 0$$

$$\Rightarrow \quad 2a\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = 2b\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\Rightarrow \quad \tan\frac{\alpha+\beta}{2} = \frac{b}{a} \qquad \left[\because \alpha \neq \beta \therefore \sin\frac{\alpha-\beta}{2} \neq 0\right]$$

$$\therefore \quad \sin(\alpha+\beta) = \frac{2\tan\dfrac{\alpha+\beta}{2}}{1+\tan^2\dfrac{\alpha+\beta}{2}} \Rightarrow \sin(\alpha+\beta) = \frac{\dfrac{2b}{a}}{1+\dfrac{b^2}{a^2}} = \frac{2ab}{a^2+b^2}$$

ALITER We have,

$$a\cos x + b\sin x = c \qquad \ldots(\text{i})$$

$$\Rightarrow \quad a\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + b\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) = c \Rightarrow a\left(1-\tan^2\frac{x}{2}\right) + 2b\tan\frac{x}{2} = c\left(1+\tan^2\frac{x}{2}\right)$$

$$\Rightarrow \quad (c+a)\tan^2\frac{x}{2} - 2b\tan\frac{x}{2} + (c-a) = 0 \qquad \ldots(\text{ii})$$

It is given that α and β are roots of the equation (i). Therefore, $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are roots of equation (ii).

$$\therefore \quad \tan\frac{\alpha}{2} + \tan\frac{\beta}{2} = \frac{2b}{c+a} \text{ and, } \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{c-a}{c+a} \qquad \ldots(\text{iii})$$

$$\text{Now,} \quad \tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) = \frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} \Rightarrow \tan\left(\frac{\alpha+\beta}{2}\right) = \frac{2b/c+a}{1-\dfrac{c-a}{c+a}} = \frac{b}{a} \qquad [\text{Using (iii)}]$$

$$\therefore \qquad \sin(\alpha+\beta) = \frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{2\frac{b}{a}}{1+\frac{b^2}{a^2}} = \frac{2ab}{a^2+b^2}$$

EXERCISE 9.1

BASIC

Prove the following identities: (1– 25)

1. $\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}} = \tan x$

2. $\dfrac{\sin 2x}{1-\cos 2x} = \cot x$

3. $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$

4. $\sqrt{2+\sqrt{2+2\cos 4x}} = 2\cos x, 0 < x < \dfrac{\pi}{4}$

5. $\dfrac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x} = \tan x$

6. $\dfrac{\sin x+\sin 2x}{1+\cos x+\cos 2x} = \tan x$

7. $\dfrac{\cos 2x}{1+\sin 2x} = \tan\left(\dfrac{\pi}{4}-x\right)$

8. $\dfrac{\cos x}{1-\sin x} = \tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$

9. $\cos^2\dfrac{\pi}{8}+\cos^2\dfrac{3\pi}{8}+\cos^2\dfrac{5\pi}{8}+\cos^2\dfrac{7\pi}{8}=2$

10. $\sin^2\dfrac{\pi}{8}+\sin^2\dfrac{3\pi}{8}+\sin^2\dfrac{5\pi}{8}+\sin^2\dfrac{7\pi}{8}=2$

11. $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2=4\cos^2\left(\dfrac{\alpha-\beta}{2}\right)$

12. $\sin^2\left(\dfrac{\pi}{8}+\dfrac{x}{2}\right)-\sin^2\left(\dfrac{\pi}{8}-\dfrac{x}{2}\right)=\dfrac{1}{\sqrt{2}}\sin x$

13. $1+\cos^2 2x = 2(\cos^4 x+\sin^4 x)$

14. $\cos^3 2x+3\cos 2x = 4(\cos^6 x-\sin^6 x)$

15. $(\sin 3x+\sin x)\sin x+(\cos 3x-\cos x)\cos x=0$

16. $\cos^2\left(\dfrac{\pi}{4}-x\right)-\sin^2\left(\dfrac{\pi}{4}-x\right) = \sin 2x$

17. $\cos 4x = 1-8\cos^2 x+8\cos^4 x$

18. $\sin 4x = 4\sin x\cos^3 x-4\cos x\sin^3 x$ [NCERT EXEMPLAR]

BASED ON LOTS

19. $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x) = 13$

20. $2(\sin^6 x+\cos^6 x)-3(\sin^4 x+\cos^4 x)+1 = 0$

21. $\cos^6 x-\sin^6 x=\cos 2x\left(1-\dfrac{1}{4}\sin^2 2x\right)$

22. $\tan\left(\dfrac{\pi}{4}+x\right)+\tan\left(\dfrac{\pi}{4}-x\right)=2\sec 2x$

23. $\cot^2 x-\tan^2 x = 4\cot 2x\,\mathrm{cosec}\,2x$

24. $\cos 4x-\cos 4\alpha = 8(\cos x-\cos\alpha)(\cos x+\cos\alpha)(\cos x-\sin\alpha)(\cos x+\sin\alpha)$

25. $\sin 3x + \sin 2x - \sin x = 4 \sin x \cos \frac{x}{2} \cos \frac{3x}{2}$ [NCERT]

26. Prove that : $\tan 82\frac{1}{2}^\circ = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1) = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

27. Prove that : $\cot \frac{\pi}{8} = \sqrt{2}+1$

28. (i) If $\cos x = -\frac{3}{5}$ and x lies in the IIIrd quadrant, find the values of $\cos \frac{x}{2}$, $\sin \frac{x}{2}$ and, $\sin 2x$.

(ii) If $\cos x = -\frac{3}{5}$ and x lies in IInd quadrant, find the values of $\sin 2x$ and $\sin \frac{x}{2}$.

29. If $\sin x = \frac{\sqrt{5}}{3}$ and x lies in IInd quadrant, find the values of $\cos \frac{x}{2}$, $\sin \frac{x}{2}$ and $\tan \frac{x}{2}$.

30. (i) If $0 \le x \le \pi$ and x lies in the IInd quadrant such that $\sin x = \frac{1}{4}$. Find the values of $\cos \frac{x}{2}$, $\sin \frac{x}{2}$ and $\tan \frac{x}{2}$.

(ii) If $\cos x = \frac{4}{5}$ and x is acute, find $\tan 2x$

(iii) If $\sin x = \frac{4}{5}$ and $0 < x < \frac{\pi}{2}$, find the value of $\sin 4x$.

31. If $\tan x = \frac{b}{a}$, then find the value of $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$. [NCERT]

32. If $\tan A = \frac{1}{7}$ and $\tan B = \frac{1}{3}$, show that $\cos 2A = \sin 4B$.

33. Prove that: $\cos 7^\circ \cos 14^\circ \cos 28^\circ \cos 56^\circ = \frac{\sin 68^\circ}{16 \cos 83^\circ}$

BASED ON HOTS

34. Prove that: $\cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{16\pi}{15} = \frac{1}{16}$

35. Prove that: $\cos \frac{\pi}{5} \cos \frac{2\pi}{5} \cos \frac{4\pi}{5} \cos \frac{8\pi}{5} = \frac{-1}{16}$

36. Prove that: $\cos \frac{\pi}{65} \cos \frac{2\pi}{65} \cos \frac{4\pi}{65} \cos \frac{8\pi}{65} \cos \frac{16\pi}{65} \cos \frac{32\pi}{65} = \frac{1}{64}$

37. If $2 \tan \alpha = 3 \tan \beta$, prove that $\tan (\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}$.

38. If $\sin \alpha + \sin \beta = a$ and $\cos \alpha + \cos \beta = b$, prove that

(i) $\sin (\alpha + \beta) = \frac{2ab}{a^2+b^2}$ (ii) $\cos (\alpha - \beta) = \frac{a^2+b^2-2}{2}$

39. If $2 \tan \frac{\alpha}{2} = \tan \frac{\beta}{2}$, prove that $\cos \alpha = \frac{3 + 5 \cos \beta}{5 + 3 \cos \beta}$.

40. If $\cos x = \frac{\cos \alpha + \cos \beta}{1 + \cos \alpha \cos \beta}$, prove that $\tan \frac{x}{2} = \pm \tan \frac{\alpha}{2} \tan \frac{\beta}{2}$

41. If $\sec(x+\alpha)+\sec(x-\alpha)=2\sec x$, prove that $\cos x=\pm\sqrt{2}\cos\frac{\alpha}{2}$

42. If $\cos\alpha+\cos\beta=\frac{1}{3}$ and $\sin\alpha+\sin\beta=\frac{1}{4}$, prove that $\cos\frac{\alpha-\beta}{2}=\pm\frac{5}{24}$.

43. If $\sin\alpha=\frac{4}{5}$ and $\cos\beta=\frac{5}{13}$, prove that $\cos\frac{\alpha-\beta}{2}=\frac{8}{\sqrt{65}}$.

44. If $a\cos 2x+b\sin 2x=c$ has α and β as its roots, then prove that

(i) $\tan\alpha+\tan\beta=\frac{2b}{a+c}$ [NCERT EXEMPLAR] (ii) $\tan\alpha\tan\beta=\frac{c-a}{c+a}$

(iii) $\tan(\alpha+\beta)=\frac{b}{a}$ [NCERT EXEMPLAR]

45. If $\cos\alpha+\cos\beta=0=\sin\alpha+\sin\beta$, then prove that $\cos 2\alpha+\cos 2\beta=-2\cos(\alpha+\beta)$.

[NCERT EXEMPLAR]

ANSWERS

28. (i) $-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{24}{25}$ (ii) $-\frac{24}{25}, \frac{2}{\sqrt{5}}$ 29. $\frac{1}{\sqrt{6}}, \sqrt{\frac{5}{6}}, \sqrt{5}$

30. (i) $\sqrt{\frac{4-\sqrt{15}}{8}}, \sqrt{\frac{4+\sqrt{15}}{8}}, 4+\sqrt{15}$ (ii) $\frac{24}{7}$ (iii) $-\frac{336}{625}$ 31. $\frac{2\cos x}{\sqrt{\cos 2x}}$

HINTS TO SELECTED PROBLEMS

25. LHS $= \sin 3x+\sin 2x-\sin x = (\sin 3x-\sin x)+\sin 2x$

$= 2\sin x\cos 2x+\sin 2x = 2\sin x\cos 2x+2\sin x\cos x$

$= 2\sin x(\cos 2x+\cos x) = 2\sin x\left(2\cos\frac{3x}{2}\cos\frac{x}{2}\right) = 4\sin x\cos\frac{x}{2}\cos\frac{3x}{2} =$ RHS

32. Use: $\cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}$, $\sin 4B=\frac{2\tan 2B}{1+\tan^2 2B}$, where $\tan 2B=\frac{2\tan B}{1-\tan^2 B}$

33. Let $A=7°$. Then, $\cos 7°\cos 14°\cos 28°\cos 56°$

$$=\cos A\cos 2A\cos 2^2A\cos 2^3A=\frac{\sin 2^4 A}{2^4\sin A}=\frac{\sin 16A}{16\sin A}=\frac{\sin 112°}{2\sin 7°}=\frac{\sin 68°}{2\cos 83°}$$

34. Let $A=\frac{2\pi}{15}$. Then,

$$\text{LHS}=\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}=\cos A\cos 2A\cos 2^2A\cos 2^3A$$

$$=\frac{\sin 2^4A}{2^4\sin A}=\frac{\sin 16A}{16\sin A}=\frac{\sin\frac{32\pi}{15}}{16\sin\frac{2\pi}{15}}=\frac{\sin\left(2\pi+\frac{2\pi}{15}\right)}{16\sin\frac{2\pi}{15}}=\frac{\sin\frac{2\pi}{15}}{16\sin\frac{2\pi}{15}}=\frac{1}{16}=\text{RHS}$$

44. We have,

$$a\cos 2x+b\sin 2x=c \quad \text{... (i)}$$

$$\Rightarrow \quad a\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)+\frac{2b\tan x}{1+\tan^2 x}=c \Rightarrow (c+a)\tan^2 x-2b\tan x+(c-a)=0 \quad \text{... (ii)}$$

It is given that α, β are roots of equation (i). Therefore, $\tan\alpha$, $\tan\beta$ are roots of equation (ii).

$\therefore \quad \tan\alpha + \tan\beta = \dfrac{2b}{c+a}$ and, $\tan\alpha \tan\beta = \dfrac{c-a}{c+a}$

Hence, $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{2b/c+a}{1-\dfrac{c-a}{c+a}} = \dfrac{b}{a}.$

45. We have, $\cos\alpha + \cos\beta = 0$ and $\sin\alpha + \sin\beta = 0$

$\therefore \quad (\cos\alpha + \cos\beta)^2 - (\sin\alpha + \sin\beta)^2 = 0^2 - 0^2$

$\Rightarrow \quad (\cos^2\alpha - \sin^2\alpha) + (\cos^2\beta - \sin^2\beta) + 2(\cos\alpha\cos\beta - \sin\alpha\sin\beta) = 0$

$\Rightarrow \quad \cos 2\alpha + \cos 2\beta + 2\cos(\alpha+\beta) = 0 \Rightarrow \cos 2\alpha + \cos 2\beta = -2\cos(\alpha+\beta)$

9.3 VALUES OF TRIGONOMETRIC FUNCTIONS AT $3x$ IN TERMS VALUES AT x

THEOREM *For the values of angle x, for which the two sides are meaningful prove that:*

(i) $\sin 3x = 3\sin x - 4\sin^3 x$ (ii) $\cos 3x = 4\cos^3 x - 3\cos x$

(iii) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

PROOF (i) Replacing y by $2x$ in $\sin(x+y) = \sin x\cos y + \cos x \sin y$, we obtain

$\sin(x+2x) = \sin x \cos 2x + \cos x \sin 2x$

$\Rightarrow \quad \sin 3x = \sin x(1 - 2\sin^2 x) + \cos x(2\sin x\cos x)$

$[\because \cos 2x = 1 - 2\sin^2 x \;\&\; \sin 2x = 2\sin x\cos x]$

$\Rightarrow \quad \sin 3x = \sin x - 2\sin^3 x + 2\sin x(1 - \sin^2 x)$

$\Rightarrow \quad \sin 3x = 3\sin x - 4\sin^3 x$

Hence, $\sin 3x = 3\sin x - 4\sin^3 x$

(ii) Replacing y by $2x$ in $\cos(x+y) = \cos x\cos y - \sin x\sin y$, we obtain

$\cos(x+2x) = \cos x\cos 2x - \sin x \sin 2x$ [Replacing y by $2x$]

$\Rightarrow \quad \cos 3x = \cos x\cos 2x - \sin x(2\sin x\cos x)$ $[\because \sin 2x = 2\sin x\cos x]$

$\Rightarrow \quad \cos 3x = \cos x(2\cos^2 x - 1) - 2\cos x(1 - \cos^2 x)$ $[\because \cos 2x = 2\cos^2 x - 1]$

$\Rightarrow \quad \cos 3x = 4\cos^3 x - 3\cos x$

Hence, $\cos 3x = 4\cos^3 x - 3\cos x$

(iii) Replacing y by $2x$ in $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$, we obtain

$$\tan(x+2x) = \frac{\tan x + \tan 2x}{1 - \tan x\tan 2x} = \frac{\tan x + \dfrac{2\tan x}{1-\tan^2 x}}{1 - \tan x \times \dfrac{2\tan x}{1-\tan^2 x}} \quad \left[\because \tan 2x = \frac{2\tan x}{1-\tan^2 x}\right]$$

$$\Rightarrow \quad \tan 3x = \frac{\tan x(1-\tan^2 x) + 2\tan x}{1 - \tan^2 x - 2\tan^2 x} = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

Hence, $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

Q.E.D.

NOTE *It should be noted the angle on the RHS of these formulae is one third of the angle on LHS.*

$\therefore\ \sin\frac{\pi}{3} = 3\sin\frac{\pi}{9} - 4\sin^3\frac{\pi}{9},\ \sin\frac{\pi}{6} = 3\sin\frac{\pi}{18} - 4\sin^3\frac{\pi}{18},\ \cos\frac{2\pi}{3} = 4\cos^3\frac{2\pi}{9} - 3\cos\frac{2\pi}{9}$ etc.

9.4 VALUES OF TRIGONOMETRIC FUNCTIONS AT x IN TERMS OF VALUES AT $\frac{x}{3}$

Replacing x by $x/3$ in the formulas in the previous section, we obtain the following formulae:

(i) $\sin x = 3\sin\frac{x}{3} - 4\sin^3\frac{x}{3}$ (ii) $\cos x = 4\cos^3\frac{x}{3} - 3\cos\frac{x}{3}$ (iii) $\tan x = \dfrac{3\tan\frac{x}{3} - \tan^3\frac{x}{3}}{1 - 3\tan^2\frac{x}{3}}$

ILLUSTRATIVE EXAMPLES

BASIC

EXAMPLE 1 *Prove that :* $8\cos^3\frac{\pi}{9} - 6\cos\frac{\pi}{9} = 1$

SOLUTION $\text{LHS} = 2\left(4\cos^3\frac{\pi}{9} - 3\cos\frac{\pi}{9}\right) = 2\cos\left(3\times\frac{\pi}{9}\right) = 2\cos\frac{\pi}{3} = 1 = \text{RHS}$

EXAMPLE 2 *Prove that:* $108\sin\frac{\pi}{18} - 144\sin^3\frac{\pi}{18} = 18$

SOLUTION $\text{LHS} = 108\sin\frac{\pi}{18} - 144\sin^3\frac{\pi}{18}$

$$= 36\left(3\sin\frac{\pi}{18} - 4\sin^3\frac{\pi}{18}\right) = 36\sin\left(3\times\frac{\pi}{18}\right) = 36\sin\frac{\pi}{6} = 36\times\frac{1}{2} = 18 = \text{RHS}$$

EXAMPLE 3 *Prove that:* $15\sin\frac{5\pi}{12} + 15\cos\frac{5\pi}{12} - 20\sin^3\frac{5\pi}{12} - 20\cos^3\frac{5\pi}{12} = 0$

SOLUTION $\text{LHS} = 15\sin\frac{5\pi}{12} + 15\cos\frac{5\pi}{12} - 20\sin^3\frac{5\pi}{12} - 20\cos^3\frac{5\pi}{12}$

$$= \left(15\sin\frac{5\pi}{12} - 20\sin^3\frac{5\pi}{12}\right) - \left(20\cos^3\frac{5\pi}{12} - 15\cos\frac{5\pi}{12}\right)$$

$$= 5\left(3\sin\frac{5\pi}{12} - 4\sin^3\frac{5\pi}{12}\right) - 5\left(4\cos^3\frac{5\pi}{12} - 3\cos\frac{5\pi}{12}\right)$$

$$= 5\sin\left(3\times\frac{5\pi}{12}\right) - 5\cos\left(3\times\frac{5\pi}{12}\right) = 5\sin\frac{5\pi}{4} - 5\cos\frac{5\pi}{4} = -5\sin\frac{\pi}{4} + 5\cos\frac{\pi}{4} = 0$$

BASED ON LOTS

EXAMPLE 4 *Prove that:* $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

SOLUTION $\text{LHS} = \cos 6x = 2\cos^2 3x - 1$ $[\because \cos 2x = 2\cos^2 x - 1]$

$$= 2(4\cos^3 x - 3\cos x)^2 - 1 = 2(16\cos^6 x + 9\cos^2 x - 24\cos^4 x) - 1$$

$$= 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1 = \text{RHS}$$

EXAMPLE 5 *Prove that:* $\cos x\cos\left(\frac{\pi}{3} - x\right)\cos\left(\frac{\pi}{3} + x\right) = \frac{1}{4}\cos 3x$

SOLUTION $\text{LHS} = \cos x\cos\left(\frac{\pi}{3} - x\right)\cos\left(\frac{\pi}{3} + x\right)$

$$= \cos x\left(\cos^2\frac{\pi}{3}-\sin^2 x\right) \qquad [\because \cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B]$$

$$= \cos x\left(\frac{1}{4}-\sin^2 x\right)=\cos x\left\{\frac{1}{4}-(1-\cos^2 x)\right\}=\cos x\left(-\frac{3}{4}+\cos^2 x\right)$$

$$= \frac{1}{4}\cos x(-3+4\cos^2 x) = \frac{1}{4}(4\cos^3 x-3\cos x) = \frac{1}{4}\cos 3x = \text{RHS}$$

EXAMPLE 6 *Prove that:* $\sin x\sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right)=\frac{1}{4}\sin 3x$

SOLUTION $\text{LHS} = \sin x\sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right)$

$$= \sin x\left(\sin^2\frac{\pi}{3}-\sin^2 x\right) \qquad [\because \sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B]$$

$$= \sin x\left(\frac{3}{4}-\sin^2 x\right)=\frac{1}{4}\sin x(3-4\sin^2 x)=\frac{1}{4}(3\sin x-4\sin^3 x)=\frac{1}{4}\sin 3x=\text{RHS}$$

NOTE *Reader is advised to learn the results derived in the above two examples as standard results. The following example is an application of the above results.*

EXAMPLE 7 *Prove that:* $\sin 20°\sin 40°\sin 60°\sin 80°=\frac{3}{16}$

SOLUTION $\text{LHS} = \frac{\sqrt{3}}{2}\left\{\sin 20°\sin(60°-20°)\sin(60+20°)\right\}$

$$= \frac{\sqrt{3}}{2}\left\{\sin x\sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right)\right\}, \text{ where } x=20°$$

$$= \frac{\sqrt{3}}{2}\times\frac{1}{4}\sin 3x=\frac{\sqrt{3}}{8}\times\sin\frac{\pi}{3}=\frac{\sqrt{3}}{8}\times\frac{\sqrt{3}}{2}=\frac{3}{16}=\text{RHS}$$

EXAMPLE 8 *Prove that:*

(i) $\tan x+\tan\left(\frac{\pi}{3}+x\right)-\tan\left(\frac{\pi}{3}-x\right)=3\tan 3x$

(ii) $\cot x+\cot\left(\frac{\pi}{3}+x\right)-\cot\left(\frac{\pi}{3}-x\right)=3\cot 3x$

SOLUTION (i) $\text{LHS}=\tan x+\tan\left(\frac{\pi}{3}+x\right)-\tan\left(\frac{\pi}{3}-x\right)=\tan x+\frac{\sqrt{3}+\tan x}{1-\sqrt{3}\tan x}-\frac{\sqrt{3}-\tan x}{1+\sqrt{3}\tan x}$

$$= \tan x+\frac{(\sqrt{3}+\tan x)(1+\sqrt{3}\tan x)-(\sqrt{3}-\tan x)(1-\sqrt{3}\tan x)}{(1-\sqrt{3}\tan x)(1+\sqrt{3}\tan x)}$$

$$= \tan x+\frac{8\tan x}{1-3\tan^2 x}=\frac{9\tan x-3\tan^3 x}{1-3\tan^2 x}=3\left(\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}\right)=3\tan 3x=\text{RHS}$$

(ii) $\text{LHS} = \cot x+\cot\left(\frac{\pi}{3}+x\right)-\cot\left(\frac{\pi}{3}-x\right)$

$$= \frac{1}{\tan x}+\frac{1}{\tan\left(\frac{\pi}{3}+x\right)}-\frac{1}{\tan\left(\frac{\pi}{3}-x\right)}=\frac{1}{\tan x}+\frac{1-\sqrt{3}\tan x}{\sqrt{3}+\tan x}-\frac{1+\sqrt{3}\tan x}{\sqrt{3}-\tan x}$$

$$= \frac{1}{\tan x}+\frac{(1-\sqrt{3}\tan x)(\sqrt{3}-\tan x)-(1+\sqrt{3}\tan x)(\sqrt{3}+\tan x)}{(\sqrt{3}+\tan x)(\sqrt{3}-\tan x)}$$

$$= \frac{1}{\tan x} - \frac{8 \tan x}{3 - \tan^2 x} = \frac{3 - 9 \tan^2 x}{3 \tan x - \tan^3 x} = 3\left(\frac{1 - 3\tan^2 x}{3 \tan x - \tan^3 x}\right) = \frac{3}{\tan 3x} = 3 \cot 3x = \text{RHS}$$

BASED ON HOTS

EXAMPLE 9 *If* $\cos \alpha + \cos \beta + \cos \gamma = 0$, *then prove that*

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 12 \cos \alpha \cos \beta \cos \gamma.$$

SOLUTION $\cos 3\alpha + \cos 3\beta + \cos 3\gamma$

$$= (4\cos^3 \alpha - 3\cos \alpha) + (4\cos^3 \beta - 3\cos\beta) + (4\cos^3\gamma - 3\cos\gamma)$$

$$= 4(\cos^3\alpha + \cos^3\beta + \cos^3\gamma) - 3(\cos\alpha + \cos\beta + \cos\gamma)$$

$$= 4(\cos^3\alpha + \cos^3\beta + \cos^3\gamma) - 3 \times 0 \qquad [\because \cos\alpha + \cos\beta + \cos\gamma = 0]$$

$$= 4 \times 3 \cos\alpha \cos\beta \cos\gamma = 12 \cos\alpha \cos\beta\cos\gamma \quad [\because a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc]$$

EXAMPLE 10 *Prove that:* $\sin 3x \sin^3 x + \cos 3x \cos^3 x = \cos^3 2x$

SOLUTION We know that

$$\sin 3x = 3\sin x - 4\sin^3 x \Rightarrow \sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

Similarly, $\cos 3x = 4\cos^3 x - 3\cos x \Rightarrow \cos^3 x = \dfrac{\cos 3x + 3\cos x}{4}$

$\therefore$ LHS $= \sin 3x \sin^3 x + \cos 3x \cos^3 x$

$$\Rightarrow \text{LHS} = \sin 3x \left\{\frac{3\sin x - \sin 3x}{4}\right\} + \cos 3x \left\{\frac{\cos 3x + 3\cos x}{4}\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left\{3(\cos x \cos 3x + \sin x \sin 3x) + (\cos^2 3x - \sin^2 3x)\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left\{3\cos(3x - x) + \cos 2(3x)\right\} = \frac{1}{4}\left\{3\cos 2x + \cos 3(2x)\right\}$$

$$= \frac{1}{4}\left\{3\cos 2x + (4\cos^3 2x - 3\cos 2x)\right\} = \cos^3 2x = \text{RHS}$$

EXAMPLE 11 *Prove that:* $\cos^3 x + \cos^3\left(\dfrac{2\pi}{3} + x\right) + \cos^3\left(\dfrac{4\pi}{3} + x\right) = \dfrac{3}{4}\cos 3x$

SOLUTION We know that $\cos 3x = 4\cos^3 x - 3\cos x$. Therefore, $\cos^3 x = \dfrac{1}{4}(\cos 3x + 3\cos x)$.

Using this, we obtain

$$\text{LHS} = \frac{1}{4}\left\{\cos 3x + 3\cos x\right\} + \frac{1}{4}\left\{\cos(2\pi + 3x) + 3\cos\left(\frac{2\pi}{3} + x\right)\right\} + \frac{1}{4}\left\{\cos(4\pi + 3x) + 3\cos\left(\frac{4\pi}{3} + x\right)\right\}$$

$$= \frac{1}{4}\left\{\cos 3x + 3\cos x\right\} + \frac{1}{4}\left\{\cos 3x + 3\cos\left(\frac{2\pi}{3} + x\right)\right\} + \frac{1}{4}\left\{\cos 3x + 3\cos\left(\frac{4\pi}{3} + x\right)\right\}$$

$$= \frac{3}{4}\cos 3x + \frac{3}{4}\left\{\cos x + \cos\left(\frac{2\pi}{3} + x\right) + \cos\left(\frac{4\pi}{3} + x\right)\right\}$$

$$= \frac{3}{4}\cos 3x + \frac{3}{4}\left\{\cos x + 2\cos(\pi + x)\cos\frac{\pi}{3}\right\}$$

$$= \frac{3}{4}\cos 3x + \frac{3}{4}\left\{\cos x - 2\cos x \times \frac{1}{2}\right\} = \frac{3}{4}\cos 3x = \text{RHS}$$

ALITER $\cos x + \cos\left(\frac{2\pi}{3}+x\right) + \cos\left(\frac{4\pi}{3}+x\right)$

$$= \cos x + 2\cos\left(\frac{\frac{4\pi}{3}+x+\frac{2\pi}{3}+x}{2}\right)\cos\left(\frac{\frac{4\pi}{3}+x-\frac{2\pi}{3}-x}{2}\right)$$

$$= \cos x + 2\cos(\pi+x)\cos\frac{\pi}{3} = \cos x - 2(\cos x)\times\frac{1}{2} = \cos x - \cos x = 0$$

$\therefore$ $\cos^3 x + \cos^3\left(\frac{2\pi}{3}+x\right) + \cos^3\left(\frac{4\pi}{3}+x\right)$

$$= 3\cos x\cos\left(\frac{2\pi}{3}+x\right)\cos\left(\frac{4\pi}{3}+x\right) \qquad [\because\ a+b+c=0 \Rightarrow a^3+b^3+c^3=3abc]$$

$$= 3\cos x\cos\left(\pi-\frac{\pi}{3}+x\right)\cos\left(\pi+\frac{\pi}{3}+x\right)$$

$$= 3\cos x\cos\left\{\pi-\left(\frac{\pi}{3}-x\right)\right\}\cos\left\{\pi+\left(\frac{\pi}{3}+x\right)\right\}$$

$$= (3\cos x)\left\{-\cos\left(\frac{\pi}{3}-x\right)\right\}\left\{-\cos\left(\frac{\pi}{3}+x\right)\right\}$$

$$= 3\cos x\cos\left(\frac{\pi}{3}-x\right)\cos\left(\frac{\pi}{3}+x\right) = 3\times\frac{1}{4}\cos 3x = \frac{3}{4}\cos 3x$$

EXAMPLE 12 *Prove that* $\frac{\tan 3x}{\tan x}$ *never lies between* $\frac{1}{3}$ *and 3.*

SOLUTION Let $y = \frac{\tan 3x}{\tan x}$. Then,

$$y = \frac{3\tan x - \tan^3 x}{\tan x(1-3\tan^2 x)} \Rightarrow y = \frac{3-\tan^2 x}{1-3\tan^2 x} \Rightarrow (3y-1)\tan^2 x = y-3 \Rightarrow \tan^2 x = \frac{y-3}{3y-1}$$

But, $\tan^2 x \geq 0$ for all x

$\therefore$ $\frac{y-3}{3y-1} \geq 0$

$\Rightarrow$ $y < \frac{1}{3}$ or, $y \geq 3$

$\Rightarrow$ y does not lie between 1/3 and 3.

Fig. 9.1 Signs of $\frac{y-3}{3y-1}$ for different values of y

Hence, $\frac{\tan 3x}{\tan x}$ never lies between $\frac{1}{3}$ and 3.

EXAMPLE 13 *Prove that:* $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$

SOLUTION $\cos 5x = \cos(3x+2x) = \cos 3x\cos 2x - \sin 3x\sin 2x$

$\Rightarrow$ $\cos 5x = (4\cos^3 x - 3\cos x)(2\cos^2 x - 1) - (3\sin x - 4\sin^3 x)(2\sin x\cos x)$

$\Rightarrow$ $\cos 5x = (4\cos^3 x - 3\cos x)(2\cos^2 x - 1) - (3 - 4\sin^2 x)(2\sin^2 x\cos x)$

$\Rightarrow$ $\cos 5x = (4\cos^3 x - 3\cos x)(2\cos^2 x - 1) - \{3 - 4(1-\cos^2 x)\}2(1-\cos^2 x)\cos x$

$\Rightarrow$ $\cos 5x = (8\cos^5 x - 10\cos^3 x + 3\cos x) - 2\cos x(1-\cos^2 x)(4\cos^2 x - 1)$

$\Rightarrow$ $\cos 5x = (8\cos^5 x - 10\cos^3 x + 3\cos x) - 2\cos x(5\cos^2 x - 4\cos^4 x - 1)$

$\Rightarrow$ $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$

EXERCISE 9.2

BASIC

Prove that:

1. $\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x$
2. $4(\cos^3 10° + \sin^3 20°) = 3(\cos 10° + \sin 20°)$
3. $\cos^3 x \sin 3x + \sin^3 x \cos 3x = \frac{3}{4}\sin 4x$

BASED ON LOTS

4. $\tan x \tan\left(x + \frac{\pi}{3}\right) + \tan x \tan\left(x - \frac{\pi}{3}\right) + \tan\left(x + \frac{\pi}{3}\right)\tan\left(x - \frac{\pi}{3}\right) = -3$
5. $\tan x + \tan\left(\frac{\pi}{3} + x\right) - \tan\left(\frac{\pi}{3} - x\right) = 3\tan 3x$
6. $\cot x + \cot\left(\frac{\pi}{3} + x\right) - \cot\left(\frac{\pi}{3} - x\right) = 3\cot 3x$
7. $\cot x + \cot\left(\frac{\pi}{3} + x\right) + \cot\left(\frac{2\pi}{3} + x\right) = 3\cot 3x$

BASED ON HOTS

8. $\sin 5x = 5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x$
9. $\sin^3 x + \sin^3\left(\frac{2\pi}{3} + x\right) + \sin^3\left(\frac{4\pi}{3} + x\right) = -\frac{3}{4}\sin 3x.$
10. $\left|\sin x \sin\left(\frac{\pi}{3} - x\right)\sin\left(\frac{\pi}{3} + x\right)\right| \le \frac{1}{4}$ for all values of x.
11. $\left|\cos x \cos\left(\frac{\pi}{3} - x\right)\cos\left(\frac{\pi}{3} + x\right)\right| \le \frac{1}{4}$ for all values of x.

9.5 VALUES OF TRIGONOMETRICAL FUNCTIONS AT SOME IMPORTANT POINTS

By using the formulae introduced in the previous sections we can now find the values of trigonometrical functions at some important points like $\frac{\pi}{10}, \frac{\pi}{5}, \frac{3\pi}{10}$ etc.

THEOREM 1 *Prove that:* $\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$.

PROOF Let $x = \frac{\pi}{10}$. Then,

$$5x = \frac{\pi}{2} \Rightarrow 2x + 3x = \frac{\pi}{2} \Rightarrow 2x = \frac{\pi}{2} - 3x \Rightarrow \sin 2x = \sin\left(\frac{\pi}{2} - 3x\right) \Rightarrow \sin 2x = \cos 3x$$

$$\Rightarrow \quad 2\sin x\cos x = 4\cos^3 x - 3\cos x \Rightarrow \cos x(2\sin x - 4\cos^2 x + 3) = 0$$

$$\Rightarrow \quad 2\sin x - 4\cos^2 x + 3 = 0 \qquad \left[\because \cos x = \cos\frac{\pi}{10} \neq 0\right]$$

$$\Rightarrow \quad 2\sin x - 4(1 - \sin^2 x) + 3 = 0 \Rightarrow 4\sin^2 x + 2\sin x - 1 = 0$$

$$\Rightarrow \quad \sin x = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

$$\Rightarrow \quad \sin x = \frac{-1+\sqrt{5}}{4} = \frac{\sqrt{5}-1}{4} \quad [\because x \text{ lies in Ist quadrant } \therefore \sin x > 0]$$

Hence, $\sin \frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$ **Q.E.D.**

THEOREM 2 *Prove that:* $\cos \frac{\pi}{10} = \frac{\sqrt{10+2\sqrt{5}}}{4}$.

PROOF Putting $x = \frac{\pi}{10}$ in $\cos x = \sqrt{1-\sin^2 x}$, we get

$$\cos \frac{\pi}{10} = \sqrt{1-\sin^2 \frac{\pi}{10}} = \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2} = \sqrt{\frac{16-(5+1-2\sqrt{5})}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

Q.E.D.

REMARK *The complement of* $\frac{\pi}{10}$ is $\frac{2\pi}{5}$.

$$\therefore \quad \sin \frac{2\pi}{5} = \sin\left(\frac{\pi}{2}-\frac{\pi}{10}\right) = \cos \frac{\pi}{10} = \frac{\sqrt{10+2\sqrt{5}}}{4} \text{ and, } \cos \frac{2\pi}{5} = \cos\left(\frac{\pi}{2}-\frac{\pi}{10}\right) = \sin \frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$$

The values of the remaining trigonometrical functions at $\frac{\pi}{10}$ *may be obtained from the above values.*

THEOREM 3 *Prove that:* $\cos \frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$.

PROOF We have, $\cos 2x = 1 - 2\sin^2 x$. Putting $x = \frac{\pi}{10}$, we obtain

$$\cos \frac{\pi}{5} = 1 - 2\sin^2 \frac{\pi}{10} = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = 1 - 2\left(\frac{6-2\sqrt{5}}{16}\right) = 1 - \left(\frac{3-\sqrt{5}}{4}\right) = \frac{\sqrt{5}+1}{4}$$

Q.E.D.

THEOREM 4 *Prove that:* $\sin \frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4}$.

PROOF Putting $x = \frac{\pi}{5}$ in $\sin x = \sqrt{1-\cos^2 x}$, we obtain

$$\sin \frac{\pi}{5} = \sqrt{1-\cos^2 \frac{\pi}{5}} = \sqrt{1-\left(\frac{\sqrt{5}+1}{4}\right)^2} = \sqrt{\frac{16-(6+2\sqrt{5})}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

Q.E.D.

REMARK *The complement of* $\frac{\pi}{5}$ *is* $\frac{3\pi}{10}$.

$$\therefore \quad \sin \frac{3\pi}{10} = \sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right) = \cos \frac{\pi}{5} = \frac{\sqrt{5}+1}{4} \text{ and, } \cos \frac{3\pi}{10} = \cos\left(\frac{\pi}{2}-\frac{\pi}{5}\right) = \sin \frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

The other trigonometrical ratios of $\frac{\pi}{5}$ *may be obtained from the above values.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Prove that:*

(i) $\sin^2 \frac{2\pi}{5} - \sin^2 \frac{\pi}{3} = \frac{\sqrt{5}-1}{8}$ (ii) $\cos^2 \frac{4\pi}{15} - \sin^2 \frac{\pi}{15} = \frac{\sqrt{5}+1}{8}$

(iii) $\sin\frac{\pi}{10}+\sin\frac{13\pi}{10}=-\frac{1}{2}$ (iv) $\sin\frac{\pi}{10}\sin\frac{13\pi}{10}=-\frac{1}{4}$

(v) $\sin^2 24°-\sin^2 6°=\frac{\sqrt{5}-1}{8}$

SOLUTION (i) LHS $=\sin^2\frac{2\pi}{5}-\sin^2\frac{\pi}{3}=\cos^2\frac{\pi}{10}-\sin^2\frac{\pi}{3}$ $\left[\because \sin\frac{\pi}{5}=\cos\frac{\pi}{10}\right]$

$$=\left\{\frac{\sqrt{10+2\sqrt{5}}}{4}\right\}^2-\left(\frac{\sqrt{3}}{2}\right)^2=\frac{10+2\sqrt{5}}{16}-\frac{3}{4}=\frac{2\sqrt{5}-2}{16}=\frac{\sqrt{5}-1}{8}=\text{RHS}$$

(ii) LHS $=\cos^2\frac{4\pi}{15}-\sin^2\frac{\pi}{15}$

$=\cos\left(\frac{4\pi}{15}+\frac{\pi}{15}\right)\cos\left(\frac{4\pi}{15}-\frac{\pi}{15}\right)$ $[\because \cos^2 A-\sin^2 B=\cos(A+B)\cos(A-B)]$

$=\cos\frac{\pi}{3}\cos\frac{\pi}{5}=\frac{1}{2}\times\frac{\sqrt{5}+1}{4}=\frac{\sqrt{5}+1}{8}=\text{RHS}$

(iii) LHS $=\sin\frac{\pi}{10}+\sin\frac{13\pi}{10}$

$=\sin\frac{\pi}{10}+\sin\left(\frac{3\pi}{2}-\frac{\pi}{5}\right)=\sin\frac{\pi}{10}-\cos\frac{\pi}{5}=\frac{\sqrt{5}-1}{4}-\frac{\sqrt{5}+1}{4}=-\frac{1}{2}=\text{RHS}$

(iv) LHS $=\sin\frac{\pi}{10}\sin\frac{13\pi}{10}=\sin\frac{\pi}{10}\sin\left(\frac{3\pi}{2}-\frac{\pi}{5}\right)$

$=-\sin\frac{\pi}{10}\cos\frac{\pi}{5}=-\frac{\sqrt{5}-1}{4}\times\frac{\sqrt{5}+1}{4}=-\left(\frac{5-1}{16}\right)=-\frac{1}{4}=\text{RHS}$

(v) LHS $=\sin^2 24°-\sin^2 6°$

$=\sin(24°+6°)\sin(24°-6°)$ $[\because \sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B]$

$=\sin 30°\sin 18°=\sin\frac{\pi}{6}\sin\frac{\pi}{10}=\frac{1}{2}\times\frac{\sqrt{5}-1}{4}=\frac{\sqrt{5}-1}{8}=\text{RHS}$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 2 *Prove that:* $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}=\frac{5}{16}$

SOLUTION If $x+y=\pi$, then $x=\pi-y \Rightarrow \sin x=\sin(\pi-y)\Rightarrow \sin x=\sin y$

$\therefore \frac{\pi}{5}+\frac{4\pi}{5}=\pi \Rightarrow \sin\frac{\pi}{5}=\sin\frac{4\pi}{5}$ and $\frac{2\pi}{5}+\frac{3\pi}{5}=\pi\Rightarrow \sin\frac{2\pi}{5}=\sin\frac{3\pi}{5}$

Using these values, we obtain

LHS $=\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}$

$=\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{2\pi}{5}\sin\frac{\pi}{5}$ $\left[\because \sin\frac{3\pi}{5}=\sin\frac{2\pi}{5}, \sin\frac{4\pi}{5}=\sin\frac{\pi}{5}\right]$

$$=\left(\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\right)^2=\left\{\sin\frac{\pi}{5}\sin\left(\frac{\pi}{2}-\frac{\pi}{10}\right)\right\}^2=\left\{\sin\frac{\pi}{5}\cos\frac{\pi}{10}\right\}^2$$

$$= \left\{ \frac{\sqrt{10-2\sqrt{5}}}{4} \times \frac{\sqrt{10+2\sqrt{5}}}{4} \right\}^2 = \frac{10-2\sqrt{5}}{16} \times \frac{10+2\sqrt{5}}{16} = \frac{100-20}{256} = \frac{80}{256} = \frac{5}{16} = \text{RHS}$$

EXAMPLE 3 *Prove that:* $16 \cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{14\pi}{15} = 1$

SOLUTION LHS $= 16 \cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{14\pi}{15} = 4\left(2 \cos \frac{2\pi}{15} \cos \frac{4\pi}{15}\right)\left(2 \cos \frac{4\pi}{15} \cos \frac{14\pi}{15}\right)$

$$= 4\left(\cos \frac{2\pi}{3} + \cos \frac{2\pi}{5}\right)\left(\cos \frac{6\pi}{5} + \cos \frac{2\pi}{3}\right) = 4\left(-\sin \frac{\pi}{6} + \sin \frac{\pi}{10}\right)\left(-\cos \frac{\pi}{5} - \sin \frac{\pi}{6}\right)$$

$$= 4\left(-\frac{1}{2} + \frac{\sqrt{5}-1}{4}\right)\left(-\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right) = 4\left(\frac{\sqrt{5}-3}{4}\right)\left(\frac{-\sqrt{5}-3}{4}\right) = 4\left(\frac{3-\sqrt{5}}{4}\right)\left(\frac{3+\sqrt{5}}{4}\right)$$

$$= \left(\frac{9-5}{4}\right) = 1 = \text{RHS}$$

EXAMPLE 4 *Prove that:* $\sin \frac{\pi}{15} \sin \frac{4\pi}{15} \sin \frac{3\pi}{10} = \frac{1}{8}$.

SOLUTION LHS $= \frac{1}{2}\left(2 \sin \frac{4\pi}{15} \sin \frac{\pi}{15}\right) \sin \frac{3\pi}{10}$

$$= \frac{1}{2}\left(\cos \frac{\pi}{5} - \cos \frac{\pi}{3}\right) \cos \frac{\pi}{5} \qquad [\because 2 \sin A \sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{1}{2}\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{1}{8} = \text{RHS}$$

EXAMPLE 5 *Prove that:* $\left(1 + \cos \frac{\pi}{10}\right)\left(1 + \cos \frac{3\pi}{10}\right)\left(1 + \cos \frac{7\pi}{10}\right)\left(1 + \cos \frac{9\pi}{10}\right) = \frac{1}{16}$.

SOLUTION If $A + B = \pi$, then $\cos A = \cos(\pi - B) = -\cos B$

$\therefore \quad \frac{\pi}{10} + \frac{9\pi}{10} = \pi \Rightarrow \cos \frac{9\pi}{10} = -\cos \frac{\pi}{10}$ and, $\frac{3\pi}{10} + \frac{7\pi}{10} = \pi \Rightarrow \cos \frac{7\pi}{10} = -\cos \frac{3\pi}{10}$

Using these values, we obtain

$$\text{LHS} = \left(1 + \cos \frac{\pi}{10}\right)\left(1 + \cos \frac{3\pi}{10}\right)\left(1 - \cos \frac{3\pi}{10}\right)\left(1 - \cos \frac{\pi}{10}\right)$$

$$= \left(1 - \cos^2 \frac{\pi}{10}\right)\left(1 - \cos^2 \frac{3\pi}{10}\right) = \sin^2 \frac{\pi}{10} \sin^2 \frac{3\pi}{10} = \sin^2 \frac{\pi}{10} \sin^2 \left(\frac{\pi}{2} - \frac{2\pi}{10}\right)$$

$$= \sin^2 \frac{\pi}{10} \cos^2 \frac{2\pi}{10} = \left(\sin \frac{\pi}{10} \cos \frac{\pi}{5}\right)^2 = \left(\frac{\sqrt{5}-1}{4} \times \frac{\sqrt{5}+1}{4}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16} = \text{RHS}$$

EXAMPLE 6 *Prove that:* $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1$.

SOLUTION LHS $= \frac{\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ}{\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ} = \frac{(2 \sin 66^\circ \sin 6^\circ)(2 \sin 78^\circ \sin 42^\circ)}{(2 \cos 66^\circ \cos 6^\circ)(2 \cos 78^\circ \cos 42^\circ)}$

$$= \frac{(\cos 60^\circ - \cos 72^\circ)(\cos 36^\circ - \cos 120^\circ)}{(\cos 60^\circ + \cos 72^\circ)(\cos 36^\circ + \cos 120^\circ)}$$

$$= \frac{(\cos 60^\circ - \sin 18^\circ)(\cos 36^\circ + \sin 30^\circ)}{(\cos 60^\circ + \sin 18^\circ)(\cos 36^\circ - \sin 30^\circ)}$$

$$= \frac{\left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} + \frac{1}{2}\right)}{\left(\frac{1}{2} + \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right)} = \frac{(3-\sqrt{5})(3+\sqrt{5})}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{9-5}{5-1} = 1 = \text{RHS}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 7 *Prove that* : $4 \sin 27° = \sqrt{(5+\sqrt{5})} - \sqrt{(3-\sqrt{5})}$.

SOLUTION $16 \sin^2 27° = 8(2\sin^2 27°) = 8\left(1 - \cos\frac{3\pi}{10}\right) = 8\left(1 - \sin\frac{\pi}{5}\right)$

$$= 8\left\{1 - \frac{\sqrt{10-2\sqrt{5}}}{4}\right\} = 2\left\{4 - \sqrt{10-2\sqrt{5}}\right\} = 8 - 2\sqrt{10-2\sqrt{5}}$$

$$= (5+\sqrt{5}) + (3-\sqrt{5}) - 2\sqrt{(5+\sqrt{5})(3-\sqrt{5})}$$

$$= \left\{\sqrt{5+\sqrt{5}}\right\}^2 + \left\{\sqrt{3-\sqrt{5}}\right\}^2 - 2\sqrt{(5+\sqrt{5})(3-\sqrt{5})}$$

$$= \left\{\sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}}\right\}^2$$

Taking square roots of both sides, we obtain

$\therefore \quad 4\sin 27° = \sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}}$ [$\because$ sin 27° is positive]

EXAMPLE 8 *Find the value of* $\tan 9° - \tan 27° - \tan 63° + \tan 81°$. [NCERT EXEMPLAR]

SOLUTION $\tan 9° - \tan 27° - \tan 63° + \tan 81° = (\tan 9° + \tan 81°) - (\tan 27° + \tan 63°)$

$= (\tan 9° + \cot 9°) - (\tan 27° + \cot 27°)$ $\left[\because \tan 81° = \tan(90° - 9°) = \cot 9°, \ \tan 63° = \tan(90° - 27°) = \cot 27°\right]$

$= \frac{1}{\sin 9° \cos 9°} - \frac{1}{\sin 27° \cos 27°}$ $\left[\because \tan x + \cot x = \frac{1}{\sin x \cos x}\right]$

$= \frac{2}{\sin 18°} - \frac{2}{\sin 54°} = \frac{2}{\sin 18°} - \frac{2}{\cos 36°} = \frac{8}{\sqrt{5}-1} - \frac{8}{\sqrt{5}+1} = \frac{8 \times 2}{5-1} = 4$

EXERCISE 9.3

BASIC

Prove that:

1. $\sin^2 \frac{2\pi}{5} - \sin^2 \frac{\pi}{3} = \frac{\sqrt{5}-1}{8}$
2. $\sin^2 24° - \sin^2 6° = \frac{\sqrt{5}-1}{8}$
3. $\sin^2 42° - \cos^2 78° = \frac{\sqrt{5}+1}{8}$

BASED ON LOTS

4. $\cos 78° \cos 42° \cos 36° = \frac{1}{8}$
5. $\cos\frac{\pi}{15} \cos\frac{2\pi}{15} \cos\frac{4\pi}{15} \cos\frac{7\pi}{15} = \frac{1}{16}$

6. $\cos 6° \cos 42° \cos 66° \cos 78° = \frac{1}{16}$

7. $\sin 6° \sin 42° \sin 66° \sin 78° = \frac{1}{16}$

8. $\cos 36° \cos 42° \cos 60° \cos 78° = \frac{1}{16}$

BASED ON HOTS

9. $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$

10. $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{128}$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If $\frac{1-\tan^2\left(\frac{\pi}{4}-x\right)}{1+\tan^2\left(\frac{\pi}{4}-x\right)} = \sin kx$, then $k =$ ....

2. If $\cos x \cos 2x \cos 2^2 x \ldots\ldots\ldots \cos 2^{n-1} x = \lambda \frac{\sin 2^n \lambda}{\sin x}$, then $\lambda =$ ....

3. The value of $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}$ is ....

4. If $\tan x = \frac{1-\cos y}{\sin y}$, then $\tan 2x =$ ....

5. If $k = \sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$, then the numerical value of k is ....

6. In a triangle ABC with $\angle C = \frac{\pi}{2}$ the equation whose roots are $\tan A$ and $\tan B$ is ....

7. The value of $\cos^2 48° - \sin^2 12°$ is ....

8. The least value of $2\sin^2\theta + 3\cos^2\theta$ is ....

9. If $\cos^6 x + \sin^6 x + k\sin^2 2x = 1$, then $k =$.............................. .

10. The value of $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is ....

11. The value of $\frac{\cot x - \tan x}{\cot 2x}$ is ....

12. If $\tan\theta = t$, then $\tan 2\theta + \sec 2\theta =$ ....

13. If $\tan\theta = \frac{a}{b}$, then $a\sin 2\theta + b\cos 2\theta$ is equal to ....

14. If $\tan x = \frac{1}{7}$, $\tan y = \frac{1}{3}$ and $\cos 2x = \sin ky$, then $k =$ ....

15. The value of $\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\cos\frac{8\pi}{5}$ is ....

16. The value of $\sin\frac{3\pi}{10}$ is ....

17. The value of $\cos^2 6° - \cos^2 24°$ is ....

18. If $\frac{\pi}{2} < x < \pi$, then $\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} =$ ....

19. If $\frac{\pi}{4} < x < \frac{\pi}{2}$, then $\sqrt{2+\sqrt{2+2\cos 4x}} =$ ....

20. The value of $108 \sin\frac{\pi}{9} - 144 \sin^3 \frac{\pi}{9}$ is ....

ANSWERS

1. 2 2. $\frac{1}{2^n}$ 3. $\frac{1}{16}$ 4. $\tan y$ 5. $\frac{1}{8}$ 6. $x^2 - \frac{2}{\sin 2A}x + 1 = 0$

7. $\frac{\sqrt{5}+1}{8}$ 8. 2 9. $\frac{3}{4}$ 10. $\frac{1}{8}$ 11. 2 12. $\frac{1+t}{1-t}$ 13. b 14. 4

15. $-\frac{1}{16}$ 16. $\frac{\sqrt{5}+1}{4}$ 17. $\frac{\sqrt{5}-1}{8}$ 18. $-\tan x$ 19. $2\sin x$ 20. $18\sqrt{3}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $\cos 4x = 1 + k \sin^2 x \cos^2 x$, then write the value of k.
2. If $\tan \frac{x}{2} = \frac{m}{n}$, then write the value of $m \sin x + n \cos x$.
3. If $\frac{\pi}{2} < x < \frac{3\pi}{2}$, then write the value of $\sqrt{\frac{1+\cos 2x}{2}}$.
4. If $\frac{\pi}{2} < x < \pi$, then write the value of $\sqrt{2+\sqrt{2+2\cos 2x}}$ in the simplest form.
5. If $\frac{\pi}{2} < x < \pi$, then write the value of $\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$.
6. If $\pi < x < \frac{3\pi}{2}$, then write the value of $\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$.
7. In a right angled triangle ABC, write the value of $\sin^2 A + \sin^2 B + \sin^2 C$.
8. Write the value of $\cos^2 76° + \cos^2 16° - \cos 76° \cos 16°$.
9. If $\frac{\pi}{4} < x < \frac{\pi}{2}$, then write the value of $\sqrt{1-\sin 2x}$.
10. Write the value of $\cos\frac{\pi}{7} \cos\frac{2\pi}{7} \cos\frac{4\pi}{7}$.
11. If $\tan A = \frac{1-\cos B}{\sin B}$, then find the value of $\tan 2A$.
12. If $\sin x + \cos x = a$, find the value of $\sin^6 x + \cos^6 x$.
13. If $\sin x + \cos x = a$, find the value of $|\sin x - \cos x|$.

ANSWERS

1. -8 2. n 3. $-\cos x$ 4. $2\sin\frac{x}{2}$ 5. $-\tan x$ 6. $\tan x$ 7. 2 8. $\frac{3}{4}$

9. $\sin x - \cos x$ 10. $-\frac{1}{8}$ 11. $\tan B$ 12. $\frac{1}{4}\{4-3(a^2-1)^2\}$ 13. $\sqrt{2-a^2}$.

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. $8\sin\frac{x}{8}\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}$ is equal to
   (a) $8\cos x$ (b) $\cos x$ (c) $8\sin x$ (d) $\sin x$

2. $\dfrac{\sec 8A-1}{\sec 4A-1}$ is equal to
   (a) $\dfrac{\tan 2A}{\tan 8A}$ (b) $\dfrac{\tan 8A}{\tan 2A}$ (c) $\dfrac{\cot 8A}{\cot 2A}$ (d) none of these

3. The value of $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}$ is
   (a) $\frac{1}{8}$ (b) $\frac{1}{16}$ (c) $\frac{1}{32}$ (d) none of these

4. If $\cos 2x + 2\cos x = 1$ then, $(2-\cos^2 x)\sin^2 x$ is equal to
   (a) 1 (b) -1 (c) $-\sqrt{5}$ (d) $\sqrt{5}$

5. For all real values of x, $\cot x - 2\cot 2x$ is equal to
   (a) $\tan 2x$ (b) $\tan x$ (c) $-\cot 3x$ (d) none of these

6. The value of $2\tan\frac{\pi}{10}+3\sec\frac{\pi}{10}-4\cos\frac{\pi}{10}$ is
   (a) 0 (b) $\sqrt{5}$ (c) 1 (d) none of these

7. If in a ΔABC, $\tan A+\tan B+\tan C=0$, then $\cot A\cot B\cot C =$
   (a) 6 (b) 1 (c) $\frac{1}{6}$ (d) none of these

8. If $\cos x=\frac{1}{2}\left(a+\frac{1}{a}\right)$, and $\cos 3x=\lambda\left(a^3+\frac{1}{a^3}\right)$, then $\lambda =$
   (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) none of these

9. If $2\tan\alpha = 3\tan\beta$, then $\tan(\alpha-\beta) =$
   (a) $\dfrac{\sin 2\beta}{5-\cos 2\beta}$ (b) $\dfrac{\cos 2\beta}{5-\cos 2\beta}$ (c) $\dfrac{\sin 2\beta}{5+\cos 2\beta}$ (d) none of these

10. If $\tan\alpha=\dfrac{1-\cos\beta}{\sin\beta}$, then
   (a) $\tan 3\alpha=\tan 2\beta$ (b) $\tan 2\alpha=\tan\beta$ (c) $\tan 2\beta=\tan\alpha$ (d) none of these

11. If $\sin\alpha+\sin\beta=a$ and $\cos\alpha-\cos\beta=b$, then $\tan\frac{\alpha-\beta}{2} =$
   (a) $-\frac{a}{b}$ (b) $-\frac{b}{a}$ (c) $\sqrt{a^2+b^2}$ (d) none of these

12. The value of $\left(\cot\frac{x}{2}-\tan\frac{x}{2}\right)^2(1-2\tan x\cot 2x)$ is

(a) 1 (b) 2 (c) 3 (d) 4

13. The value of $\tan x\sin\left(\frac{\pi}{2}+x\right)\cos\left(\frac{\pi}{2}-x\right)$ is

(a) 1 (b) -1 (c) $\frac{1}{2}\sin 2x$ (d) none of these

14. The value of $\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{\pi}{9}\right)+\sin^2\left(\frac{7\pi}{18}\right)+\sin^2\left(\frac{4\pi}{9}\right)$ is

(a) 1 (b) 2 (c) 4 (d) none of these

15. If $5\sin\alpha=3\sin(\alpha+2\beta)\neq 0$, then $\tan(\alpha+\beta)$ is equal to

(a) $2\tan\beta$ (b) $3\tan\beta$ (c) $4\tan\beta$ (d) $6\tan\beta$

16. The value of $2\cos x-\cos 3x-\cos 5x-16\cos^3 x\sin^2 x$ is

(a) 2 (b) 1 (c) 0 (d) -1

17. If $A=2\sin^2 x-\cos 2x$, then A lies in the interval

(a) $[-1, 3]$ (b) $[1, 2]$ (c) $[-2, 4]$ (d) none of these

18. The value of $\frac{\cos 3x}{2\cos 2x-1}$ is equal to

(a) $\cos x$ (b) $\sin x$ (c) $\tan x$ (d) none of these

19. If $\tan(\pi/4+x)+\tan(\pi/4-x)=\lambda\sec 2x$, then

(a) 3 (b) 4 (c) 1 (d) 2

20. The value of $\cos^2\left(\frac{\pi}{6}+x\right)-\sin^2\left(\frac{\pi}{6}-x\right)$ is

(a) $\frac{1}{2}\cos 2x$ (b) 0 (c) $-\frac{1}{2}\cos 2x$ (d) $\frac{1}{2}$

21. $\frac{\sin 3x}{1+2\cos 2x}$ is equal to

(a) $\cos x$ (b) $\sin x$ (c) $-\cos x$ (d) $\sin x$

22. The value of $2\sin^2 B+4\cos(A+B)\sin A\sin B+\cos 2(A+B)$ is

(a) 0 (b) $\cos 3A$ (c) $\cos 2A$ (d) none of these

23. The value of $\frac{2(\sin 2x+2\cos^2 x-1)}{\cos x-\sin x-\cos 3x+\sin 3x}$ is

(a) $\cos x$ (b) $\sec x$ (c) $\operatorname{cosec} x$ (d) $\sin x$

24. $2(1-2\sin^2 7x)\sin 3x$ is equal to

(a) $\sin 17x-\sin 11x$ (b) $\sin 11x-\sin 17x$
(c) $\cos 17x-\cos 11x$ (d) $\cos 17x+\cos 11x$

25. If α and β are acute angles satisfying $\cos 2\alpha=\frac{3\cos 2\beta-1}{3-\cos 2\beta}$, then $\tan\alpha=$

(a) $\sqrt{2}\tan\beta$ (b) $\frac{1}{\sqrt{2}}\tan\beta$ (c) $\sqrt{2}\cot\beta$ (d) $\frac{1}{\sqrt{2}}\cot\beta$

26. If $\tan\frac{x}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\alpha}{2}$, then $\cos\alpha =$

(a) $1 - e\cos(\cos x + e)$ (b) $\frac{1+e\cos x}{\cos x - e}$ (c) $\frac{1-e\cos x}{\cos x - e}$ (d) $\frac{\cos x - e}{1 - e\cos x}$

27. If $(2^n + 1)x = \pi$, then $2^n \cos x \cos 2x \cos 2^2 x \ldots \cos 2^{n-1} x =$

(a) -1 (b) 1 (c) 1/2 (d) none of these

28. If $\tan x = t$ then $\tan 2x + \sec 2x$ is equal to

(a) $\frac{1+t}{1-t}$ (b) $\frac{1-t}{1+t}$ (c) $\frac{2t}{1-t}$ (d) $\frac{2t}{1+t}$

29. The value of $\cos^4 x + \sin^4 x - 6\cos^2 x \sin^2 x$ is

(a) $\cos 2x$ (b) $\sin 2x$ (c) $\cos 4x$ (d) none of these

30. The value of $\cos(36° - A)\cos(36° + A) + \cos(54° - A)\cos(54° + A)$ is

(a) $\cos 2A$ (b) $\sin 2A$ (c) $\cos A$ (d) 0

31. The value of $\tan x \tan\left(\frac{\pi}{3} - x\right)\tan\left(\frac{\pi}{3} + x\right)$ is

(a) $\cot 3x$ (b) $2\cot 3x$ (c) $\tan 3x$ (d) $3\tan 3x$

32. The value of $\tan x + \tan\left(\frac{\pi}{3} + x\right) + \tan\left(\frac{2\pi}{3} + x\right)$ is

(a) $3\tan 3x$ (b) $\tan 3x$ (c) $3\cot 3x$ (d) $\cot 3x$

33. The value of $\frac{\sin 5\alpha - \sin 3\alpha}{\cos 5\alpha + 2\cos 4\alpha + \cos 3\alpha}$ is

(a) $\cot\alpha/2$ (b) $\cot\alpha$ (c) $\tan\alpha/2$ (d) none of these

34. $\frac{\sin 5x}{\sin x}$ is equal to

(a) $16\cos^4 x - 12\cos^2 x + 1$ (b) $16\cos^4 x + 12\cos^2 x + 1$

(c) $16\cos^4 x - 12\cos^2 x - 1$ (d) $16\cos^4 x + 12\cos^2 x - 1$

35. If $n = 1, 2, 3, \ldots$, then $\cos\alpha\cos 2\alpha\cos 4\alpha \ldots \cos 2^{n-1}\alpha$ is equal to

(a) $\frac{\sin 2n\alpha}{2n\sin\alpha}$ (b) $\frac{\sin 2^n\alpha}{2^n\sin 2^{n-1}\alpha}$ (c) $\frac{\sin 4^{n-1}\alpha}{4^{n-1}\sin\alpha}$ (d) $\frac{\sin 2^n\alpha}{2^n\sin\alpha}$

36. If $\tan x = \frac{a}{b}$, then $b\cos 2x + a\sin 2x$ is equal to

(a) a (b) b (c) $\frac{a}{b}$ (d) $\frac{b}{a}$

[NCERT EXEMPLAR]

37. If $\tan\alpha = \frac{1}{7}$, $\tan\beta = \frac{1}{3}$, then $\cos 2\alpha$ is equal to

(a) $\sin 2\beta$ (b) $\sin 4\beta$ (c) $\sin 3\beta$ (d) $\cos 2\beta$

[NCERT EXEMPLAR]

38. The value of $\cos^2 48° - \sin^2 12°$ is

(a) $\frac{\sqrt{5}+1}{8}$ (b) $\frac{\sqrt{5}-1}{8}$ (c) $\frac{\sqrt{5}+1}{5}$ (d) $\frac{\sqrt{5}+1}{2\sqrt{2}}$

[NCERT EXEMPLAR]

39. The value of $\frac{1-\tan^2 15°}{1+\tan^2 15°}$ is

(a) 1 (b) $\sqrt{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) 2

[NCERT EXEMPLAR]

40. The value of $\tan 75° - \cot 75°$ is

(a) $2\sqrt{3}$ (b) $2+\sqrt{3}$ (c) $2-\sqrt{3}$ (d) 1

[NCERT EXEMPLAR]

41. $\cos 2\theta \cos 2\phi + \sin^2(\theta-\phi) - \sin^2(\theta+\phi)$ is equal to

(a) $\sin 2(\theta+\phi)$ (b) $\cos 2(\theta+\phi)$ (c) $\sin 2(\theta-\phi)$ (d) $\cos 2(\theta-\phi)$

[NCERT EXEMPLAR]

42. If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, then $\tan(2A+B)$ is equal

(a) 1 (b) 2 (c) 3 (d) 4

[NCERT EXEMPLAR]

43. If $\sin\theta + \cos\theta = 1$, then the value of $\sin 2\theta$ is equal to

(a) 1 (b) $\frac{1}{2}$ (c) 0 (d) -1

[NCERT EXEMPLAR]

44. The value of $\sin\frac{\pi}{10}\sin\frac{13\pi}{10}$ is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $-\frac{1}{4}$ (d) 1

[NCERT EXEMPLAR]

45. If $\sin\theta = -\frac{4}{5}$ and θ lies in third quadrant, then the value of $\cos\frac{\theta}{2}$ is

(a) $\frac{1}{5}$ (b) $-\frac{1}{\sqrt{10}}$ (c) $-\frac{1}{\sqrt{5}}$ (d) $\frac{1}{\sqrt{10}}$

[NCERT EXEMPLAR]

46. The value of $\cos 12° + \cos 84° + \cos 156° + \cos 132°$ is

(a) $\frac{1}{2}$ (b) 1 (c) $-\frac{1}{2}$ (d) $-\frac{1}{8}$

ANSWERS

1. (d)	2. (b)	3. (d)	4. (a)	5. (b)	6. (a)	7. (d)	8. (b)
9. (a)	10. (b)	11. (b)	12. (d)	13. (d)	14. (b)	15. (c)	16. (c)
17. (a)	18. (a)	19. (d)	20. (a)	21. (b)	22. (c)	23. (c)	24. (a)
25. (a)	26. (d)	27. (b)	28. (a)	29. (c)	30. (a)	31. (c)	32. (a)
33. (c)	34. (a)	35. (d)	36. (b)	37. (b)	38. (a)	39. (c)	40. (a)
41. (b)	42. (c)	43. (c)	44. (c)	45. (c)	46. (c)		

SUMMARY

1. (i) $\sin 2x = 2 \sin x \cos x$ (ii) $\cos 2x = \cos^2 x - \sin^2 x$

(iii) $\cos 2x = 2 \cos^2 x - 1$ or, $1 + \cos 2x = 2 \cos^2 x$

(iv) $\cos 2x = 1 - 2 \sin^2 x$ or, $1 - \cos 2x = 2 \sin^2 x$

(v) $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$ (vi) $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}$ (vii) $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

2. (i) $\sin 3x = 3 \sin x - 4 \sin^3 x$ (ii) $\cos 3x = 4 \cos^3 x - 3 \cos x$

(iii) $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

3. (i) $\sin \dfrac{x}{2} = \sqrt{\dfrac{1 - \cos x}{2}}$ (ii) $\cos \dfrac{x}{2} = \sqrt{\dfrac{1 + \cos x}{2}}$ (iii) $\tan \dfrac{x}{2} = \sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$

4. (i) $\cos x \cos 2x \cos 2^2 x \cos 2^3 x \ldots \cos 2^{n-1} x = \dfrac{\sin 2^n x}{2^n \sin x}$

(ii) $\sin x \sin\left(\dfrac{\pi}{3} - x\right) \sin\left(\dfrac{\pi}{3} + x\right) = \dfrac{1}{4} \sin 3x$

(iii) $\cos x \cos\left(\dfrac{\pi}{3} - x\right) \cos\left(\dfrac{\pi}{3} + x\right) = \dfrac{1}{4} \cos 3x$

5. (i) $\sin \dfrac{\pi}{10} = \dfrac{\sqrt{5} - 1}{4}$ (ii) $\cos \dfrac{\pi}{5} = \dfrac{\sqrt{5} + 1}{4}$

(iii) $\cos \dfrac{\pi}{10} = \dfrac{\sqrt{10 + 2\sqrt{5}}}{4}$ (iv) $\sin \dfrac{\pi}{5} = \dfrac{\sqrt{10 - 2\sqrt{5}}}{4}$

CHAPTER 10

TRIGONOMETRIC EQUATIONS

10.1 SOME DEFINITIONS

TRIGONOMETRIC EQUATIONS *The equations containing trigonometric functions of unknown angles are known as trigonometric equations.*

$\cos x = \frac{1}{2}$, $\sin x = 0$, $\tan x = \sqrt{3}$ etc. are trigonometric equations.

SOLUTION OF A TRIGONOMETRIC EQUATION *A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.*

Consider the equation $\sin x = \frac{1}{2}$. This equation is clearly satisfied by $x = \frac{\pi}{6}, \frac{5\pi}{6}$ etc. So, these are its solutions.

Solving an equation means to find the set of all values of the unknown angle which satisfy the given equation.

Consider the equation $2\cos x + 1 = 0$ or $\cos x = -1/2$. This equation is clearly satisfied by $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ etc.

Since the trigonometric functions are periodic. Therefore, if a trigonometric equation has a solution, it will have infinitely many solutions. For example, $x = \frac{2\pi}{3}$, $2\pi \pm \frac{2\pi}{3}$, $4\pi \pm \frac{2\pi}{3}$, ... are solutions of $2\cos x + 1 = 0$. These solutions can be put together in compact form as $2n\pi \pm \frac{2\pi}{3}$, where n is an integer. This solution is known as the general solution.

Thus, a solution generalised by means of periodicity is known as the general solution.

It also follows from the above discussion that solving an equation means to find its general solution.

10.2 GENERAL SOLUTIONS OF TRIGONOMETRIC EQUATIONS

In this section, we shall obtain the general solutions of the trigonometric equations $\sin x = 0$, $\cos x = 0$, $\tan x = 0$ and $\cot x = 0$.

THEOREM 1 *Prove that the general solution of* $\sin x = 0$ *is given by* $x = n\pi, n \in Z$.

PROOF In ΔOMP, we obtain

$$\sin x = \frac{PM}{OP}$$

$\therefore$ $\sin x = 0$

$\Rightarrow$ $\frac{PM}{OP} = 0$

$\Rightarrow$ $PM = 0$

$\Rightarrow$ OP coincides with OX or, OX'

$\Rightarrow$ $x = 0, \pi, 2\pi, \ldots, -\pi, -2\pi, -3\pi, \ldots$

$\Rightarrow$ $x = n\pi, n \in Z$.

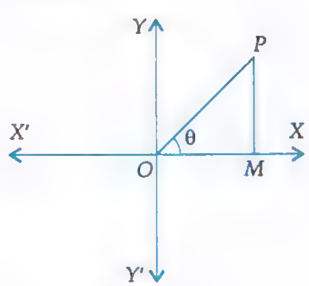


Fig. 10.1

Hence, $x = n\pi, n \in Z$ is the general solution of $\sin x = 0$. Q.E.D.

ALITER In chapter 6, we have learnt that the curve $y = \sin x$ cuts x-axis at points $(0, 0)$, $(\pm \pi, 0)$, $(\pm 2\pi, 0)$ etc. Thus, $\sin x = 0$ at $x = n\pi$, $n \in Z$.

THEOREM 2 *Prove that the general solution of* $\tan x = 0$ *is* $x = n\pi$, $n \in Z$.

PROOF By definition,

$$\tan x = \frac{PM}{OM} \qquad \text{[See Fig. 10.1]}$$

$$\therefore \quad \tan x = 0$$

$$\Rightarrow \quad \frac{PM}{OM} = 0$$

$$\Rightarrow \quad PM = 0$$

$\Rightarrow$ OP coincides with OX or, $OX' \Rightarrow x = 0, \ \pi, \ 2\pi, \ ...., \ -\pi, \ -2\pi, \ ..... \Rightarrow x = n\pi, \ n \in Z$.

Hence, $x = n\pi$, $n \in Z$ is general solution of $\tan x = 0$. Q.E.D.

THEOREM 3 *Prove that the general solution of* $\cos x = 0$ is $x = (2n+1)\frac{\pi}{2}$, $n \in Z$.

PROOF By definition

$$\cos x = \frac{OM}{OP} \qquad \text{[See Fig. 10.1]}$$

$$\therefore \quad \cos x = 0$$

$$\Rightarrow \quad \frac{OM}{OP} = 0$$

$$\Rightarrow \quad OM = 0$$

$\Rightarrow$ OP coincides with OY or, $OY' \Rightarrow x = \pm\frac{\pi}{2}, \ \pm\frac{3\pi}{2}, \ \pm\frac{5\pi}{2}, \ ..... \Rightarrow x = (2n+1)\frac{\pi}{2}, \ n \in Z$.

Hence, the general solution of $\cos x = 0$ is $x = (2n+1)\frac{\pi}{2}$, $n \in Z$. Q.E.D.

THEOREM 4 *Prove that the general solution of* $\cot x = 0$ *is* $x = (2n+1)\frac{\pi}{2}$, $n \in Z$.

PROOF By definition,

$$\cot x = \frac{OM}{PM} \qquad \text{[See Fig. 10.1]}$$

$$\therefore \quad \cot x = 0$$

$$\Rightarrow \quad \frac{OM}{PM} = 0$$

$$\Rightarrow \quad OM = 0$$

$\Rightarrow$ OP coincides with OY or, $OY' \Rightarrow x = \pm\frac{\pi}{2}, \ \pm\frac{3\pi}{2}, \ \pm\frac{5\pi}{2}, \ ..... \Rightarrow x = (2n+1)\frac{\pi}{2}, \ n \in Z$.

Hence, $x = (2n+1)\frac{\pi}{2}$, $n \in Z$ is the general solution of $\cot x = 0$. Q.E.D.

NOTE *Since* $\sec x \geq 1$, or $\sec x \leq -1$, *therefore* $\sec x = 0$ *does not have any solution. Similarly,* $\operatorname{cosec} x = 0$ *has no solution.*

ILLUSTRATIVE EXAMPLES

BASIC

EXAMPLE 1 *Find the general solutions of the following equations:*

(i) $\sin 2x = 0$ (ii) $\sin \frac{3x}{2} = 0$ (iii) $\sin^2 2x = 0$

SOLUTION (i) We have,

$$\sin 2x = 0$$

$\Rightarrow \quad 2x = n\pi, \quad \text{where } n \in Z \qquad [\because \sin x = 0 \Rightarrow x = n\pi]$

$\Rightarrow \quad x = \frac{n\pi}{2}, \quad n \in Z.$

(ii) $\sin \frac{3x}{2} = 0$

$\Rightarrow \quad \frac{3x}{2} = n\pi, \; n \in Z \qquad [\sin x = 0 \Rightarrow x = n\pi]$

$\Rightarrow \quad x = \frac{2n\pi}{3}, n \in Z.$

(iii) $\sin^2 2x = 0 \Rightarrow \sin 2x = 0 \Rightarrow 2x = n\pi \Rightarrow x = \frac{n\pi}{2}, \quad n \in Z.$

EXAMPLE 2 *Find the general solutions of the following equations:*

(i) $\cos 3x = 0$ (ii) $\cos \frac{3x}{2} = 0$ (iii) $\cos^2 3x = 0$

SOLUTION We know that the general solution of the equation $\cos x = 0$ is $x = (2n+1)\frac{\pi}{2}, n \in Z$.

Therefore,

(i) $\cos 3x = 0 \Rightarrow 3x = (2n+1)\frac{\pi}{2} \Rightarrow x = (2n+1)\frac{\pi}{6}, \; n \in Z$

(ii) $\cos \frac{3x}{2} = 0 \Rightarrow \frac{3x}{2} = (2n+1)\frac{\pi}{2} \Rightarrow x = (2n+1)\frac{\pi}{3}, \; n \in Z$

(iii) $\cos^2 3x = 0 \Rightarrow \cos 3x = 0 \Rightarrow 3x = (2n+1)\frac{\pi}{2} \Rightarrow x = (2n+1)\frac{\pi}{6}, \; n \in Z.$

EXAMPLE 3 *Find the general solutions of the following equations:*

(i) $\tan 2x = 0$ (ii) $\tan \frac{x}{2} = 0$ (iii) $\tan \frac{3x}{4} = 0$

SOLUTION We know that the general solution of the equation $\tan x = 0$ is $x = n\pi, n \in Z$.
Therefore,

(i) $\tan 2x = 0 \Rightarrow 2x = n\pi \Rightarrow x = \frac{n\pi}{2}, \; n \in Z$

(ii) $\tan \frac{x}{2} = 0 \Rightarrow \frac{x}{2} = n\pi \Rightarrow x = 2n\pi, \; n \in Z$

(iii) $\tan \frac{3x}{4} = 0 \Rightarrow \frac{3x}{4} = n\pi \Rightarrow x = \frac{4n\pi}{3}, n \in Z$

THEOREM 5 *Prove that the general solution of* $\sin x = \sin \alpha$ *is given by:* $x = n\pi + (-1)^n \alpha$, $n \in Z$.

PROOF We have,

$$\sin x = \sin \alpha$$

$$\Leftrightarrow \quad \sin x - \sin \alpha = 0$$

$$\Leftrightarrow \quad 2 \sin\left(\frac{x-\alpha}{2}\right) \cos\left(\frac{x+\alpha}{2}\right) = 0$$

$$\Leftrightarrow \quad \sin\left(\frac{x-\alpha}{2}\right) = 0 \quad \text{or,} \quad \cos\left(\frac{x+\alpha}{2}\right) = 0$$

$$\Leftrightarrow \quad \frac{x-\alpha}{2} = m\pi, \quad \text{or,} \quad \frac{x+\alpha}{2} = (2m+1)\frac{\pi}{2}, m \in Z$$

$$\Leftrightarrow \quad x = 2m\pi + \alpha \quad \text{or,} \quad x = (2m+1)\pi - \alpha, m \in Z$$

$$\Leftrightarrow \quad x = (\text{Any even multiple of } \pi) + \alpha \quad \text{or,} \quad x = (\text{Any odd multiple of } \pi) - \alpha$$

$$\Leftrightarrow \quad x = n\pi + (-1)^n \alpha, \quad \text{where } n \in Z.$$

Q.E.D.

REMARK 1 *The equation* cosec x = cosec α *is equivalent to* sin x = sin α. *Thus,* cosec x = cosec α *and* sin x = sin α *have the same general solution.*

THEOREM 6 *Prove that the general solution of* $\cos x = \cos \alpha$ *is given by:* $x = 2n\pi \pm \alpha$, where $n \in Z$.

PROOF We have,

$$\cos x = \cos \alpha$$

$$\Leftrightarrow \quad \cos x - \cos \alpha = 0$$

$$\Leftrightarrow \quad -2 \sin\left(\frac{x+\alpha}{2}\right) \sin\left(\frac{x-\alpha}{2}\right) = 0$$

$$\Leftrightarrow \quad \sin\left(\frac{x+\alpha}{2}\right) = 0 \text{ or, } \sin\left(\frac{x-\alpha}{2}\right) = 0$$

$$\Leftrightarrow \quad \frac{x+\alpha}{2} = n\pi, \text{ or } \frac{x-\alpha}{2} = n\pi, \; n \in Z$$

$$\Leftrightarrow \quad x = 2n\pi - \alpha \text{ or, } x = 2n\pi + \alpha, n \in Z \Leftrightarrow x = 2n\pi \pm \alpha, n \in Z.$$

Q.E.D.

REMARK 2 *Since* $\sec x = \sec \alpha \Leftrightarrow \cos x = \cos \alpha$. *So, the general solutions of* $\cos x = \cos \alpha$ *and* $\sec x = \sec \alpha$ *are same.*

THEOREM 7 *Prove that the general solution of* $\tan x = \tan \alpha$ *is given by:* $x = n\pi + \alpha, n \in Z$.

PROOF We have,

$$\tan x = \tan \alpha \Leftrightarrow \frac{\sin x}{\cos x} = \frac{\sin \alpha}{\cos \alpha} \Leftrightarrow \sin x \cos \alpha - \cos x \sin \alpha = 0$$

$$\Leftrightarrow \quad \sin(x - \alpha) = 0 \Leftrightarrow x - \alpha = n\pi, n \in Z \Leftrightarrow x = n\pi + \alpha, n \in Z$$

Q.E.D.

REMARK 3 *Since* $\tan x = \tan \alpha \Leftrightarrow \cot x = \cot \alpha$. *So, general solutions of* $\cot x = \cot \alpha$ *and* $\tan x = \tan \alpha$ *are same.*

In order to find the general solutions of trigonometrical equations of the form $\sin x = \sin \alpha$, $\cos x = \cos \alpha$ and $\tan x = \tan \alpha$, we may use the following algorithm.

ALGORITHM

Step II *Find a value of x, preferably between 0 and* 2π or between $-\pi$ and π, *satisfying the given equation and call it* α.

Step II *If the equation is* $\sin x = \sin \alpha$, write $x = n\pi + (-1)^n \alpha, n \in Z$ *as the general solution.*

For the equation $\cos x = \cos \alpha$, write $x = 2n\pi \pm \alpha, n \in Z$ *as the general solution.*

For the equation $\tan x = \tan \alpha$, write $x = n\pi + \alpha, n \in Z$ *as the general solution.*

Following examples illustrate the algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I ON FINDING THE GENERAL SOLUTIONS OF THE EQUATIONS OF THE FORM $\sin x = \sin \alpha$, $\cos x = \cos \alpha$, $\tan x = \tan \alpha$

EXAMPLE 1 *Find the general solutions of the following equations:*

(i) $\sin x = \frac{\sqrt{3}}{2}$ (ii) $2 \sin x + 1 = 0$ (iii) $\text{cosec } x = 2$

SOLUTION (i) A value of x satisfying $\sin x = \frac{\sqrt{3}}{2}$ is $\frac{\pi}{3}$.

$$\therefore \quad \sin x = \frac{\sqrt{3}}{2} \Rightarrow \sin x = \sin\frac{\pi}{3} \Rightarrow x = n\pi + (-1)^n \frac{\pi}{3}, n \in Z$$

(ii) We have, $2 \sin x + 1 = 0 \Rightarrow \sin x = -\frac{1}{2}$. A value of x satisfying this equation is $-\pi/6$.

$\therefore \quad \sin x = -\frac{1}{2} \Rightarrow \sin x = \sin\left(-\frac{\pi}{6}\right) \Rightarrow x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right) \Rightarrow x = n\pi + (-1)^{n+1}\frac{\pi}{6}, n \in Z.$

(iii) We have, $\operatorname{cosec} x = 2 \Rightarrow \sin x = \frac{1}{2} \Rightarrow \sin x = \sin\frac{\pi}{6} \Rightarrow x = n\pi + (-1)^n \frac{\pi}{6}, n \in Z.$

EXAMPLE 2 *Find the general solutions of the following equations:*

(i) $\cos x = \frac{1}{2}$ (ii) $\cos 3x = -\frac{1}{2}$ (iii) $\sqrt{3} \sec 2x = 2$

(iv) $\sec x \cos 5x + 1 = 0,\ 0 < x \le \frac{\pi}{2}$ [NCERT EXEMPLAR]

SOLUTION (i) $\cos x = \frac{1}{2} \Rightarrow \cos x = \cos\frac{\pi}{3} \Rightarrow x = 2n\pi \pm \frac{\pi}{3},\ n \in Z$

(ii) $\cos 3x = -\frac{1}{2} \Rightarrow \cos 3x = \cos\frac{2\pi}{3} \Rightarrow 3x = 2n\pi \pm \frac{2\pi}{3} \Rightarrow x = \frac{2n\pi}{3} \pm \frac{2\pi}{9}, n \in Z$

(iii) $\sqrt{3}\sec 2x = 2 \Rightarrow \cos 2x = \frac{\sqrt{3}}{2} \Rightarrow \cos 2x = \cos\frac{\pi}{6} \Rightarrow 2x = 2n\pi \pm \frac{\pi}{6} \Rightarrow x = n\pi \pm \frac{\pi}{12},\ n \in Z$

(iv) We have, $\sec x \cos 5x + 1 = 0,\ 0 < x \le \frac{\pi}{2}$

$\Rightarrow \quad \frac{\cos 5x}{\cos x} + 1 = 0$

$\Rightarrow \quad \cos 5x + \cos x = 0$

$\Rightarrow \quad 2\cos\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right) = 0$

$\Rightarrow \quad 2\cos 3x \cos 2x = 0$

$\Rightarrow \quad \cos 3x = 0$ or, $\cos 2x = 0$

$\Rightarrow \quad 3x = \frac{\pi}{2}, \frac{3\pi}{2}$ or, $2x = \frac{\pi}{2}$ $\left[\because\ 0 < x \le \frac{\pi}{2}\ \therefore\ 0 < 3x \le \frac{3\pi}{2} \text{ and } 0 < 2x \le \pi\right]$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{\pi}{2}$ or, $x = \frac{\pi}{4} \Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{4}$

EXAMPLE 3 *Solve the following trigonometric equations:*

(i) $\tan x = \frac{1}{\sqrt{3}}$ (ii) $\tan 2x = \sqrt{3}$ (iii) $\tan 3x = -1$

(iv) $2\tan^2 x \sec^2 x + 1 = 2,\ 0 \le x \le 2\pi$ [NCERT EXEMPLAR]

SOLUTION (i) $\tan x = \frac{1}{\sqrt{3}} \Rightarrow \tan x = \tan\frac{\pi}{6} \Rightarrow x = n\pi + \frac{\pi}{6}, n \in Z$

(ii) $\tan 2x = \sqrt{3} \Rightarrow \tan 2x = \tan\frac{\pi}{3} \Rightarrow 2x = n\pi + \frac{\pi}{3} \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{6}, n \in Z$

(iii) $\tan 3x = -1 \Rightarrow \tan 3x = \tan\left(-\frac{\pi}{4}\right) \Rightarrow 3x = n\pi + \left(-\frac{\pi}{4}\right) \Rightarrow x = \frac{n\pi}{3} - \frac{\pi}{12}, n \in Z.$

(iv) We have, $\cot x + \tan x = 2\operatorname{cosec} x$

$\Rightarrow \quad \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{2}{\sin x}$ and $x \ne n\pi, n \in Z$

$$\Rightarrow \quad \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{2}{\sin x}$$

$$\Rightarrow \quad \frac{1}{\sin x \cos x} = \frac{2}{\sin x} \Rightarrow \frac{1}{\cos x} = 2 \qquad [\because x \neq n\pi \therefore \sin x \neq 0]$$

$$\Rightarrow \quad \cos x = \frac{1}{2} \Rightarrow \cos x = \cos\frac{\pi}{3} \Rightarrow x = 2n\pi \pm \frac{\pi}{3}, \quad n \in Z.$$

EXAMPLE 4 *Solve the following trigonometric equations:*

(i) $\sin\frac{x}{2} = -1$ (ii) $\cos\frac{3x}{2} = \frac{1}{2}$ (iii) $\tan\left(\frac{2}{3}x\right) = \sqrt{3}$

(iv) $\cot x + \tan x = 2 \operatorname{cosec} x$ [NCERT EXEMPLAR]

SOLUTION (i) $\sin\frac{x}{2} = -1$

$$\Rightarrow \quad \sin\frac{x}{2} = \sin\left(-\frac{\pi}{2}\right) \Rightarrow \frac{x}{2} = n\pi + (-1)^n\left(-\frac{\pi}{2}\right) \Rightarrow x = 2n\pi + (-1)^{n+1}\pi, n \in Z$$

(ii) $\cos\frac{3x}{2} = \frac{1}{2} \Rightarrow \cos\frac{3x}{2} = \cos\frac{\pi}{3} \Rightarrow \frac{3x}{2} = 2n\pi \pm \frac{\pi}{3} \Rightarrow x = \frac{4n\pi}{3} \pm \frac{2\pi}{9}, n \in Z$

(iii) $\tan\left(\frac{2x}{3}\right) = \sqrt{3} \Rightarrow \tan\left(\frac{2x}{3}\right) = \tan\frac{\pi}{3} \Rightarrow \frac{2x}{3} = n\pi + \frac{\pi}{3} \Rightarrow x = \frac{3n\pi}{2} + \frac{\pi}{2}, n \in Z.$

(iv) We have, $2\tan^2 x + \sec^2 x = 2, \; 0 \leq x \leq 2\pi$

$$\Rightarrow \quad 2\tan^2 x + 1 + \tan^2 x = 2 \Rightarrow 3\tan^2 x = 1 \Rightarrow \tan^2 x = \frac{1}{3} \Rightarrow \tan x = \pm\frac{1}{\sqrt{3}}$$

Case I *When* $\tan x = \frac{1}{\sqrt{3}}$: In this case, we obtain $x = \frac{\pi}{6}, \frac{7\pi}{6}$

Case II *When* $\tan x = -\frac{1}{\sqrt{3}}$: In this case, we obtain $x = \frac{5\pi}{6}, \frac{11\pi}{6}$

Hence, the possible solutions of the given equation are: $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type II ON FINDING THE GENERAL SOLUTION OF THE EQUATIONS REDUCIBLE TO THE FORMS $\sin x = \sin\alpha$, $\cos x = \cos\alpha$, $\tan x = \tan\alpha$

EXAMPLE 5 *Solve the equation:* $\sin x + \sin 3x + \sin 5x = 0.$ [NCERT EXEMPLAR]

SOLUTION We have,

$$\sin x + \sin 3x + \sin 5x = 0 \Rightarrow (\sin 5x + \sin x) + \sin 3x = 0$$

$$\Rightarrow \quad 2\sin 3x \cos 2x + \sin 3x = 0$$

$$\Rightarrow \quad \sin 3x(2\cos 2x + 1) = 0 \Rightarrow \sin 3x = 0 \text{ or } 2\cos 2x + 1 = 0 \Rightarrow \sin 3x = 0 \text{ or, } \cos 2x = -\frac{1}{2}$$

Now, $\sin 3x = 0 \Rightarrow 3x = n\pi \Rightarrow x = \frac{n\pi}{3}, n \in Z$

And, $\cos 2x = -\frac{1}{2} \Rightarrow \cos 2x = \cos\frac{2\pi}{3} \Rightarrow 2x = 2m\pi \pm \frac{2\pi}{3} \Rightarrow x = m\pi \pm \frac{\pi}{3}, m \in Z.$

$$\Rightarrow \quad x = (3m \pm 1)\frac{\pi}{3}, \quad m \in Z.$$

These values of x are contained in $x = \frac{n\pi}{3}$, $n \in Z$. Hence, the general solution of the given equation is : $x = \frac{n\pi}{3}$, $n \in Z$.

EXAMPLE 6 *Solve the equation:* $\cos x + \cos 3x - 2 \cos 2x = 0$

SOLUTION We have,

$$\cos x + \cos 3x - 2\cos 2x = 0$$

$$\Leftrightarrow \quad 2\cos 2x \cos x - 2\cos 2x = 0 \Leftrightarrow 2\cos 2x(\cos x - 1) = 0 \Rightarrow \cos 2x = 0 \text{ or, } \cos x - 1 = 0$$

Now, $\cos 2x = 0 \Rightarrow 2x = (2n+1)\frac{\pi}{2} \Rightarrow x = (2n+1)\frac{\pi}{4}, n \in Z$

And, $\cos x - 1 = 0 \Rightarrow \cos x = 1 \Rightarrow \cos x = \cos 0 \Rightarrow x = 2m\pi \pm 0 \Rightarrow x = 2m\pi, m \in Z$

Hence, $x = (2n+1)\frac{\pi}{4}$ or, $x = 2m\pi$, where $m, n \in Z$.

EXAMPLE 7 *Solve the equation:* $\sin mx + \sin nx = 0$.

SOLUTION We have,

$$\sin mx + \sin nx = 0$$

$$\Rightarrow \quad 2\sin\left(\frac{m+n}{2}\right)x \cos\left(\frac{m-n}{2}\right)x = 0 \Rightarrow \sin\left(\frac{m+n}{2}\right)x = 0 \text{ or, } \cos\left(\frac{m-n}{2}\right)x = 0$$

Now, $\sin\left(\frac{m+n}{2}\right)x = 0 \Rightarrow \left(\frac{m+n}{2}\right)x = r\pi \Rightarrow x = \frac{2r\pi}{m+n}, \quad r \in Z$

And, $\cos\left(\frac{m-n}{2}\right)x = 0 \Rightarrow \left(\frac{m-n}{2}\right)x = (2s+1)\frac{\pi}{2} \Rightarrow x = \frac{(2s+1)\pi}{m-n}, s \in Z$

Hence, $x = \frac{2r\pi}{m+n}$ or, $x = \frac{(2s+1)\pi}{m-n}$, where $r, s \in Z$.

EXAMPLE 8 *Solve the following equations:*

(i) $\sin 2x + \cos x = 0$ [NCERT] (ii) $\sin 3x + \cos 2x = 0$ (iii) $\sin 2x + \sin 4x + \sin 6x = 0$

SOLUTION (i) $\sin 2x + \cos x = 0$

$$\Rightarrow \quad \cos x = -\sin 2x \Rightarrow \cos x = \cos\left(\frac{\pi}{2} + 2x\right) \Rightarrow x = 2n\pi \pm \left(\frac{\pi}{2} + 2x\right), \ n \in Z$$

Taking positive sign, we obtain

$$x = 2n\pi + \frac{\pi}{2} + 2x \Rightarrow -x = 2n\pi + \frac{\pi}{2} \Rightarrow x = -2n\pi - \frac{\pi}{2} \Rightarrow x = 2m\pi - \frac{\pi}{2}, \text{ where } m = -n \in Z.$$

Taking negative sign, we obtain

$$x = 2n\pi - \left(\frac{\pi}{2} + 2x\right) \Rightarrow 3x = 2n\pi - \frac{\pi}{2} \Rightarrow x = \frac{2n\pi}{3} - \frac{\pi}{6}, n \in Z.$$

Hence, $x = 2m\pi - \frac{\pi}{2}$, or, $x = \frac{2n\pi}{3} - \frac{\pi}{6}$, where $m, n \in Z$.

(ii) $\sin 3x + \cos 2x = 0$

$$\Rightarrow \cos 2x = -\sin 3x \Rightarrow \cos 2x = \cos\left(\frac{\pi}{2} + 3x\right) \Rightarrow 2x = 2n\pi \pm \left(\frac{\pi}{2} + 3x\right), \ n \in Z$$

Taking positive sign, we obtain

$$2x = 2n\pi + \frac{\pi}{2} + 3x \Rightarrow -x = 2n\pi + \frac{\pi}{2} \Rightarrow x = -2n\pi - \frac{\pi}{2} \Rightarrow x = 2m\pi - \frac{\pi}{2}, \text{ where } -n = m.$$

Taking negative sign, we obtain

$$2x = 2n\pi - \frac{\pi}{2} - 3x \Rightarrow 5x = 2n\pi - \frac{\pi}{2} \Rightarrow x = \frac{2n\pi}{5} - \frac{\pi}{10},\ n \in Z$$

Hence, $x = \dfrac{2n\pi}{5} - \dfrac{\pi}{10}$ or, $x = 2m\pi - \dfrac{\pi}{2}$, where $m, n \in Z$.

(iii) We have,

$$\sin 2x + \sin 4x + \sin 6x = 0$$

$$\Rightarrow \sin 4x + (\sin 2x + \sin 6x) = 0$$

$$\Rightarrow \sin 4x + 2\sin 4x \cos 2x = 0$$

$$\Rightarrow \sin 4x(1 + 2\cos 2x) = 0$$

$$\Rightarrow \sin 4x = 0 \text{ or, } 1 + 2\cos 2x = 0 \Rightarrow \sin 4x = 0 \text{ or, } \cos 2x = -\frac{1}{2}$$

Now, $\sin 4x = 0 \Rightarrow 4x = n\pi \Rightarrow x = \dfrac{n\pi}{4},\ n \in Z$

And, $\cos 2x = -\dfrac{1}{2}$

$$\Rightarrow \cos 2x = \cos\frac{2\pi}{3} \Rightarrow 2x = 2m\pi \pm \frac{2\pi}{3} \Rightarrow x = m\pi \pm \frac{\pi}{3},\ m \in Z$$

Hence, $x = \dfrac{n\pi}{4}$ or, $x = m\pi \pm \dfrac{\pi}{3}$, where $m, n \in Z$.

EXAMPLE 9 *Find the sum of all solutions of* $\cos x \cos\left(x + \dfrac{\pi}{3}\right)\cos\left(\dfrac{\pi}{3} - x\right) = \dfrac{1}{4},\ x \in [0, 6\pi]$.

SOLUTION We have,

$$\cos x \cos\left(\frac{\pi}{3} + x\right)\cos\left(\frac{\pi}{3} - x\right) = \frac{1}{4}$$

$$\Rightarrow \frac{1}{4}\cos 3x = \frac{1}{4} \qquad \left[\because \cos x \cos\left(\frac{\pi}{3} + x\right)\cos\left(\frac{\pi}{3} - x\right) = \frac{1}{4}\cos 3x\right]$$

$$\Rightarrow \cos 3x = 1 \Rightarrow \cos 3x = \cos 0 \Rightarrow 3x = 2n\pi \Rightarrow x = \frac{2n\pi}{3},\ n \in Z$$

But, $x \in [0, 6\pi]$. Therefore, $x = 0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{6\pi}{3}, \dfrac{8\pi}{3}, \ldots, \dfrac{18\pi}{3}$.

$$\text{Sum of all these solutions} = 0 + \frac{2\pi}{3} + \frac{4\pi}{3} + \ldots + \frac{18\pi}{3} = \frac{2\pi}{3}(1 + 2 + 3 + \ldots + 9) = \frac{2\pi}{3} \times \frac{9 \times 10}{2} = 30\pi$$

Hence, required sum $= 30\pi$.

EXAMPLE 10 *Solve:* $\sin x - 2\sin 2x + \sin 3x = \cos x - 2\cos 2x + \cos 3x$

SOLUTION We have,

$$\sin x - 2\sin 2x + \sin 3x = \cos x - 2\cos 2x + \cos 3x$$

$$\Rightarrow (\sin 3x + \sin x) - 2\sin 2x = (\cos 3x + \cos x) - 2\cos 2x$$

$$\Rightarrow 2\sin 2x \cos x - 2\sin 2x = 2\cos 2x \cos x - 2\cos 2x$$

$$\Rightarrow 2\sin 2x(\cos x - 1) = 2\cos 2x(\cos x - 1)$$

$$\Rightarrow 2(\cos x - 1)(\sin 2x - \cos 2x) = 0$$

$$\Rightarrow 2(\cos x - 1) = 0 \text{ or, } \sin 2x - \cos 2x = 0$$

$$\Rightarrow \cos x = 1 \text{ or, } \sin 2x = \cos 2x$$

$$\Rightarrow \cos x = \cos 0 \text{ or, } \tan 2x = 1 \Rightarrow \cos x = \cos 0 \text{ or, } \tan 2x = \tan\frac{\pi}{4}$$

$\Rightarrow \quad x = 2n\pi$ or, $2x = n\pi + \frac{\pi}{4} \Rightarrow x = 2n\pi$ or, $x = n\pi + \frac{\pi}{8}, n \in Z$

EXAMPLE 11 *Find the general solution of the equation*

$\sin x - 3 \sin 2x + \sin 3x = \cos x - 3 \cos 2x + \cos 3x$ [NCERT EXEMPLAR]

SOLUTION We have,

$\sin x - 3 \sin 2x + \sin 3x = \cos x - 3 \cos 2x + \cos 3x$

$\Rightarrow \quad (\sin x + \sin 3x) - 3 \sin 2x = (\cos x + \cos 3x) - 3 \cos 2x$

$\Rightarrow \quad 2 \sin 2x \cos x - 3 \sin 2x = 2 \cos 2x \cos x - 3 \cos 2x$

$\Rightarrow \quad \sin 2x (2 \cos x - 3) = \cos 2x (2 \cos x - 3)$

$\Rightarrow \quad \sin 2x = \cos 2x \qquad [\because 2 \cos x - 3 \neq 0]$

$\Rightarrow \quad \tan 2x = 1 \Rightarrow \tan 2x = \tan \frac{\pi}{4} \Rightarrow 2x = n\pi + \frac{\pi}{4} \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{8}, \ n \in Z.$

EXAMPLE 12 *Solve the following equations:*

(i) $2 \cos^2 x + 3 \sin x = 0$ [NCERT] (ii) $2 \sin^2 x = 3 \cos x, 0 \leq x \leq 2\pi$ [NCERT EXEMPLAR]

(iii) $\cot^2 x + \frac{3}{\sin x} + 3 = 0$ (iv) $2 \tan x - \cot x = -1$

(v) $4 \cos x - 3 \sec x = \tan x$ (vi) $\tan^2 x + (1 - \sqrt{3}) \tan x - \sqrt{3} = 0$

(vii) $\sec^2 2x = 1 - \tan 2x$ [NCERT]

SOLUTION (i) $2 \cos^2 x + 3 \sin x = 0$

$\Rightarrow \quad 2(1 - \sin^2 x) + 3 \sin x = 0$

$\Rightarrow \quad 2 \sin^2 x - 3 \sin \theta - 2 = 0$

$\Rightarrow \quad 2 \sin^2 x - 4 \sin x + \sin x - 2 = 0$

$\Rightarrow \quad 2 \sin x (\sin x - 2) + 1 (\sin x - 2) = 0$

$\Rightarrow \quad (\sin x - 2)(2 \sin x + 1) = 0$

$\Rightarrow \quad 2 \sin x + 1 = 0 \qquad [\because \sin x \neq 2 \ \therefore \sin x - 2 \neq 0]$

$\Rightarrow \quad \sin x = -\frac{1}{2}$

$\Rightarrow \quad \sin x = \sin\left(-\frac{\pi}{6}\right) \Rightarrow x = n\pi + (-1)^n \left(-\frac{\pi}{6}\right) \Rightarrow x = n\pi + (-1)^{n+1} \frac{\pi}{6}, \quad n \in Z.$

(ii) We have, $2 \sin^2 x = 3 \cos x, \ 0 \leq x \leq 2\pi$

$\Rightarrow \quad 2(1 - \cos^2 x) = 3 \cos x$

$\Rightarrow \quad 2 \cos^2 x + 3 \cos x - 2 = 0$

$\Rightarrow \quad 2 \cos^2 x + 4 \cos x - \cos x - 2 = 0$

$\Rightarrow \quad 2 \cos x (\cos x + 2) - (\cos x + 2) = 0$

$\Rightarrow \quad (\cos x + 2)(2 \cos x - 1) = 0$

$\Rightarrow \quad 2 \cos x - 1 = 0 \qquad [\because \cos x + 2 \neq 0]$

$\Rightarrow \quad \cos x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{\pi}{3}, \frac{5\pi}{3}$ $[\because\ 0 \le x \le 2\pi]$

(iii) $\cot^2 x + \frac{3}{\sin x} + 3 = 0$

$\Rightarrow \quad \operatorname{cosec}^2 x - 1 + 3 \operatorname{cosec} x + 3 = 0$

$\Rightarrow \quad \operatorname{cosec}^2 x + 3 \operatorname{cosec} x + 2 = 0$

$\Rightarrow \quad (\operatorname{cosec} x + 2)(\operatorname{cosec} x + 1) = 0 \Rightarrow \operatorname{cosec} x + 2 = 0$ or, $\operatorname{cosec} x + 1 = 0$

Now, $\operatorname{cosec} x + 2 = 0$

$\Rightarrow \quad \frac{1}{\sin x} + 2 = 0$

$\Rightarrow \quad \sin x = -\frac{1}{2}$

$\Rightarrow \quad \sin x = \sin\left(-\frac{\pi}{6}\right) \Rightarrow x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right), \Rightarrow x = n\pi + (-1)^{n+1}\frac{\pi}{6},\ n \in Z$

And, $\operatorname{cosec} x + 1 = 0$

$\Rightarrow \frac{1}{\sin x} + 1 = 0$

$\Rightarrow \sin x = -1$

$\Rightarrow \sin x = \sin\left(-\frac{\pi}{2}\right) \Rightarrow x = m\pi + (-1)^m\left(-\frac{\pi}{2}\right) \Rightarrow x = m\pi + (-1)^{m+1}\frac{\pi}{2}, m \in Z$

Hence, $x = n\pi + (-1)^{n+1}\frac{\pi}{6}$ or, $x = m\pi + (-1)^{m+1}\frac{\pi}{2}, m, n \in Z$

(iv) $2\tan x - \cot x = -1$

$\Rightarrow 2\tan x - \frac{1}{\tan x} = -1$

$\Rightarrow 2\tan^2 x + \tan x - 1 = 0$

$\Rightarrow 2\tan^2 x + 2\tan x - \tan x - 1 = 0$

$\Rightarrow 2\tan x(\tan x + 1) - (\tan x + 1) = 0 \Rightarrow (\tan x + 1)(2\tan x - 1) = 0 \Rightarrow \tan x = -1$ or, $\tan x = \frac{1}{2}$

Now,

$\tan x = -1 \Rightarrow \tan x = \tan\left(-\frac{\pi}{4}\right) \Rightarrow x = n\pi + \left(-\frac{\pi}{4}\right), \Rightarrow x = n\pi - \frac{\pi}{4},\ n \in Z$

And, $\tan x = \frac{1}{2}$

$\Rightarrow \tan x = \tan\alpha$, where $\tan\alpha = \frac{1}{2} \Rightarrow x = m\pi + \alpha$, where $\tan\alpha = \frac{1}{2}$ and $m \in Z$

Hence, $x = n\pi - \frac{\pi}{4}$ or, $x = m\pi + \alpha$, where $m,\ n \in Z$ and $\tan\alpha = \frac{1}{2}$

(v) $4\cos x - 3\sec x = \tan x$

$\Rightarrow 4\cos x - \frac{3}{\cos x} = \frac{\sin x}{\cos x}$

$\Rightarrow 4\cos^2 x - 3 = \sin x$

$\Rightarrow 4(1 - \sin^2 x) - 3 = \sin x$

$\Rightarrow 4\sin^2 x + \sin x - 1 = 0$

$\Rightarrow \sin x = \dfrac{-1 \pm \sqrt{1+16}}{8} \Rightarrow \sin x = \dfrac{-1 \pm \sqrt{17}}{8} \Rightarrow \sin x = \dfrac{-1+\sqrt{17}}{8}$ or, $\sin x = \dfrac{-1-\sqrt{17}}{8}$

Now, $\sin x = \dfrac{-1+\sqrt{17}}{8}$

$\Rightarrow \sin x = \sin \alpha$, where $\sin \alpha = \dfrac{-1+\sqrt{17}}{8}$

$\Rightarrow x = n\pi + (-1)^n \alpha$, where $\sin \alpha = \dfrac{-1+\sqrt{17}}{8}$ and $n \in Z$

And, $\sin x = \dfrac{-1-\sqrt{17}}{8}$

$\Rightarrow \sin x = \sin \beta$, where $\sin \beta = \dfrac{-1-\sqrt{17}}{8}$

$\Rightarrow x = n\pi + (-1)^n \beta$, where $\sin \beta = \dfrac{-1-\sqrt{17}}{8}$

(vi) $\tan^2 x + (1-\sqrt{3}) \tan x - \sqrt{3} = 0$

$\Rightarrow \tan^2 x + \tan x - \sqrt{3} \tan x - \sqrt{3} = 0$

$\Rightarrow \tan (\tan x + 1) - \sqrt{3} (\tan x + 1) = 0$

$\Rightarrow (\tan x + 1)(\tan x - \sqrt{3}) = 0$

$\Rightarrow \tan x + 1 = 0$ or $\tan x - \sqrt{3} = 0 \Rightarrow \tan x = -1$ or, $\tan x = \sqrt{3}$

Now, $\tan x = -1 \Rightarrow \tan x = \tan\left(-\dfrac{\pi}{4}\right) \Rightarrow x = n\pi - \dfrac{\pi}{4}$, $n \in Z$

And, $\tan x = \sqrt{3} \Rightarrow \tan x = \tan \dfrac{\pi}{3} \Rightarrow x = m\pi + \dfrac{\pi}{3}$, $m \in Z$

Hence, $x = n\pi - \dfrac{\pi}{4}$ or, $x = m\pi + \dfrac{\pi}{3}$, where $m, n \in Z$.

(vii) $\sec^2 2x = 1 - \tan 2x$

$\Rightarrow 1 + \tan^2 2x = 1 - \tan 2x$

$\Rightarrow \tan^2 2x + \tan 2x = 0$

$\Rightarrow \tan 2x (\tan 2x + 1) = 0$

$\Rightarrow \tan 2x = 0$ or, $\tan 2x + 1 = 0$

$\Rightarrow \tan 2x = 0$ or, $\tan 2x = -1 \Rightarrow \tan 2x = 0$ or, $\tan 2x = \tan \dfrac{3\pi}{4}$

$\Rightarrow 2x = n\pi$ or, $2x = n\pi + \dfrac{3\pi}{4} \Rightarrow x = \dfrac{n\pi}{2}$ or, $x = \dfrac{n\pi}{2} + \dfrac{3\pi}{8}$, $n \in Z$

EXAMPLE 13 *Solve :* $\tan\left(x + \dfrac{\pi}{12}\right) = 3 \tan\left(x - \dfrac{\pi}{12}\right)$

SOLUTION We have,

$$\tan\left(x + \frac{\pi}{12}\right) = 3 \tan\left(x - \frac{\pi}{12}\right) \Rightarrow \frac{\tan\left(x + \frac{\pi}{12}\right)}{\tan\left(x - \frac{\pi}{12}\right)} = \frac{3}{1}$$

$$\Rightarrow \quad \frac{\tan\left(x+\frac{\pi}{12}\right)+\tan\left(x-\frac{\pi}{12}\right)}{\tan\left(x+\frac{\pi}{12}\right)-\tan\left(x-\frac{\pi}{12}\right)}=\frac{3+1}{3-1} \qquad \text{[Applying Componendo and dividendo]}$$

$$\Rightarrow \quad \frac{\sin\left(x+\frac{\pi}{12}+x-\frac{\pi}{12}\right)}{\sin\left(x+\frac{\pi}{12}-x+\frac{\pi}{12}\right)}=2 \qquad \left[\because \frac{\tan A+\tan B}{\tan A-\tan B}=\frac{\sin(A+B)}{\sin(A-B)}\right]$$

$$\Rightarrow \quad \frac{\sin 2x}{\sin\frac{\pi}{6}}=2 \Rightarrow \sin 2x=1 \Rightarrow \sin 2x=\sin\frac{\pi}{2} \Rightarrow 2x=n\pi+(-1)^n\frac{\pi}{2} \Rightarrow x=\frac{n\pi}{2}+(-1)^n\frac{\pi}{4}, n\in Z$$

EXAMPLE 14 *Solve the following equations:*

(i) $\tan x+\tan 2x+\tan x\tan 2x=1$ (ii) $\tan x+\tan 2x+\tan 3x=\tan x\tan 2x\tan 3x$

(iii) $\tan x+\tan 2x+\sqrt{3}\tan x\tan 2x=\sqrt{3}$ (iv) $\tan x+\tan\left(x+\frac{\pi}{3}\right)+\tan\left(x+\frac{2\pi}{3}\right)=3$

SOLUTION (i) $\tan x+\tan 2x+\tan x\tan 2x=1$

$$\Rightarrow \quad \tan x+\tan 2x=1-\tan x\tan 2x$$

$$\Rightarrow \quad \frac{\tan x+\tan 2x}{1-\tan x\tan 2x}=1$$

$$\Rightarrow \quad \tan 3x=1 \Rightarrow \tan 3x=\tan\frac{\pi}{4} \Rightarrow 3x=n\pi+\frac{\pi}{4} \Rightarrow x=\frac{n\pi}{3}+\frac{\pi}{12},\ n\in Z$$

(ii) $\tan x+\tan 2x+\tan 3x=\tan x\tan 2x\tan 3x$

$$\Rightarrow \quad \tan x+\tan 2x=-\tan 3x+\tan x\tan 2x\tan 3x$$

$$\Rightarrow \quad \tan x+\tan 2x=-\tan 3x(1-\tan x\tan 2x)$$

$$\Rightarrow \quad \frac{\tan x+\tan 2x}{1-\tan x\tan 2x}=-\tan 3x$$

$$\Rightarrow \quad \tan(x+2x)=-\tan 3x \Rightarrow \tan 3x=-\tan 3x \Rightarrow 2\tan 3x=0 \Rightarrow \tan 3x=0$$

$$\Rightarrow \quad 3x=n\pi \Rightarrow x=\frac{n\pi}{3},\quad n\in Z.$$

(iii) $\tan x+\tan 2x+\sqrt{3}\tan x\tan 2x=\sqrt{3}$

$$\Rightarrow \quad \tan x+\tan 2x=\sqrt{3}(1-\tan x\tan 2x)$$

$$\Rightarrow \quad \frac{\tan x+\tan 2x}{1-\tan x\tan 2x}=\sqrt{3}$$

$$\Rightarrow \quad \tan 3x=\sqrt{3} \Rightarrow \tan 3x=\tan\frac{\pi}{3} \Rightarrow 3x=n\pi+\frac{\pi}{3} \Rightarrow x=\frac{n\pi}{3}+\frac{\pi}{9},\ n\in Z$$

(iv) $\tan x+\tan\left(x+\frac{\pi}{3}\right)+\tan\left(x+\frac{2\pi}{3}\right)=3$

$$\Rightarrow \quad \tan x+\frac{\tan x+\tan\frac{\pi}{3}}{1-\tan x\tan\frac{\pi}{3}}+\frac{\tan x+\tan\frac{2\pi}{3}}{1-\tan x\tan\frac{2\pi}{3}}=3$$

$$\Rightarrow \quad \tan x+\frac{\tan x+\sqrt{3}}{1-\sqrt{3}\tan x}+\frac{\tan x-\sqrt{3}}{1+\sqrt{3}\tan x}=3$$

$$\Rightarrow \quad \tan x + \frac{8\tan x}{1-3\tan^2 x} = 3 \Rightarrow \frac{\tan x - 3\tan^3 x + 8\tan x}{1-3\tan^2 x} = 3 \Rightarrow \frac{3(3\tan x - \tan^3 x)}{1-3\tan^2 x} = 3$$

$$\Rightarrow \quad 3\tan 3x = 3 \Rightarrow \tan 3x = 1 \Rightarrow \tan 3x = \tan\frac{\pi}{4} \Rightarrow 3x = n\pi + \frac{\pi}{4} \Rightarrow x = \frac{n\pi}{3} + \frac{\pi}{12}, n \in Z.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 15 *Solve :* $\tan^2 x \tan^2 3x \tan 4x = \tan^2 x - \tan^2 3x + \tan 4x.$

SOLUTION We have,

$$\tan^2 x \tan^2 3x \tan 4x = \tan^2 x - \tan^2 3x + \tan 4x$$

$$\Rightarrow \quad \tan^2 3x - \tan^2 x = \tan 4x - \tan^2 x \tan^2 3x \tan 4x$$

$$\Rightarrow \quad \tan^2 3x - \tan^2 x = \tan 4x(1 - \tan^2 x \tan^2 3x)$$

$$\Rightarrow \quad \tan 4x = \frac{\tan^2 3x - \tan^2 x}{1 - \tan^2 x \tan^2 3x}$$

$$\Rightarrow \quad \tan 4x = \frac{\tan 3x + \tan x}{1 - \tan 3x \tan x} \times \frac{\tan 3x - \tan x}{1 + \tan 3x \tan x}$$

$$\Rightarrow \quad \tan 4x = \tan(3x+x)\tan(3x-x) \Rightarrow \tan 4x = \tan 4x \tan 2x \Rightarrow \tan 4x(\tan 2x - 1) = 0$$

$$\Rightarrow \quad \tan 4x = 0 \text{ or, } \tan 2x - 1 = 0 \Rightarrow \tan 4x = 0 \text{ or, } \tan 2x = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad 4x = n\pi \text{ or, } 2x = n\pi + \frac{\pi}{4} \Rightarrow x = \frac{n\pi}{4} \text{ or, } x = \frac{n\pi}{2} + \frac{\pi}{8}, n \in Z$$

EXAMPLE 16 *Solve :* $\sec x - 1 = (\sqrt{2} - 1)\tan x,\ x \neq (2n-1)\frac{\pi}{2}, n \in Z$

SOLUTION We have,

$$\sec x - 1 = (\sqrt{2}-1)\tan x$$

$$\Rightarrow \quad \frac{1}{\cos x} - 1 = \tan\frac{\pi}{8}\tan x \qquad \left[\because \tan\frac{\pi}{8} = \sqrt{2} - 1\right]$$

$$\Rightarrow \quad \frac{1-\cos x}{\cos x} = \frac{\sin x \sin\frac{\pi}{8}}{\cos x \cos\frac{\pi}{8}} \qquad \left[\because \cos x \neq 0 \text{ as } x \neq (2n-1)\frac{\pi}{2}\right]$$

$$\Rightarrow \quad \cos\frac{\pi}{8} - \cos x \cos\frac{\pi}{8} = \sin x \sin\frac{\pi}{8} \Rightarrow \sin x \sin\frac{\pi}{8} + \cos x \cos\frac{\pi}{8} = \cos\frac{\pi}{8}$$

$$\Rightarrow \quad \cos\left(x - \frac{\pi}{8}\right) = \cos\frac{\pi}{8} \Rightarrow x - \frac{\pi}{8} = 2n\pi \pm \frac{\pi}{8} \Rightarrow x = 2n\pi + \frac{\pi}{4} \text{ or, } x = 2n\pi, n \in Z$$

EXAMPLE 17 *Find the value of* $x \in (-\pi, \pi)$ *satisfying the equation*

$$8^{1 + |\cos x| + \cos^2 x + |\cos^3 x| + \ldots \text{to } \infty} = 4^3.$$

SOLUTION We have,

$$8^{1 + |\cos x| + \cos^2 x + |\cos^3 x| + \cos^4 x \ldots \text{to } \infty} = 4^3$$

$$\Rightarrow \quad 8^{1 + |\cos x| + \cos^2 x + |\cos^3 x| + \cos^4 x \ldots \text{to } \infty} = 64$$

$$\Rightarrow \quad 8^{\frac{1}{1 - |\cos x|}} = 8^2$$

$$\Rightarrow \quad \frac{1}{1-|\cos x|}=2 \Rightarrow 1-|\cos x|=\frac{1}{2} \Rightarrow |\cos x|=\frac{1}{2} \Rightarrow \cos x=\pm\frac{1}{2} \Rightarrow x=\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$$

10.3 GENERAL SOLUTIONS OF TRIGONOMETRICAL EQUATIONS OF THE FORM $\sin^2 x = \sin^2 \alpha$, $\cos^2 x = \cos^2 \alpha$, $\tan^2 x = \tan^2 \alpha$

THEOREM *Prove that:*

(i) $\sin^2 x = \sin^2 \alpha \Rightarrow x = n\pi \pm \alpha, n \in Z$ (ii) $\cos^2 x = \cos^2 \alpha \Rightarrow x = n\pi \pm \alpha, n \in Z$

(iii) $\tan^2 x = \tan^2 \alpha \Rightarrow x = n\pi \pm \alpha, n \in Z$

PROOF (i) $\sin^2 x = \sin^2 \alpha$

$\Rightarrow \quad 2\sin^2 x = 2\sin^2 \alpha$

$\Rightarrow \quad 1-\cos 2x = 1-\cos 2\alpha \Rightarrow \cos 2x = \cos 2\alpha \Rightarrow 2x = 2n\pi \pm 2\alpha \Rightarrow x = n\pi \pm \alpha, \ n \in Z$

(ii) $\cos^2 x = \cos^2 \alpha$

$\Rightarrow \quad 2\cos^2 x = 2\cos^2 \alpha$

$\Rightarrow \quad 1+\cos 2x = 1+\cos 2\alpha \Rightarrow \cos 2x = \cos 2\alpha \Rightarrow 2x = 2n\pi \pm 2\alpha \Rightarrow x = n\pi \pm \alpha, n \in Z$

(iii) $\tan^2 x = \tan^2 \alpha$

$$\Rightarrow \quad \frac{1-\tan^2 x}{1+\tan^2 x} = \frac{1-\tan^2 \alpha}{1+\tan^2 \alpha} \Rightarrow \cos 2x = \cos 2\alpha \Rightarrow 2x = 2n\pi \pm 2\alpha \Rightarrow x = n\pi \pm \alpha, \ n \in Z$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve:* $7\cos^2 x + 3\sin^2 x = 4$

SOLUTION We have,

$$7\cos^2 x + 3\sin^2 x = 4 \Rightarrow 7(1-\sin^2 x) + 3\sin^2 x = 4 \Rightarrow 4\sin^2 x = 3$$

$$\Rightarrow \quad 4\sin^2 x = 3 \Rightarrow \sin^2 x = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \sin^2 x = \sin^2\frac{\pi}{3} \Rightarrow x = n\pi \pm \frac{\pi}{3}, \ n \in Z$$

EXAMPLE 2 *Solve:* $5\cos^2 x + 7\sin^2 x - 6 = 0$ [NCERT EXEMPLAR]

SOLUTION We have, $5\cos^2 x + 7\sin^2 x - 6 = 0$

$$\Rightarrow \quad 5(1-\sin^2 x) + 7\sin^2 x - 6 = 0 \Rightarrow 2\sin^2 x - 1 = 0 \Rightarrow \sin^2 x = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2$$

$$\Rightarrow \quad \sin^2 x = \sin^2\frac{\pi}{4} \Rightarrow x = n\pi \pm \frac{\pi}{4}, \quad n \in Z.$$

EXAMPLE 3 *Solve:* $2\sin^2 x + \sin^2 2x = 2$

SOLUTION We have,

$$2\sin^2 x + \sin^2 2x = 2$$

$$\Rightarrow \quad 2\sin^2 x + (2\sin x\cos x)^2 = 2$$

$\Rightarrow \quad 4\sin^2 x\cos^2 x + 2\sin^2 x = 2$

$\Rightarrow \quad 2\sin^2 x\cos^2 x + \sin^2 x = 1$

$\Rightarrow \quad 2\sin^2 x\cos^2 x - (1-\sin^2 x) = 0$

$\Rightarrow \quad 2\sin^2 x\cos^2 x - \cos^2 x = 0$

$\Rightarrow \quad \cos^2 x(2\sin^2 x - 1) = 0$

$\Rightarrow \quad \cos^2 x = 0$ or, $2\sin^2 x - 1 = 0 \Rightarrow \cos^2 x = 0$ or, $\sin^2 x = \frac{1}{2}$

Now, $\cos^2 x = 0 \Rightarrow \cos^2 x = \cos^2\frac{\pi}{2} \Rightarrow x = n\pi \pm \frac{\pi}{2}$, $n \in Z$

And, $\sin^2 x = \frac{1}{2} \Rightarrow \sin^2 x = \sin^2\frac{\pi}{4} \Rightarrow x = m\pi \pm \frac{\pi}{4}$, $m \in Z$

Hence, $x = n\pi \pm \frac{\pi}{2}$ or $x = m\pi \pm \frac{\pi}{4}$, where $m, n \in Z$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Solve:* $\sin 3\alpha = 4\sin\alpha\sin(x+\alpha)\sin(x-\alpha)$, where $\alpha \neq n\pi, n \in Z$

SOLUTION We have,

$\sin 3\alpha = 4\sin\alpha\sin(x+\alpha)\sin(x-\alpha)$

$\Rightarrow \sin 3\alpha = 4\sin\alpha(\sin^2 x - \sin^2\alpha)$

$\Rightarrow 3\sin\alpha - 4\sin^3\alpha = 4\sin^2 x\sin\alpha - 4\sin^3\alpha$

$\Rightarrow 3\sin\alpha = 4\sin^2 x\sin\alpha \Rightarrow \sin^2 x = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \sin^2 x = \sin^2\frac{\pi}{3} \Rightarrow x = n\pi \pm \frac{\pi}{3}$, $n \in Z$

EXAMPLE 5 *Solve:* $4\sin x\sin 2x\sin 4x = \sin 3x$

SOLUTION We have,

$4\sin x\sin 2x\sin 4x = \sin 3x$

$\Rightarrow 4\sin x\sin(3x-x).\sin(3x+x) = \sin 3x$

$\Rightarrow 4[\sin x(\sin^2 3x - \sin^2 x)] = 3\sin x - 4\sin^3 x$

$\Rightarrow 4\sin x\sin^2 3x - 4\sin^3 x = 3\sin x - 4\sin^3 x$

$\Rightarrow 4\sin x\sin^2 3x = 3\sin x$

$\Rightarrow \sin x(4\sin^2 3x - 3) = 0 \Rightarrow \sin x = 0$ or, $4\sin^2 3x - 3 = 0 \Rightarrow \sin x = 0$ or, $\sin^2 3x = \frac{3}{4}$

Now, $\sin x = 0 \Rightarrow x = n\pi$, $n \in Z$

And, $\sin^2 3x = \frac{3}{4}$

$\Rightarrow \quad \sin^2 3x = \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \sin^2 3x = \sin^2\frac{\pi}{3} \Rightarrow 3x = m\pi \pm \frac{\pi}{3} \Rightarrow x = \frac{m\pi}{3} \pm \frac{\pi}{9}, m \in Z$

Hence, $x = n\pi$ or, $x = \frac{m\pi}{3} \pm \frac{\pi}{9}$, where $m, n \in Z$.

EXAMPLE 6 *Solve:* $81^{\sin^2 x} + 81^{\cos^2 x} = 30, 0 \le x \le \pi$.

SOLUTION We have,

$$81^{\sin^2 x} + 81^{\cos^2 x} = 30 \Rightarrow 81^{\sin^2 x} + 81^{1-\sin^2 x} = 30 \Rightarrow 81^{\sin^2 x} + \frac{81}{81^{\sin^2 x}} = 30$$

$\Rightarrow\ y + \dfrac{81}{y} = 30$, where $y = 81^{\sin^2 x}$

$\Rightarrow\ y^2 - 30y + 81 = 0 \Rightarrow (y-27)(y-3) = 0 \Rightarrow y = 27$ or, $y = 3 \Rightarrow 81^{\sin^2 x} = 27$ or, $81^{\sin^2 x} = 3$

Now, $\quad 81^{\sin^2 x} = 27$

$\Rightarrow \quad (3^4)^{\sin^2 x} = 3^3$

$\Rightarrow \quad 3^{4\sin^2 x} = 3^3$

$\Rightarrow \quad 4\sin^2 x = 3 \Rightarrow \sin^2 x = \dfrac{3}{4} = \left(\dfrac{\sqrt{3}}{2}\right)^2 \Rightarrow \sin^2 x = \sin^2 \dfrac{\pi}{3} \Rightarrow x = n\pi \pm \dfrac{\pi}{3}, n \in Z$

and,

$\quad 81^{\sin^2 x} = 3$

$\Rightarrow \quad (3^4)^{\sin^2 x} = 3$

$\Rightarrow \quad 3^{4\sin^2 x} = 3^1$

$\Rightarrow \quad 4\sin^2 x = 1 \Rightarrow \sin^2 x = \dfrac{1}{4} = \left(\dfrac{1}{2}\right)^2 \Rightarrow \sin^2 x = \sin^2 \dfrac{\pi}{6} \Rightarrow x = n\pi \pm \dfrac{\pi}{6}, n \in Z$

Hence, $x = n\pi \pm \dfrac{\pi}{3}$ or $x = n\pi \pm \dfrac{\pi}{6}, n \in Z$

10.4 TRIGONOMETRIC EQUATIONS OF THE FORM

$a \cos x + b \sin x = c$, where $a, b, c \in R$ such that $|c| \leq \sqrt{a^2 + b^2}$

To solve this type of equations, we first reduce them in the form $\cos x = \cos \alpha$, or $\sin x = \sin \alpha$.

The following algorithm provides the method of solution.

ALGORITHM

Step I *Obtain the equation* $a \cos x + b \sin x = c$.

Step II *Put* $a = r \cos \alpha$ *and* $b = r \sin \alpha$, *where* $r = \sqrt{a^2 + b^2}$ *and* $\tan \alpha = b/a$ i.e. $\alpha = \tan^{-1}(b/a)$.

Step III *Using the substitution in step II, the equation reduces to*

$$r \cos(x - \alpha) = c \Rightarrow \cos(x-\alpha) = \frac{c}{r} = \cos\beta \text{ (say)}.$$

Step IV *Solve the equation obtained in step III by using the formulas discussed earlier.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve:* $\sqrt{3} \cos x + \sin x = \sqrt{2}$ [NCERT EXEMPLAR]

SOLUTION We have, $\sqrt{3} \cos x + \sin x = \sqrt{2}$ ...(i)

This is of the form $a \cos x + b \sin x = c$, where $a = \sqrt{3}, b = 1$ and $c = \sqrt{2}$. Let $a = r\cos\alpha$ and $b = r \sin \alpha$. Then,

$$\sqrt{3} = r\cos\alpha \quad \text{and} \quad 1 = r \sin\alpha \Rightarrow r = \sqrt{a^2+b^2} = \sqrt{(\sqrt{3})^2 + 1^2} = 2 \text{ and}$$

$\tan\alpha = \dfrac{1}{\sqrt{3}} \Rightarrow \alpha = \dfrac{\pi}{6}$

Substituting $a = \sqrt{3} = r\cos\alpha$ and $b = 1 = r\sin\alpha$ in the equation (i) it reduces to

$$r\cos\alpha\cos x + r\sin\alpha\sin x = \sqrt{2}$$

$$\Rightarrow \quad r\cos(x-\alpha) = \sqrt{2}$$

$$\Rightarrow \quad 2\cos\left(x - \frac{\pi}{6}\right) = \sqrt{2}$$

$$\Rightarrow \quad \cos\left(x - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}$$

$$\Rightarrow \quad x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}, n \in Z$$

$$\Rightarrow \quad x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}, \; n \in Z$$

$$\Rightarrow \quad x = 2n\pi + \frac{\pi}{4} + \frac{\pi}{6} \text{ or, } x = 2n\pi - \frac{\pi}{4} + \frac{\pi}{6} \;\Rightarrow\; x = 2n\pi + \frac{5\pi}{12} \text{ or, } x = 2n\pi - \frac{\pi}{12}$$

Hence, $x = 2n\pi + \frac{5\pi}{12}$ or, $x = 2n\pi - \frac{\pi}{12}$, where $n \in Z$

ALITER We have, $\sqrt{3}\cos x + \sin x = \sqrt{2}$. Dividing both sides by $\sqrt{(\sqrt{3})^2 + 1^2} = 2$, we obtain

$$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \quad \cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}$$

$$\Rightarrow \quad x - \frac{\pi}{6} = 2n\pi + \frac{\pi}{4}, \text{ where } n \in Z$$

$$\Rightarrow \quad x - \frac{\pi}{6} = 2n\pi + \frac{\pi}{4} \text{ or, } x - \frac{\pi}{6} = 2n\pi - \frac{\pi}{4}, \; n \in Z$$

$$\Rightarrow \quad x = 2n\pi + \frac{5\pi}{12} \text{ or, } x = 2n\pi - \frac{\pi}{12}, \; n \in Z$$

EXAMPLE 2 *If $\sin x + \cos x = 1$, then find the general value of x.* [NCERT EXEMPLAR]

SOLUTION We have, $\sin x + \cos x = 1$. Dividing throughout by $\sqrt{1^2 + 1^2} = \sqrt{2}$, we obtain

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin x\sin\frac{\pi}{4} + \cos x\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$$

$$\Rightarrow \quad x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4} \;\Rightarrow\; x = \left(2n\pi \pm \frac{\pi}{4}\right) + \frac{\pi}{4} \;\Rightarrow\; x = 2n\pi + \frac{\pi}{2} \text{ or, } x = 2n\pi, \quad n \in Z$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Solve:* $\sqrt{2}\sec x + \tan x = 1$

SOLUTION We have, $\sqrt{2}\sec x + \tan x = 1$

$$\Rightarrow \frac{\sqrt{2}}{\cos x} + \frac{\sin x}{\cos x} = 1 \Rightarrow \sqrt{2} + \sin x = \cos x \Rightarrow \cos x - \sin x = \sqrt{2} \quad ...(i)$$

This is of the form, $a\cos x - b\sin x = c$, where $a = 1$, $b = 1$ and $c = \sqrt{2}$

Let $a = r\cos\alpha$, and $b = r\sin\alpha$. Then,

$$1 = r\cos\alpha \text{ and } 1 = r\sin\alpha$$

$$\Rightarrow r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2} \text{ and, } \tan\alpha = \frac{r\sin\alpha}{r\cos\alpha} = 1 \Rightarrow r = \sqrt{2} \text{ and, } \alpha = \frac{\pi}{4}$$

Substituting $a = 1 = r\cos\alpha$ and $b = 1 = r\sin\alpha$ in (i), we get

$$r\cos x\cos\alpha - r\sin x\sin\alpha = \sqrt{2}$$

$$\Rightarrow r\cos(x+\alpha) = \sqrt{2}$$

$$\Rightarrow \cos\left(x+\frac{\pi}{4}\right) = 1$$

$$\Rightarrow \cos\left(x+\frac{\pi}{4}\right) = \cos 0^\circ \Rightarrow x + \frac{\pi}{4} = 2n\pi \pm 0 \Rightarrow x = 2n\pi - \frac{\pi}{4},\ n \in Z$$

ALITER The given equation reduces to $\cos x - \sin x = \sqrt{2}$. Dividing throughout by $\sqrt{(1)^2+(-1)^2} = \sqrt{2}$, we obtain

$$\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = 1$$

$$\Rightarrow \cos x\cos\frac{\pi}{4} - \sin x\sin\frac{\pi}{4} = 1 \Rightarrow \cos\left(x+\frac{\pi}{4}\right) = \cos 0 \Rightarrow x + \frac{\pi}{4} = 2n\pi \pm 0 \Rightarrow x = 2n\pi - \frac{\pi}{4},\ n \in Z.$$

EXAMPLE 4 *Solve:* $\cot x + \operatorname{cosec} x = \sqrt{3}$.

SOLUTION We observe that the LHS of the given equation is meaningful for all $x \neq n\pi$, $n \in Z$.

Now, $\cot x + \operatorname{cosec} x = \sqrt{3}$

$$\Rightarrow \frac{\cos x}{\sin x} + \frac{1}{\sin x} = \sqrt{3} \Rightarrow \cos x + 1 = \sqrt{3}\sin x \Rightarrow \sqrt{3}\sin x - \cos x = 1 \quad ...(i)$$

This is of the form $a\sin x + b\cos x = c$, where $a = \sqrt{3}, b = -1$ and $c = 1$.

Let $\sqrt{3} = r\sin\alpha$ and $1 = r\cos\alpha$. Then,

$$r = \sqrt{a^2+b^2} = \sqrt{3+1} = 2 \text{ and, } \tan\alpha = \frac{\sqrt{3}}{1} = \sqrt{3} \Rightarrow r = 2 \text{ and } \alpha = \pi/3$$

Substituting $a = \sqrt{3} = r\sin\alpha$ and $b = 1 = r\cos\alpha$ in (i), we get

$$r\sin\alpha\sin x - r\cos\alpha\cos x = 1$$

$$\Rightarrow -r\cos(x+\alpha) = 1 \Rightarrow -2\cos\left(x+\frac{\pi}{3}\right) = 1 \Rightarrow \cos\left(x+\frac{\pi}{3}\right) = -\frac{1}{2} \Rightarrow \cos\left(x+\frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$\Rightarrow x + \frac{\pi}{3} = 2n\pi \pm \frac{2\pi}{3},\ n \in Z$$

$$\Rightarrow x = 2n\pi \pm \frac{2\pi}{3} - \frac{\pi}{3} \Rightarrow x = 2n\pi + \frac{\pi}{3} \text{ or, } x = 2n\pi - \pi = (2n-1)\pi,\ n \in Z$$

But, x cannot be equal to $(2n-1)\pi$ as it makes $\sin x = 0$. Hence, $x = 2n\pi + \frac{\pi}{3}$, $n \in Z$

ALITER The given equation reduces to the equation $\sqrt{3}\sin x-\cos x=1$.

Dividing throughout by $\sqrt{(\sqrt{3})^2+(-1)^2}=2$, we obtain

$$\frac{\sqrt{3}}{2}\sin x-\frac{1}{2}\cos x=\frac{1}{2} \Rightarrow \sin\frac{\pi}{3}\sin x-\cos\frac{\pi}{3}\cos x=\frac{1}{2} \Rightarrow \cos x\cos\frac{\pi}{3}-\sin x\sin\frac{\pi}{3}=-\frac{1}{2}$$

$$\Rightarrow \cos\left(x+\frac{\pi}{3}\right)=\cos\frac{2\pi}{3} \Rightarrow x+\frac{\pi}{3}=2n\pi\pm\frac{2\pi}{3} \Rightarrow x=2n\pi+\frac{\pi}{3} \text{ or } x=2n\pi-\pi, n\in Z$$

$$\Rightarrow x=2n\pi+\frac{\pi}{3}, n\in Z \qquad [\because x\neq(2n-1)\pi, n\in Z]$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 5 *Solve:* $(\sqrt{3}-1)\cos x+(\sqrt{3}+1)\sin x=2$. [NCERT EXEMPLAR]

SOLUTION We have, $(\sqrt{3}-1)\cos x+(\sqrt{3}+1)\sin x=2$. ...(i)

Let $\sqrt{3}-1=r\sin\alpha$ and $\sqrt{3}+1=r\cos\alpha$. Then, $r=\sqrt{(\sqrt{3}-1)^2+(\sqrt{3}+1)^2}=2\sqrt{2}$

and $$\tan\alpha=\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}}=\tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\tan\frac{\pi}{12}$$

$$\Rightarrow \quad r=2\sqrt{2} \text{ and } \alpha=\frac{\pi}{12}$$

Putting $\sqrt{3}-1=r\sin\alpha$ and $\sqrt{3}+1=r\cos\alpha$ in (i), we obtain

$$r\sin\alpha\cos x+r\cos\alpha\sin x=2$$

$$r\sin(x+\alpha)=2 \Rightarrow 2\sqrt{2}\sin\left(x+\frac{\pi}{12}\right)=2 \Rightarrow \sin\left(x+\frac{\pi}{12}\right)=\frac{1}{\sqrt{2}} \Rightarrow \sin\left(x+\frac{\pi}{12}\right)=\sin\frac{\pi}{4}$$

$$\Rightarrow \quad x+\frac{\pi}{12}=n\pi+(-1)^n\frac{\pi}{4} \Rightarrow x=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{12}, n\in Z$$

$$\Rightarrow \quad x=\begin{cases} n\pi+\frac{\pi}{4}-\frac{\pi}{12}, & \text{if } n \text{ is even} \\ n\pi-\frac{\pi}{4}-\frac{\pi}{12}, & \text{if } n \text{ is odd}\end{cases} \Rightarrow x=\begin{cases} n\pi+\frac{\pi}{6}, & \text{if } n \text{ is even} \\ n\pi-\frac{\pi}{3}, & \text{if } n \text{ is odd}\end{cases}$$

ALITER We have,

$$(\sqrt{3}-1)\cos x+(\sqrt{3}+1)\sin x=2$$

Dividing throuhout by $\sqrt{(\sqrt{3}-1)^2+(\sqrt{3}+1)^2}=2\sqrt{2}$, we obtain

$$\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\cos x+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\sin x=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \left(\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}-\frac{1}{2}\times\frac{1}{\sqrt{2}}\right)\cos x+\left(\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}\right)\sin x=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos x\left(\cos\frac{\pi}{6}\cos\frac{\pi}{4}-\sin\frac{\pi}{6}\sin\frac{\pi}{4}\right)+\left(\sin\frac{\pi}{6}\cos\frac{\pi}{4}+\cos\frac{\pi}{6}\sin\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos x\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)+\sin x\sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos x\cos\frac{5\pi}{12}+\sin x\sin\frac{5\pi}{12}=\cos\frac{\pi}{4}$$

$$\Rightarrow \quad \cos\left(x-\frac{5\pi}{12}\right)=\cos\frac{\pi}{4} \Rightarrow x-\frac{5\pi}{12}=2n\pi\pm\frac{\pi}{4}, n\in Z \Rightarrow x=2n\pi+\frac{2\pi}{3} \text{ or, } x=2n\pi+\frac{\pi}{6}, n\in Z$$

EXERCISE 10.1

BASIC

1. Find the general solutions of the following equations:
   (i) $\sin x = \frac{1}{2}$ (ii) $\cos x = -\frac{\sqrt{3}}{2}$ (iii) $\text{cosec } x = -\sqrt{2}$
   (iv) $\sec x = \sqrt{2}$ (v) $\tan x = -\frac{1}{\sqrt{3}}$ (vi) $\sqrt{3} \sec x = 2$
2. Find the general solutions of the following equations:
   (i) $\sin 2x = \frac{\sqrt{3}}{2}$ (ii) $\cos 3x = \frac{1}{2}$ (iii) $\sin 9x = \sin x$
   (iv) $\sin 2x = \cos 3x$ (v) $\tan x + \cot 2x = 0$ (vi) $\tan 3x = \cot x$
   (vii) $\tan 2x \tan x = 1$ (viii) $\tan mx + \cot nx = 0$ (ix) $\tan px = \cot qx$
   (x) $\sin 2x + \cos x = 0$ (xi) $\sin x = \tan x$ (xii) $\sin 3x + \cos 2x = 0$
3. Solve the following equations:
   (i) $\sin^2 x - \cos x = \frac{1}{4}$ (ii) $2\cos^2 x - 5\cos x + 2 = 0$
   (iii) $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$ (iv) $4\sin^2 x - 8\cos x + 1 = 0$
   (v) $\tan^2 x + (1-\sqrt{3})\tan x - \sqrt{3} = 0$ (vi) $3\cos^2 x - 2\sqrt{3}\sin x\cos x - 3\sin^2 x = 0$
   (vii) $\cos 4x = \cos 2x$

BASED ON LOTS

4. Solve the following equations:
   (i) $\cos x + \cos 2x + \cos 3x = 0$ (ii) $\cos x + \cos 3x - \cos 2x = 0$ [NCERT]
   (iii) $\sin x + \sin 5x = \sin 3x$ (iv) $\cos x \cos 2x \cos 3x = \frac{1}{4}$
   (v) $\cos x + \sin x = \cos 2x + \sin 2x$ (vi) $\sin x + \sin 2x + \sin 3x = 0$
   (vii) $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$ (viii) $\sin 3x - \sin x = 4\cos^2 x - 2$
   (ix) $\sin 2x - \sin 4x + \sin 6x = 0$ [NCERT]
5. Solve the following equations:
   (i) $\tan x + \tan 2x + \tan 3x = 0$ (ii) $\tan x + \tan 2x = \tan 3x$
   (iii) $\tan 3x + \tan x = 2\tan 2x$
6. Solve the following equations:
   (i) $\sin x + \cos x = \sqrt{2}$ (ii) $\sqrt{3}\cos x + \sin x = 1$
   (iii) $\sin x + \cos x = 1$ (iv) $\text{cosec } x = 1 + \cot x$
7. Solve the following equations:
   (i) $4\sin x\cos x + 2\sin x + 2\cos x + 1 = 0$ (ii) $\cos x + \sin x = \cos 2x + \sin 2x$
   (iii) $\sin x \tan x - 1 = \tan x - \sin x$ (iv) $3\tan x + \cot x = 5\text{ cosec } x$

BASED ON HOTS

Solve the following equations: (8 – 10)

8. $3 - 2\cos x - 4\sin x - \cos 2x + \sin 2x = 0$
9. $3\sin^2 x - 5\sin x\cos x + 8\cos^2 x = 2$

10. $2^{\sin^2 x} + 2^{\cos^2 x} = 2\sqrt{2}$

11. Find the most general value of x satisfying the equations $\tan x = -1$ and $\cos x = \frac{1}{\sqrt{2}}$.

[NCERT EXEMPLAR]

ANSWERS

1. (i) $x = n\pi + (-1)^n \frac{\pi}{6},\ n \in Z$ (ii) $x = 2n\pi \pm \frac{7\pi}{6},\ n \in Z$

(iii) $x = n\pi + (-1)^{n+1} \frac{\pi}{4},\ n \in Z$ (iv) $x = 2n\pi \pm \frac{\pi}{4},\ n \in Z$

(v) $x = n\pi - \frac{\pi}{6},\ n \in Z$ (vi) $x = 2n\pi \pm \frac{\pi}{6},\ n \in Z$

2. (i) $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{6},\ n \in Z$ (ii) $x = \frac{2n\pi}{3} \pm \frac{\pi}{9},\ n \in Z$

(iii) $x = \frac{r\pi}{4}$ or $x = (2r+1)\frac{\pi}{10}$, where $r \in Z$

(iv) $x = (4n+1)\frac{\pi}{10}$ or $x = (4n-1)\frac{\pi}{2}$, where $n \in Z$

(v) $x = n\pi - \frac{\pi}{2},\ n \in Z$ (vi) $x = \frac{n\pi}{4} + \frac{\pi}{8},\ n \in Z$

(vii) $x = \frac{n\pi}{3} + \frac{\pi}{6},\ n \in Z$ (viii) $x = \frac{(2r+1)\pi}{m-n},\ r \in Z$

(ix) $x = \left(\frac{2n+1}{p+q}\right)\frac{\pi}{2},\ n \in Z$

(x) $x = (4n-1)\frac{\pi}{2}$ or $x = (4m-1)\frac{\pi}{6}$, where $m,\ n \in Z$

(xi) $x = m\pi$ or $x = 2n\pi$, where $m,\ n \in Z$

(xii) $x = (4n-1)\frac{\pi}{10}$ or $x = (4m-1)\frac{\pi}{2},\ m,\ n \in Z$

3. (i) $x = 2n\pi \pm \frac{\pi}{3},\ n \in Z$ (ii) $x = 2n\pi \pm \frac{\pi}{3},\ n \in Z$

(iii) $x = 2n\pi \pm \frac{5\pi}{6},\ n \in Z$ (iv) $x = 2n\pi \pm \frac{\pi}{3},\ n \in Z$

(v) $x = n\pi - \frac{\pi}{4}$ or $x = m\pi + \frac{\pi}{3}$, where $m,\ n \in Z$

(vi) $x = n\pi - \frac{\pi}{3}$ or $x = m\pi + \frac{\pi}{6}$, where $m,\ n \in Z$

(vii) $x = n\pi,\ x = \frac{n\pi}{3},\ n \in Z$

4. (i) $x = (2n+1)\frac{\pi}{4}$ or $x = 2m\pi \pm \frac{2\pi}{3},\ m, n \in Z$

(ii) $x = (2n+1)\frac{\pi}{4}$ or $x = 2m\pi \pm \frac{\pi}{3},\ m, n \in Z$

(iii) $x = \frac{n\pi}{3}$ or $x = m\pi \pm \frac{\pi}{6},\ m, n \in Z$

(iv) $x = 2n+1\frac{\pi}{8}$ or $x = m\pi \pm \frac{\pi}{3},\ m, n \in Z$

(v) $x=\frac{(2n\pi)}{3}+\frac{\pi}{6}$ or $x=2m\pi,\ m,n\in Z$

(vi) $x=\frac{n\pi}{2}$ or $x=2n\pi\pm\frac{2\pi}{3},\ m,n\in Z$

(vii) $x=n\pi+\frac{\pi}{2},\ x=(2m+1)\,\pi,\ x=\frac{2r\pi}{5}, m,n\,r\in Z$

(viii) $x=n\pi+(-1)^n\frac{\pi}{2}$ or $x=(2m+1)\frac{\pi}{4},\ m,n\in Z$

(ix) $x=\frac{n\pi}{4},\ x=n\pi\pm\frac{\pi}{6},\ n\in Z$

5. (i) $x=\frac{m\pi}{3}$ or $x=n\pi\pm\alpha$, where $\alpha = \tan^{-1}\frac{1}{\sqrt{2}}$ and $m,n\in Z$

(ii) $x=m\pi$ or $x=\frac{n\pi}{3}$, where $m,n\in Z$ (iii) $x=n\pi$, where $n\in Z$

6. (i) $x=(8n+)\frac{\pi}{4},\ n\in Z$

(ii) $x=(4n+1)\frac{\pi}{2}$ or $x=(12m-1)\frac{\pi}{6}, m,n\in Z$

(iii) $x=2n\pi$ or $x=2m\pi+\frac{\pi}{2},\ m,n\in Z$ (iv) $x=2m\pi+\frac{\pi}{2},\ m,n\in Z$

7. (i) $x=2n\pi+\frac{2\pi}{3}, x=n\pi+(-1)^n\frac{7\pi}{6}$ (ii) $x=2n\pi$ or, $x=\frac{21\pi}{3}+\frac{\pi}{6}, n\in Z$

(iii) $x=n\pi+x=\frac{3\pi}{4}, n\in Z$ (iv) $x=2n\pi\pm\frac{\pi}{3}, n\in Z$

8. $x=n\pi+(-1)^n\frac{\pi}{2}, x=2n\pi+\frac{\pi}{2}, n\in Z$

9. $x=n\pi+\alpha, x=n\pi+\beta$, where $\tan\alpha=2$, $\tan\beta=2, n\in Z$

10. $x=n\pi\pm\frac{\pi}{4}, n\in Z$

HINTS TO SELECTED PROBLEMS

11. A value of x satisfying $\tan x=-1$ and $\cos x=\frac{1}{\sqrt{2}}$ is $x=\frac{7\pi}{4}$. Hence, the most general value of x is $x=2n\pi+\frac{7\pi}{4}, n\in Z$.

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. In a ΔABC, if $\frac{\sin(A-B)}{\sin(A+B)}=\frac{a^2-b^2}{k}$, then $k=$ ....

2. In a ΔABC, if $c^2+a^2-b^2=ac$, then the measure of angle B is ....

3. In a ΔABC, if $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$ and $a=2$, then area of ΔABC is equal to ....

4. In a triangle ABC, if $a=2, b=4$ and $A+B=\frac{2\pi}{3}$, then area of ΔABC is ....

5. The angles A, B, C of a ΔABC are in AP and the sides a, b, c are in G.P. If $a^2 + c^2 = \lambda b^2$, then $\lambda =$ ....

6. In a ΔABC, if $\angle C = 60°$, $a = 47$ cm and $b = 94$ cm, then $c^2 =$ ....

7. In a ΔABC, if $\angle C = \frac{\pi}{2}$, $\angle A = \frac{\pi}{6}$, $c = 20$, then $a =$ ....

8. In a ΔABC, if $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = k\,(a^2 + b^2 + c^2)$, then $k =$ ....

9. In a ΔABC, if $c^2 \sin A \sin B = ab$, then $A + B =$ ....

10. In a ΔABC, if $a = 8, b = 9$ and $3 \cos C = 2$, then $C =$ ....

11. In a ΔABC, if $b = \sqrt{3}$, $c = 1$ and $B - C = \frac{\pi}{2}$, then $A =$ ....

12. If angles of a triangle are in A.P. and $b : c = \sqrt{3} : \sqrt{2}$, then $C =$ ....

13. If the sides of a ΔABC are $a, b, \sqrt{a^2 + ab + b^2}$, then the measure of the largest angle is ....

14. In a ΔABC, if $a^4 + b^4 + c^4 = 2a^2b^2 + 2b^2 c^2$, then $B =$ ....

15. In a ΔABC, if $a = 4, b = 3, A = \frac{\pi}{3.}$ Then side C is given by ....

ANSWERS

1. c^2 2. $\frac{\pi}{3}$ 3. $\sqrt{3}$ sq. units 4. $\frac{\sqrt{3}}{2}$ sq. units 5. 2 6. 6627
7. 10 8. $\frac{1}{2abc}$ 9. $\frac{\pi}{2}$ 10. 7 11. $\frac{\pi}{6}$ 12. $\frac{\pi}{4}$ 13. $\frac{2\pi}{3}$ 14. $\frac{\pi}{4}$ or $\frac{3\pi}{4}$
15. $c^2 - 3c - 7 = 0$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Find the area of the triangle ΔABC in which $a = 1, b = 2$ and $\angle c = 60°$.
2. In a ΔABC, if $b = \sqrt{3}$, $c = 1$ and $\angle A = 30°$, find a.
3. In a ΔABC, if $\cos A = \frac{\sin B}{2 \sin C}$, then show that $c = a$.
4. In a ΔABC, if $b = 20, c = 21$ and $\sin A = \frac{3}{5}$, find a.
5. In a ΔABC, if $\sin A$ and $\sin B$ are the roots of the equation $c^2x^2 - c\,(a + b)\,x + ab = 0$, then find $\angle C$.
6. In ΔABC, if $a = 8, b = 10, c = 12$ and $C = \lambda A$, find the value of λ.
7. If the sides of a triangle are proportional to $2, \sqrt{6}$ and $\sqrt{3} - 1$, find the measure of its greatest angle.
8. If in a ΔABC, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then find the measures of angles A, B, C.

9. In any triangle ABC, find the value of $a \sin(B-C)+b\sin(C-A)+c\sin(A-B)$.
10. In any ΔABC, find the value of $\Sigma\, a(\sin B-\sin C)$

ANSWERS

1. $\sqrt{3}$ sq. units 2. 1 4. 13 5. 90° 6. 2 7. 120°
8. $A=B=C=60°$ 9. 0 10. 0

MULTIPLE CHOICES QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The smallest value of x satisfying the equation $\sqrt{3}(\cot x+\tan x)=4$ is
   (a) $2\pi/3$ (b) $\pi/3$ (c) $\pi/6$ (d) $\pi/12$
2. If $\cos x+\sqrt{3}\sin x=2$, then $x=$
   (a) $\pi/3$ (b) $2\pi/3$ (c) $4\pi/3$ (d) $5\pi/3$
3. If $\tan px-\tan qx=0$, then the values of θ form a series in
   (a) AP (b) GP (c) HP (d) none of these
4. If a is any real number, the number of roots of $\cot x-\tan x=a$ in the first quadrant is (are).
   (a) 2 (b) 0 (c) 1 (d) none of these
5. The general solution of the equation $7\cos^2 x+3\sin^2 x=4$ is
   (a) $x=2n\pi\pm\frac{\pi}{6}, n\in Z$ (b) $x=2n\pi\pm\frac{2\pi}{3}, n\in Z$
   (c) $x=n\pi\pm\frac{\pi}{3}, n\in Z$ (d) none of these
6. A solution of the equation $\cos^2 x+\sin x+1=0$, lies in the interval
   (a) $(-\pi/4, \pi/4)$ (b) $(\pi/4, 3\pi/4)$ (c) $(3\pi/4, 5\pi/4)$ (d) $(5\pi/4, 7\pi/4)$
7. The number of solution in $[0, \pi/2]$ of the equation $\cos 3x\tan 5x = \sin 7x$ is
   (a) 5 (b) 7 (c) 6 (d) none of these
8. The general value of x satisfying the equation $\sqrt{3}\sin x+\cos x = \sqrt{3}$ is given by
   (a) $x = n\pi+(-1)^n\frac{\pi}{4}+\frac{\pi}{3}, n\in Z$ (b) $x = n\pi+(-1)^n\frac{\pi}{3}-\frac{\pi}{6}, n\in Z$
   (c) $x = n\pi\pm\frac{\pi}{6}, n\in Z$ (d) $x = n\pi\pm\frac{\pi}{3}, n\in Z$
9. The smallest positive angle which satisfies the equation $2\sin^2 x+\sqrt{3}\cos x+1=0$ is
   (a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$
10. If $4\sin^2 x=1$, then the values of x are
   (a) $2n\pi\pm\frac{\pi}{3}, n\in Z$ (b) $n\pi\pm\frac{\pi}{3}, n\in Z$ (c) $n\pi\pm\frac{\pi}{6}, n\in Z$ (d) $2n\pi\pm\frac{\pi}{6}, n\in Z$
11. If $\cot x-\tan x = \sec x$, then, x is equal to
   (a) $2n\pi+\frac{3\pi}{2}, n\in Z$ (b) $n\pi+(-1)^n\frac{\pi}{6}, n\in Z$
   (c) $n\pi+\frac{\pi}{2}, n\in Z$ (d) none of these

12. A value of x satisfying $\cos x + \sqrt{3} \sin x = 2$ is
(a) $\frac{5\pi}{3}$ (b) $\frac{4\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{3}$

13. In $(0, \pi)$, the number of solutions of the equation $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$ is
(a) 7 (b) 5 (c) 4 (d) 2

14. The number of values of x in $[0, 2\pi]$ that satisfy the equation $\sin^2 x - \cos x = \frac{1}{4}$
(a) 1 (b) 2 (c) 3 (d) 4

15. If $e^{\sin x} - e^{-\sin x} - 4 = 0$, then $x =$
(a) 0 (b) $\sin^{-1}\{\log_e(2-\sqrt{5})\}$
(c) 1 (d) none of these

16. The equation $3 \cos x + 4 \sin x = 6$ has .... solution
(a) finite (b) infinite (c) one (d) no

17. If $\sqrt{3} \cos x + \sin x = \sqrt{2}$, then general value of θ is
(a) $n\pi + (-1)^n \frac{\pi}{4}, n \in Z$ (b) $(-1)^n \frac{\pi}{4} - \frac{\pi}{3}, n \in Z$
(c) $n\pi + \frac{\pi}{4} - \frac{\pi}{3}, n \in Z$ (d) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3}, n \in Z$

18. General solution of $\tan 5x = \cot 2x$ is
(a) $\frac{n\pi}{7} + \frac{\pi}{2}, n \in Z$ (b) $x = \frac{n\pi}{7} + \frac{\pi}{3}, n \in Z$
(c) $x = \frac{n\pi}{7} + \frac{\pi}{14}, n \in Z$ (d) $x = \frac{n\pi}{7} - \frac{\pi}{14}, n \in Z$

19. The solution of the equation $\cos^2 x + \sin x + 1 = 0$ lies in the interval
(a) $(-\pi/4, \pi/4)$ (b) $(\pi/4, 3\pi/4)$ (c) $(3\pi/4, 5\pi/4)$ (d) $(5\pi/4, 7\pi/4)$

20. If $\cos x = -\frac{1}{2}$ and $0 < x < 2\pi$, then the solutions are
(a) $x = \frac{\pi}{3}, \frac{4\pi}{3}$ (b) $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ (c) $x = \frac{2\pi}{3}, \frac{7\pi}{6}$ (d) $\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$

21. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3 \sin^2 x - 7 \sin x + 2 = 0$ is
(a) 0 (b) 5 (c) 6 (d) 10

22. Number of solutions of the equation $\tan x + \sec x = 2 \cos x$ lying in the interval $[0, 2\pi]$ is
(a) 0 (b) 1 (c) 2 (d) 3

[NCERT EXEMPLAR]

ANSWERS

1. (c)	2. (a)	3. (a)	4. (c)	5. (a)	6. (d)	7. (c)	8. (b)	9. (a)
10. (c)	11. (b)	12. (d)	13. (d)	14. (b)	15. (d)	16. (d)	17. (d)	18. (c)
19. (d)	20. (b)	21. (c)	22. (c)					

SUMMARY

1. An equation containing trigonometric functions of unknown angles is known as a trigonometric equation.
2. A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.
3. Following are the general solutions of trigonometric equations in standar.' forms:

	Trigonometric equation	*General solution*
(i)	$\sin\theta = 0$	$\theta = n\pi, n \in Z$
(ii)	$\cos\theta = 0$	$\theta = (2n+1)\frac{\pi}{2}, n \in Z$
(iii)	$\tan\theta = 0$	$\theta = n\pi, n \in Z$
(iv)	$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n \alpha, n \in Z$
(v)	$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha, n \in Z$
(vi)	$\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha, n \in Z$
(vii)	$\left.\begin{array}{l}\sin^2\theta = \sin^2\alpha \\ \cos^2\theta = \cos^2\alpha \\ \tan^2\theta = \tan^2\alpha\end{array}\right\}$	$\theta = n\pi \pm \alpha, n \in Z$

4. The equation $a\cos\theta + b\sin\theta = c$ is solvable for $|c| \leq \sqrt{a^2+b^2}$.

CHAPTER 11

MATHEMATICAL INDUCTION

11.1 STATEMENTS

A sentence or description which can be judged to be true or false is called a statement.

Following are some examples of statements:

EXAMPLE 1 *2 divides 6.*

EXAMPLE 2 *Jaipur is the capital of Rajasthan.*

EXAMPLE 3 *There are 5 days in a week.*

EXAMPLE 4 $(x+1)$ *is a factor of* $x^2 - 3x + 2$.

EXAMPLE 5 $A \cup B = B \cup A$.

Clearly, statements in Examples 1, 2 and 5 are true statements whereas statements in Examples 3 and 4 are false.

MATHEMATICAL STATEMENTS *Statements involving mathematical relations are known as the mathematical statements.*

Clearly, statements in examples 1, 4 and 5 are mathematical statements. In this chapter, we shall be mainly discussing mathematical statements concerning natural numbers. We shall be using notations $P(n)$ or $P_1(n)$ or $P_2(n)$ etc. to denote such statements.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Let* $P(n)$ *be the statement* "$10n + 3$ *is prime*". *Then,*
$P(2)$ is the statement "$10 \times 2 + 3$ is prime" i.e. "23 is prime".
Clearly, $P(2)$ is true.
$P(3)$ is the statement "$10 \times 3 + 3$ is prime" i.e. "33 is prime".
Clearly $P(3)$ is not true.

EXAMPLE 2 *If* $P(n)$ *is the statement* "$n^3 + n$ *is divisible by 3*", *is the statement* $P(3)$ *true ? Is the statement* $P(4)$ *true ?*

SOLUTION $P(3)$ is the statement "$3^3 + 3 = 30$ is divisible by 3".

Clearly, it is true.
$P(4)$ is the statement "$4^3 + 4 = 68$ is divisible by 3".

Clearly, it is not true.

EXAMPLE 3 *If* $P(n)$ *is the statement* "$n(n+1)(n+2)$ *is divisible by 12*", *prove that the statements* $P(3)$ *and* $P(4)$ *are true, but that* $P(5)$ *is not true.*

SOLUTION $P(3)$ is the statement "$3(3+1)(3+2) = 60$ is divisible by 12".
It is true.
$P(4)$ is the statement "$4(4+1)(4+2) = 120$ is divisible by 12".

It is also true.
P (5) is the statement "5 (5 + 1)(5 + 2) = 210 is divisible by 12".
Clearly it is not true.

EXAMPLE 4 *Let $P(n)$ be the statement "7 divides $(2^{3n} - 1)$". What is $P(n+1)$?*

SOLUTION $P(n+1)$ is the statement "7 divides $(2^{3(n+1)} - 1)$".

Clearly, $P(n+1)$ is obtained by replacing n by $(n+1)$ in $P(n)$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *If $P(n)$ is the statement "$n^2 > 100$", prove that whenever $P(r)$ is true, $P(r+1)$ is also true.*

SOLUTION The statement $P(n)$ is "$n^2 > 100$". Let $P(r)$ be true. Then $r^2 > 100$.

We wish to prove that the statement $P(r+1)$ is true i.e. "$(r+1)^2 > 100$".

Now,

$P(r)$ is true

$\Rightarrow \quad r^2 > 100$

$\Rightarrow \quad r^2 + 2r + 1 > 100 + 2r + 1$ [Adding $(2r+1)$ on both sides]

$\Rightarrow \quad (r+1)^2 > 100 + 2r + 1$

$\Rightarrow \quad (r+1)^2 > 100 \Rightarrow P(r+1)$ is true [$\because$ $100 + 2r + 1 > 100$ for every natural number r]

Thus, whenever $P(r)$ is true, $P(r+1)$ is also true.

EXAMPLE 6 *Let $P(n)$ be the statement "$3^n > n$". If $P(n)$ is true, prove that $P(n+1)$ is true.*

SOLUTION We are given that $P(n)$ is true i.e. $3^n > n$, and we wish to prove that $P(n+1)$ is true i.e. $3^{(n+1)} > (n+1)$.

Now,

$P(n)$ is true

$\Rightarrow \quad 3^n > n$

$\Rightarrow \quad 3.3^n > 3n$ [Multiplying both sides by 3]

$\Rightarrow \quad 3^{n+1} > n + 2n$

$\Rightarrow \quad 3^{n+1} > n + 1$ [$\because$ $2n > 1$ for every $n \in N \Rightarrow 2n + n > n + 1$ for every $n \in N$]

$\Rightarrow \quad P(n+1)$ is true

EXAMPLE 7 *If $P(n)$ is the statement "$2^{3n} - 1$ is an integral multiple of 7", and if $P(r)$ is true, prove that $P(r+1)$ is true.*

SOLUTION Let $P(r)$ be true. Then, $2^{3r} - 1$ is an integral multiple of 7.

We wish to prove that $P(r+1)$ is true i.e. $2^{3(r+1)} - 1$ is an integral multiple of 7.

Now,

$P(r)$ is true

$\Rightarrow \quad 2^{3r} - 1$ is an integral multiple of 7

$\Rightarrow \quad 2^{3r} - 1 = 7\lambda$ for some $\lambda \in N$.

$\Rightarrow \quad 2^{3r} = 7\lambda + 1$ ...(i)

Now, $2^{3(r+1)} - 1 = 2^{3r} \times 2^3 - 1 = (7\lambda + 1) \times 8 - 1$ [Using (i)]

$\Rightarrow \quad 2^{3(r+1)}-1 = 56\lambda+8-1 = 56\lambda+7 = 7(8\lambda+1)$

$\Rightarrow \quad 2^{3(r+1)}-1 = 7\mu$, where $\mu = 8\lambda+1 \in N$

$\Rightarrow \quad 2^{3(r+1)}-1$ is an integral multiple of 7

$\Rightarrow \quad P(r+1)$ is true

EXERCISE 11.1

BASIC

1. If $P(n)$ is the statement "$n(n+1)$ is even", then what is $P(3)$?
2. If $P(n)$ is the statement "n^3+n is divisible by 3", prove that $P(3)$ is true but $P(4)$ is not true.
3. If $P(n)$ is the statement "n^2-n+41 is prime", prove that $P(1)$, $P(2)$ and $P(3)$ are true. Prove also that $P(41)$ is not true.

BASED ON LOTS

4. If $P(n)$ is the statement "$2^n \geq 3n$", and if $P(r)$ is true, prove that $P(r+1)$ is true.
5. If $P(n)$ is the statement "n^2+n is even", and if $P(r)$ is true, then $P(r+1)$ is true.
6. Given an example of a statement $P(n)$ such that it is true for all $n \in N$.
7. Give an example of a statement $P(n)$ which is true for all $n \geq 4$ but $P(1)$, $P(2)$ and $P(3)$ are not true. Justify your answer.

ANSWERS

1. $P(3): 3(3+1)$ is even
6. $P(n): 1+2+\ldots+n = \dfrac{n(n+1)}{2}$
7. $P(n): 2n < n!$

HINTS TO SELECTED PROBLEMS

4. Let $P(r)$ be true. Then,

$2^r \geq 3r$

$\Rightarrow 2.2^r \geq 6r \Rightarrow 2^{r+1} \geq 3r+3r \Rightarrow 2^{r+1} \geq 3r+3 \qquad [\because 3r \geq 3 \Rightarrow 3r+3r \geq 3r+3]$

$\Rightarrow 2^{r+1} \geq 3(r+1) \Rightarrow P(r+1)$ is true

6. See the statement in Q. No. 4

11.2 THE PRINCIPLES OF MATHEMATICAL INDUCTION

FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

Let $P(n)$ be a statement involving the natural number n such that

(I) $P(1)$ is true i.e. $P(n)$ is true for $n=1$.

and, (II) $P(m+1)$ is true, whenever $P(m)$ is true.

i.e. $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Then, $P(n)$ is true for all natural numbers n.

SECOND PRINCIPLE OF MATHEMATICAL INDUCTION

Let $P(n)$ be a statement involving the natural number n such that

(I) $P(1)$ is true i.e. $P(n)$ is true for $n=1$.

and, (II) $P(m+1)$ is true, whenever $P(n)$ is true for all n, where $1 \leq n \leq m$.

Then, $P(n)$ is true for all natural numbers.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I PROBLEMS BASED UPON FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

Recall that the first principle of mathematical induction consists of two parts. First we must show that the given statement $P(n)$ is true for $n = 1$. The second part has two steps. The first step is to assume that the statement $P(n)$ is true for some $m \in N$. The second step is to use this assumption to prove that the statement $P(n)$ is true for $n = m + 1$.

In order to prove that a statement is true for all natural numbers using first principle of mathematical induction, we may use the following algorithm:

ALGORITHM

Step I *Obtain $P(n)$ and understand its meaning.*

Step II *Prove that the statement $P(1)$ is true i.e. $P(n)$ is true for $n = 1$.*

Step III *Assume that the statement $P(n)$ is true for $n = m$ (say) i.e. $P(m)$ is true.*

Step IV *Using assumption in step III prove that $P(m+1)$ is true.*

Step V *Combining the results of step II and step IV, conclude by the first principle of mathematical induction that $P(n)$ is true for all $n \in N$.*

The following examples illustrate the above algorithm.

BASIC

EXAMPLE 1 *Prove by the principle of mathematical induction that for all $n \in N$, $n^2 + n$ is even natural number*

SOLUTION Let $P(n)$ be the statement "$n^2 + n$ is even".

Step I We have, $P(n): n^2 + n$ is even

$\because \quad 1^2 + 1 = 2$, which is even

$\therefore \quad P(1)$ is true

Step II Let $P(m)$ be true. Then,

$P(m)$ is ture $\Rightarrow m^2 + m$ is even $\Rightarrow m^2 + m = 2\lambda$ for some $\lambda \in N$ ...(i)

Now, we shall show that $P(m+1)$ is true. For this we have to show that $(m+1)^2 + (m+1)$ is an even natural number.

Now,

$$(m+1)^2 + (m+1) = (m^2 + 2m + 1) + (m+1) = (m^2 + m) + (2m + 2)$$
$$= m^2 + m + 2(m+1) = 2\lambda + 2(m+1) \quad \text{[Using (i)]}$$
$$= 2(\lambda + m + 1) = 2\mu, \text{ where } \mu = \lambda + m + 1 \in N$$

$\therefore \quad (m+1)^2 + (m+1)$ is an even natural number $\Rightarrow P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$ i.e. $n^2 + n$ is even for all $n \in N$.

EXAMPLE 2 *Prove by the principle of mathematical induction that : $n(n+1)(2n+1)$ is divisible by 6 for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement "$n(n+1)(2n+1)$ is divisible by 6".

i.e. $P(n): n(n+1)(2n+1)$ is divisible by 6

Step I We have, $P(1): 1(1+1)(2+1)$ is divisible by 6.

$\because \quad 1\,(1+1)\,(2+1) = 6$ which is divisible by 6 $\therefore$ $P(1)$ is true

Step II Let $P\,(m)$ be true. Then,

$m\,(m+1)\,(2m+1)$ is divisible by 6 $\Rightarrow m\,(m+1)\,(2m+1) = 6\lambda$ for some $\lambda \in N$ ...(i)

Now, we shall show that $P\,(m+1)$ is true. For this we have to show that

$(m+1)\,(m+1+1)\,\{2(m+1)+1\}$ is divisible by 6.

Now,

$$\begin{aligned}(m+1)\,(m+1+1)\,\{2\,(m+1)+1\} &= (m+1)(m+2)\,\{(2m+1)+2\}\\ &= (m+1)\,(m+2)\,(2m+1) + 2\,(m+1)\,(m+2)\\ &= m\,(m+1)\,(2m+1) + 2\,(m+1)\,(2m+1) + 2\,(m+1)\,(m+2)\\ &= m\,(m+1)\,(2m+1) + 2\,(m+1)\,(2m+1+m+2)\\ &= m\,(m+1)\,(2m+1) + 2\,(m+1)\,(3m+3)\\ &= m\,(m+1)\,(2m+1) + 6\,(m+1)^2 = 6\lambda + 6\,(m+1)^2 \quad \text{[Using (i)]}\\ &= 6\,\{\lambda + (m+1)^2\}, \text{ which is divisible by 6}\end{aligned}$$

$\Rightarrow \quad P\,(m+1)$ is true.

Thus, $P\,(m)$ is true $\Rightarrow P\,(m+1)$ is true

Hence, by the principle of mathematical induction, the given statement is true for all $n \in N$.

EXAMPLE 3 *Prove by the principle of mathematical induction that for all $n \in N$:*

$$1 + 4 + 7 + \ldots + (3n-2) = \frac{1}{2}\,n\,(3n-1)$$

SOLUTION Let $P\,(n)$ be the statement given by

$$P\,(n) : 1 + 4 + 7 + \ldots + (3n-2) = \frac{1}{2}\,n\,(3n-1)$$

Step I We have, $P(1) : 1 = \frac{1}{2} \times (1) \times (3 \times 1 - 1)$.

$\because \quad 1 = \frac{1}{2} \times (1) \times (3 \times 1 - 1)$. So, $P\,(1)$ is true

Step II Let $P\,(m)$ be true. Then,

$$1 + 4 + 7 + \ldots + (3m-2) = \frac{1}{2}\,m\,(3m-1) \quad \text{...(i)}$$

We wish to show that $P\,(m+1)$ is true. For this we have to show that

$$1 + 4 + 7 + \ldots + (3m-2) + \{3(m+1) - 2\} = \frac{1}{2}\,(m+1)\,\{3(m+1) - 1\}$$

Now, $1 + 4 + 7 + \ldots + (3m-2) + \{3\,(m+1) - 2\}$

$$\begin{aligned}&= \frac{1}{2}\,m\,(3m-1) + \{3\,(m+1) - 2\} \quad \text{[Using (i)]}\\ &= \frac{1}{2}\,m\,(3m-1) + (3m+1) = \frac{1}{2}\,\{3m^2 - m + 6m + 2\}\\ &= \frac{1}{2}\,(3m^2 + 5m + 2) = \frac{1}{2}\,(m+1)\,(3m+2) = \frac{1}{2}\,(m+1)\,\{3\,(m+1) - 1\}\end{aligned}$$

$\therefore \quad P\,(m+1)$ is true

Thus, $P\,(m)$ is true $\Rightarrow P\,(m+1)$ is true.

Hence, by the principle of mathematical induction, the given result is true for all $n \in N$.

EXAMPLE 4 *Prove by the principle of mathematical induction that for all $n \in N$:*

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}\,n\,(n+1)\,(2n+1)$$

SOLUTION Let $P\,(n)$ be the statement given by

$$P\,(n) : 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}\,n\,(n+1)\,(2n+1)$$

Step I We have, $P(1): 1^2 = \frac{1}{6}(1)(1+1)(2\times 1+1)$

$\because \quad 1^2 = 1 = \frac{1}{6}(1)(1+1)(2\times 1+1)$. So, $P(1)$ is true

Step II Let $P(m)$ be true. Then,

$$1^2+2^2+3^2+\ldots+m^2 = \frac{1}{6}m(m+1)(2m+1) \quad \ldots(i)$$

We wish to show that $P(m+1)$ is true. For this we have to show that

$$1^2+2^2+3^2+\ldots+m^2+(m+1)^2 = \frac{1}{6}(m+1)\{(m+1)+1\}\ \{2(m+1)+1)\}$$

Now, $1^2+2^2+3^2+\ldots+m^2+(m+1)^2$

$$= \{1^2+2^2+3^2+\ldots+m^2\}+(m+1)^2$$

$$= \frac{1}{6}m(m+1)(2m+1)+(m+1)^2 \quad \text{[Using (i)]}$$

$$= \frac{1}{6}(m+1)\{m(2m+1)+6(m+1)\} = \frac{1}{6}(m+1)\{2m^2+7m+6\}$$

$$= \frac{1}{6}(m+1)(m+2)(2m+3) = \frac{1}{6}(m+1)\{(m+1)+1\}\{2(m+1)+1\}$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction, the given result is true for all $n \in N$.

EXAMPLE 5 *Using the principle of mathematical induction prove that:*

$$1^3+2^3+3^3+\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 \text{ for all } n \in N$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1^3+2^3+3^3+\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Step I We have, $P(1): 1^3 = \left\{\frac{1(1+1)}{2}\right\}^2$. Clearly, $1^3 = 1 = \left\{\frac{1(1+1)}{2}\right\}^2$. Therefore, $P(1)$ is true.

Step II Let $P(m)$ be true. Then,

$$1^3+2^3+3^3+\ldots+m^3 = \left\{\frac{m(m+1)}{2}\right\}^2 \quad \ldots(i)$$

We shall now prove that $P(m+1)$ is true. For this we have to prove that

$$1^3+2^3+3^3+\ldots+m^3+(m+1)^3 = \left\{\frac{(m+1)\{(m+1)+1\}}{2}\right\}^2$$

Now,

$$1^3+2^3+3^3+\ldots+m^3+(m+1)^3$$

$$= \left\{1^3+2^3+\ldots+m^3\right\}+(m+1)^3 = \left\{\frac{m(m+1)}{2}\right\}^2+(m+1)^3 \quad \text{[Using (i)]}$$

$$= (m+1)^2\left\{\frac{m^2}{4}+(m+1)\right\}$$

$$= (m+1)^2\left\{\frac{m^2+4m+4}{4}\right\} = \frac{(m+1)^2(m+2)^2}{4} = \left\{\frac{(m+1)\{(m+1)+1\}}{2}\right\}^2$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, the given result is true for all $n \in N$.

EXAMPLE 6 *Using the principle of mathematical induction, prove that*

$$1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4} \text{ for all } n \in N.$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n) : 1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

Step I We have,

$$P(1) : 1.2.3 = \frac{1(1+1)(1+2)(1+3)}{4}$$

$$\because \quad 1.2.3 = 6 \text{ and } \frac{1(1+1)(1+2)(1+3)}{4} = \frac{2 \times 3 \times 4}{4} = 6$$

$$\therefore \quad 1.2.3 = \frac{1(1+1)(1+2)(1+3)}{4}. \text{ So, } P(1) \text{ is true.}$$

Step II Let $P(m)$ be true. Then,

$$1.2.3 + 2.3.4 + \ldots + m(m+1)(m+2) = \frac{m(m+1)(m+2)(m+3)}{4} \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true. For this we will prove that

$$1.2.3 + 2.3.4 + \ldots + m(m+1)(m+2) + (m+1)(m+2)(m+3) = \frac{(m+1)(m+2)(m+3)(m+4)}{4}$$

Now, $1.2.3 + 2.3.4 + \ldots + m(m+1)(m+2) + (m+1)(m+2)(m+3)$

$$= \frac{m(m+1)(m+2)(m+3)}{4} + (m+1)(m+2)(m+3) \quad \text{[Using (i)]}$$

$$= (m+1)(m+2)(m+3)\left(\frac{m}{4}+1\right) = \frac{(m+1)(m+2)(m+3)(m+4)}{4}$$

$\therefore$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 7 *Using the principle of mathematical induction prove that*

$$1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \frac{(2n-1)\,3^{n+1} + 3}{4} \text{ for all } n \in N$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n) : 1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \frac{(2n-1)\,3^{n+1} + 3}{4}$$

Step I $P(1) : 1.3 = \dfrac{(2 \times 1 - 1) \times 3^{1+1} + 3}{4}$

$$\because \quad 1.3 = 3 \text{ and } \frac{(2 \times 1 - 1) \times 3^{1+1} + 3}{4} = \frac{9+3}{4} = 3$$

$$\therefore \quad 1.3 = \frac{(2 \times 1 - 1) \times 3^{1+1} + 3}{4}. \text{ So, } P(1) \text{ is true.}$$

Step II Let $P(m)$ be true. Then,

$$1.3 + 2.3^2 + 3.3^3 + \ldots + m.3^m = \frac{(2m-1)\,3^{m+1} + 3}{4} \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true.

i.e. $\quad 1.3 + 2.3^2 + 3.3^3 + \ldots + m.3^m + (m+1).3^{m+1} = \dfrac{\{2(m+1)-1\}\, 3^{(m+1)+1} + 3}{4}$

Now,

$$1.3 + 2.3^2 + 3.3^3 + \ldots + m.3^m + (m+1).3^{m+1}$$

$$= \frac{(2m-1)\, 3^{m+1} + 3}{4} + (m+1)\, 3^{m+1}$$

$$= \frac{(2m-1)\, 3^{m+1} + 3 + (4m+4)\, 3^{m+1}}{4}$$

$$= \frac{(2m-1) \times 3^{m+1} + (4m+4) \times 3^{m+1} + 3}{4}$$

$$= \frac{(2m-1+4m+4)\, 3^{m+1} + 3}{4}$$

$$= \frac{(6m+3)\, 3^{m+1} + 3}{4} = \frac{(2m+1)\, 3^{m+2} + 3}{4} = \frac{\{2(m+1)-1\}\, 3^{(m+1)+1} + 3}{4}$$

$\therefore \quad P(m+1)$ is true.

Hence, by the principal of mathematical induction $P(n)$ is true for all $n \in N$ i.e., the given result is true for all $n \in N$.

EXAMPLE 8 *Prove by the principle of mathematical induction that for all $n \in N$:*

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Step I We have, $P(1): \dfrac{1}{1.2} = \dfrac{1}{1+1}$. We find that $\dfrac{1}{1.2} = \dfrac{1}{1+1} = \dfrac{1}{2}$. So, $P(1)$ is true

Step II Let $P(m)$ be true. Then, $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots + \dfrac{1}{m(m+1)} = \dfrac{m}{m+1}$ ...(i)

We shall now show that $P(m+1)$ is true. For this we have to show that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{m(m+1)} + \frac{1}{(m+1)(m+1+1)} = \frac{(m+1)}{(m+1)+1}$$

Now, $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots + \dfrac{1}{m(m+1)} + \dfrac{1}{(m+1)((m+1)+1)}$

$$= \left\{\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{m(m+1)}\right\} + \frac{1}{(m+1)((m+1)+1)}$$

$$= \frac{m}{m+1} + \frac{1}{(m+1)((m+1)+1)} = \frac{m}{m+1} + \frac{1}{(m+1)(m+2)} \qquad \text{[Using (i)]}$$

$$= \frac{1}{(m+1)}\left\{\frac{m}{1} + \frac{1}{m+2}\right\} = \frac{1}{(m+1)} \times \frac{(m^2+2m+1)}{(m+2)} = \frac{(m+1)^2}{(m+1)(m+2)}$$

$$= \frac{m+1}{m+2} = \frac{(m+1)}{(m+1)+1}$$

$\therefore \quad P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction, the given statement is true for all $n \in N$.

EXAMPLE 9 *Using the principle of mathematical induction prove that*

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}+\ldots+\frac{1}{1+2+3+\ldots+n}=\frac{2n}{n+1} \text{ for all } n \in N.$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+n}=\frac{2n}{n+1}$$

Step I We have, $P(1): 1=\frac{2\times 1}{1+1}$. Clearly, $\frac{2\times 1}{1+1}=\frac{2}{2}=1$. Therefore, $1=\frac{2\times 1}{1+1}$. So, $P(1)$ is true.

Step II Let $P(m)$ be true. Then,

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+m}=\frac{2m}{m+1} \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true. For this we will prove that

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+m}+\frac{1}{1+2+3+\ldots+(m+1)}=\frac{2(m+1)}{(m+1)+1}$$

Now, $1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3\ldots+m}+\frac{1}{1+2+3+\ldots+(m+1)}$

$$=\frac{2m}{m+1}+\frac{1}{1+2+3+\ldots+(m+1)} \quad \text{[Using (i)]}$$

$$=\frac{2m}{m+1}+\frac{1}{\frac{(m+1)(m+2)}{2}} \quad \left[\because 1+2+\ldots+m+(m+1)=\frac{(m+1)(m+2)}{2}\right]$$

$$=\frac{2m}{m+1}+\frac{2}{(m+1)(m+2)} \quad \text{[Using (i)]}$$

$$=\frac{2}{m+1}\left\{m+\frac{1}{(m+2)}\right\}=\frac{2}{m+1}\left\{\frac{m^2+2m+1}{(m+2)}\right\}=\frac{2}{m+1}\times\frac{(m+1)^2}{m+2}=\frac{2(m+1)}{(m+1)+1}$$

$\therefore$ $P(m+1)$ is true.

Hence, by the principle of of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 10 *Prove by induction that the sum $S_n = n^3+3n^2+5n+3$ is divisible by 3 for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by

$P(n): S_n = n^3+3n^2+5n+3$ is divisible by 3

Step I We have, $P(1): S_1 = 1^3+3(1)^2+5(1)+3$ is divisible by 3.

Since $1^3+3(1)^2+5(1)+3=12$, which is divisible by 3. Therefore, $P(1)$ is true

Step II Let $P(m)$ be true. Then,

$S_m = m^3+3m^2+5m+3$ is divisible by 3

$\Rightarrow$ $S_m = m^3+3m^2+5m+3 = 3\lambda$ for some $\lambda \in N$ ...(i)

We now wish to show that $P(m+1)$ is true. For this we have to show that $(m+1)^3+3(m+1)^2+5(m+1)+3$ is divisible by 3.

Now, $(m+1)^3+3(m+1)^2+5(m+1)+3=(m^3+3m^2+5m+3)+3m^2+9m+9$

$$=3\lambda+3(m^2+3m+3)$$

$$= 3(\lambda + m^2 + 3m + 3) \quad \text{[Using (i)]}$$

$$= 3\mu, \quad \text{where } \mu = \lambda + m^2 + 3m + 3 \in N$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction the statement is true for all $n \in N$.

EXAMPLE 11 *Prove by the principle of mathematical induction that for all $n \in N$:*

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n) : \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Step I We have, $P(1) : \frac{1}{1.3} = \frac{1}{(2 \times 1 + 1)}$. Clearly, $\frac{1}{1.3} = \frac{1}{(2 \times 1 + 1)}$. So, $P(1)$ is true.

Step II Let $P(m)$ be true. Then,

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2m-1)(2m+1)} = \frac{m}{2m+1} \quad \text{...(i)}$$

We shall now show that $P(m+1)$ is true. For this we shall show that

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2m-1)(2m+1)} + \frac{1}{(2m+1)(2m+3)} = \frac{m+1}{2m+3}$$

Now,

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2m-1)(2m+1)} + \frac{1}{(2m+1)(2m+3)}$$

$$= \frac{m}{2m+1} + \frac{1}{(2m+1)(2m+3)} \quad \text{[Using (i)]}$$

$$= \frac{2m^2 + 3m + 1}{(2m+1)(2m+3)} = \frac{(2m+1)(m+1)}{(2m+1)(2m+3)} = \frac{m+1}{2m+3}$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction, the given result is true for all $n \in N$.

EXAMPLE 12 *Using the principle of mathematical induction, prove that*

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)} \textit{ for all } n \in N.$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n) : \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

Step I We have, $P(1) = \frac{1}{1.2.3} = \frac{1(1+3)}{4(1+1)(1+2)}$

$$\because \quad \frac{1}{1.2.3} = \frac{1}{6} \text{ and } \frac{1(1+3)}{4(1+1)(1+2)} = \frac{4}{4 \times 2 \times 3} = \frac{1}{6}$$

$\therefore$ $\frac{1}{1.2.3} = \frac{1(1+3)}{4(1+1)(1+2)}$. So, $P(1)$ is true.

Step II Let $P(m)$ be true. Then,

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{m(m+1)(m+2)}=\frac{m(m+3)}{4(m+1)(m+2)} \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true.

i.e. $$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\ldots+\frac{1}{m(m+1)(m+2)}+\frac{1}{(m+1)(m+2)(m+3)}=\frac{(m+1)(m+4)}{4(m+2)(m+3)}$$

Now,

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{m(m+1)(m+2)}+\frac{1}{(m+1)(m+2)(m+3)}$$

$$=\frac{m(m+3)}{4(m+1)(m+2)}+\frac{1}{(m+1)(m+2)(m+3)} \quad \text{[Using (i)]}$$

$$=\frac{m(m+3)^2+4}{4(m+1)(m+2)(m+3)}$$

$$=\frac{m^3+6m^2+9m+4}{4(m+1)(m+2)(m+3)}=\frac{(m+1)^2(m+4)}{4(m+1)(m+2)(m+3)}=\frac{(m+1)(m+4)}{4(m+2)(m+3)}$$

$\therefore$ $P(m+1)$ is true.

Hence, $P(n)$ is true for all $n \in N$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 13 *If x and y are any two distinct integers, then prove by mathematical induction that $(x^n - y^n)$ is divisible by $(x-y)$ for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by: $P(n)$: $(x^n - y^n)$ is divisible by $(x-y)$

Step I $P(1):(x^1-y^1)$ is divisible by $(x-y)$.

$\because$ $x^1-y^1 = (x-y)$ is divisible by $(x-y)$. So, $P(1)$ is true

Step II Let $P(m)$ be true. Then,

(x^m-y^m) is divisible by $(x-y) \Rightarrow (x^m-y^m) = \lambda(x-y)$ for some $\lambda \in Z$ ...(i)

We shall now show that $P(m+1)$ is true. For this it is sufficient to show that $(x^{m+1}-y^{m+1})$ is divisible by $(x-y)$.

Now,

$$\begin{aligned} x^{m+1}-y^{m+1} &= x^{m+1}-x^m y+x^m y-y^{m+1} \\ &= x^m(x-y)+y(x^m-y^m) \\ &= x^m(x-y)+y\lambda(x-y) \quad \text{[Using (i)]} \\ &= (x-y)(x^m+y\lambda) \text{ which is divisible by } (x-y) \end{aligned}$$

So, $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

i.e. (x^n-y^n) is divisible by $(x-y)$ for all $n \in N$.

EXAMPLE 14 *Using principle of mathematical induction, prove that $x^{2n}-y^{2n}$ is divisible by $x+y$ for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by $P(n):(x^{2n}-y^{2n})$ is divisible by $(x+y)$.

Step I $P(1):(x^2-y^2)$ is divisible by $(x+y)$.

$\because$ $(x^2 - y^2) = (x - y)(x + y)$, which is divisible by $(x + y)$. So, $P(1)$ is true.

Step II Let $P(m)$ be true. Then,

$x^{2m} - y^{2m}$ is divisible by $(x + y) \Rightarrow x^{2m} - y^{2m} = \lambda(x + y)$ ...(i)

We shall now show that $P(m + 1)$ is true i.e., $x^{2m+2} - y^{2m+2}$ is divisible by $(x + y)$.

Now,

$$
\begin{aligned}
x^{2m+2} - y^{2m+2} &= x^{2m+2} - x^{2m} y^2 + x^{2m} y^2 - y^{2m+2} \\
&= x^{2m}(x^2 - y^2) + y^2(x^{2m} - y^{2m}) \\
&= x^{2m}(x^2 - y^2) + y^2 \lambda(x + y) \quad \text{[Using (i)]} \\
&= (x + y)\left\{x^{2m}(x - y) + \lambda y^2\right\}, \text{ which is divisible by } (x + y).
\end{aligned}
$$

$\therefore$ $P(m + 1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m + 1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$ i.e., $x^{2n} - y^{2n}$ is divisible by $(x + y)$ for all $n \in N$.

EXAMPLE 15 *Using principle of mathematical induction, prove that*

(i) $41^n - 14^n$ *is a multiple of* 27 (ii) $7^n - 3^n$ *is divisible by* 4.

SOLUTION (i) Let $P(n)$ be the statement given by $P(n) : 41^n - 14^n$ is a multiple of 27.

Step I $P(1) : 41^1 - 14^1$ is a multiple of 27.

$\because$ $41^1 - 14^1 = 41 - 14 = 27$, which is a multiple of 27. So, $P(1)$ is true.

Step II $P(m)$ be true. Then,

$41^m - 14^m$ is a multiple of 27 $\Rightarrow$ $41^m - 14^m = 27\lambda$ for some $\lambda \in N$ ...(i)

Now,

$$
\begin{aligned}
41^{m+1} - 14^{m+1} &= 41^{m+1} - 41 \times 14^m + 41 \times 14^m - 14^{m+1} \\
&= 41(41^m - 14^m) + (41 - 14)14^m \\
&= 41 \times 27\lambda + 27 \times 14^m \quad \text{[Using (i)]} \\
&= 27(41\lambda + 14^m), \text{ which is a multiple of 27.}
\end{aligned}
$$

$\therefore$ $P(m + 1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m + 1)$ is true. Hence, $P(n)$ is true for all $n \in N$.

(ii) Proceed as in (i).

EXAMPLE 16 *Using the principle of mathematical induction, prove that* $(2^{3n} - 1)$ *is divisible by 7 for all* $n \in N$. [NCERT EXEMPLAR]

SOLUTION Let $P(n)$ be the statement given by $P(n)$: $2^{3n} - 1$ is divisible by 7

Step I $P(1) : 2^{3 \times 1} - 1$ is divisible by 7.

Clearly, $2^{3 \times 1} - 1 = 8 - 1 = 7$, which is divisible by 7. So, $P(1)$ is true

Step II Let $P(m)$ be true. Then,

$2^{3m} - 1$ is divisible by 7 $\Rightarrow$ $2^{3m} - 1 = 7\lambda$, for some $\lambda \in N$ ...(i)

We shall now show that $P(m + 1)$ is true. For this we have to show that $2^{3(m+1)} - 1$ is divisible by 7.

Now,

$$2^{3(m+1)} - 1 = 2^{3m} \times 2^3 - 1 = (7\lambda + 1)\, 2^3 - 1 \qquad \text{[Using (i)]}$$

$$= 56\lambda + 8 - 1 = 7(8\lambda + 1), \text{ which is divisible by } 7$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$ i.e. $2^{3n} - 1$ is divisible by 7 for all $n \in N$.

EXAMPLE 17 *Prove by the principle of induction that for all $n \in N$, $(10^{2n-1} + 1)$ is divisible by 11.*

SOLUTION Let $P(n)$ be the statement given by $P(n)$: $10^{2n-1} + 1$ is divisible by 11

Step I We have, $P(1) : 10^{2 \times 1 - 1} + 1$ is divisible by 11.

Since $10^{2 \times 1 - 1} + 1 = 11$, which is divisible by 11. So, $P(1)$ is true.

Step II Let $P(m)$ be true. Then,

$10^{2m-1} + 1$ is divisible by 11 $\Rightarrow 10^{2m-1} + 1 = 11\lambda$, for some $\lambda \in N$ ...(i)

We shall now show that $P(m+1)$ is true. For this we have to show that $10^{2(m+1)} + 1$ is divisible by 11.

Now, $10^{2(m+1)-1} + 1 = 10^{2m+1} + 1 = 10^{2m-1} \times 10^2 + 1$

$$= (11\lambda - 1)\, 100 + 1 \qquad \text{[Using (i)]}$$

$$= 1100\lambda - 99 = 11(100\lambda - 9) = 11\mu, \text{ where } \mu = 100\lambda - 9 \in N$$

$\therefore$ $10^{2(m+1)} - 1$ is divisible by 11 $\Rightarrow P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction $P(m)$ is true for all $n \in N$ i.e. $10^{2n-1} + 1$ is divisible by 11 for all $n \in N$.

EXAMPLE 18 *Prove that $10^n + 3.4^{n+2} + 5$ is divisible by 9 for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by $P(n)$: $10^n + 3 . 4^{n+2} + 5$ is divisible by 9

Step I $P(1) : 10^1 + 3(4^{1+2}) + 5$ is divisible by 9.

$\because$ $10^1 + 3(4^{1+2}) + 5 = 10 + 192 + 5 = 207$, which is divisible by 9

$\therefore$ $P(1)$ is true.

Step II Let $P(m)$ be true. Then,

$10^m + 3(4^{m+2}) + 5$ is divisible by 9 $\Rightarrow 10^m + 3(4^{m+2}) + 5 = 9\lambda,\ \lambda \in N$ ...(i)

We shall now show that $P(m+1)$ is true for which we have to show that $10^{(m+1)} + 3(4^{m+3}) + 5$ is divisible by 9.

Now,

$$10^{m+1} + 3(4^{m+3}) + 5 = 10^m\,(10) + 3(4^{m+3}) + 5$$

$$= \{9\lambda - 3(4^{m+2}) - 5\} \times 10 + 3 \times 4^{m+3} + 5 \qquad \text{[Using (i)]}$$

$$= 90\lambda - 30 \times 4^{m+2} - 50 + 3 \times 4 \times 4^{m+2} + 5$$

$$= 90\lambda - 30 \times 4^{m+2} + 12 \times 4^{m+2} - 45$$

$$= 90\lambda - 18 \times 4^{m+2} - 45$$

$$= 9(10\lambda - 2 \times 4^{m+2} - 5) = 9\mu, \text{ where } \mu = 10\lambda - 2 \times 4^{m+2} - 5$$

$\therefore$ $10^{m+1} + 3.4^{m+3} + 5$ is divisible by 9 $\Rightarrow P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 19 *Prove by induction that the sum of the cubes of three consecutive natural numbers is divisible by 9.*

SOLUTION Let $P(n)$ be the statement given by

$P(n)$: Sum of the cubes of three consecutive natural numbers starting from n is divisible by 9.

Step I $P(1)$: Sum of the cubes of first three consecutive natural numbers is divisible by 9. Since $1^3 + 2^3 + 3^3 = 36$, which is divisible by 9. Therefore, $P(1)$ is true.

Step II Let $P(m)$ be true. Then, sum of the cubes of three consecutive natural numbers starting with m is divisible by 9. i.e. $m^3 + (m+1)^3 + (m+2)^3$ is divisible by 9

$$\therefore \quad m^3 + (m+1)^3 + (m+2)^3 = 9\lambda, \lambda \in N \qquad \text{... (i)}$$

We shall now show that $P(m+1)$ is true for which we have to show that

$(m+1)^3 + (m+2)^3 + (m+3)^3$ is divisible by 9.

Now,
$$\begin{aligned}(m+1)^3 + (m+2)^3 + (m+3)^3 &= (m+1)^3 + (m+2)^3 + m^3 + 9m^2 + 27m + 27\\ &= m^3 + (m+1)^3 + (m+2)^3 + 9(m^2 + 3m + 3)\\ &= 9\lambda + 9(m^2 + 3m + 3) \qquad \text{[Using (i)]}\\ &= 9(\lambda + m^2 + 3m + 3), \text{ which is divisible by 9.}\end{aligned}$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 20 *Using principle of mathematical induction prove that $4^n + 15n - 1$ is divisible by 9 for all natural numbers n.*

SOLUTION Let $P(n)$ be the statement given by $P(n): 4^n + 15n - 1$ is divisible by 9

Step I $P(1): 4^1 + 15 \times 1 - 1$ is divisible by 9.

$\because \quad 4^1 + 15 \times 1 - 1 = 18$, which is divisible by 9. Therefore, $P(1)$ is true

Step II Let $P(m)$ be true. Then,

$4^m + 15m - 1$ is divisible by 9 $\Rightarrow 4^m + 15m - 1 = 9\lambda$, for some $\lambda \in N$

We shall now show that $P(m+1)$ is true, for this we have to show that $4^{m+1} + 15(m+1) - 1$ is divisible by 9.

Now,
$$\begin{aligned}4^{m+1} + 15(m+1) - 1 &= 4^m . 4 + 15(m+1) - 1\\ &= (9\lambda - 15m + 1) \times 4 + 15(m+1) - 1\\ &= 36\lambda - 45m + 18 = 9(4\lambda - 5m + 2), \text{ which is divisible by 9.}\end{aligned}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$ i.e., $4^n + 15n - 1$ is divisible by 9.

EXAMPLE 21 *Prove that: $2.7^n + 3.5^n - 5$ is divisible by 24 for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by $P(n): 2.7^n + 3.5^n - 5$ is divisible by 24.

Step I We have, $P(1): 2 \times 7^1 + 3 \times 5^1 - 5$ is divisible by 24

$\because$ $2 \times 7^1 + 3 \times 5^1 - 5 = 14 + 15 - 5 = 24$, which is divisible by 24. Therefore, $P(1)$ is true.

Step II Let $P(m)$ be true. Then,

$2 \times 7^m + 3 \times 5^m - 5$ is divisible by 24

$$\Rightarrow \quad 2 \times 7^m + 3 \times 5^m - 5 = 24\lambda \text{ for some } \lambda \in N \Rightarrow 3 \times 5^m = 24\lambda + 5 - 2 \times 7^m \quad \text{...(i)}$$

Now,
$$\begin{aligned} 2 \times 7^{m+1} + 3 \times 5^{m+1} - 5 &= 2 \times 7^{m+1} + (3 \times 5^m)\,5 - 5 \\ &= 2 \times 7^{m+1} + (24\lambda + 5 - 2 \times 7^m)\,5 - 5 \quad \text{[Using (i)]} \\ &= 2 \times 7^{m+1} + 120\lambda + 25 - 10 \times 7^m - 5 \\ &= (2 \times 7^{m+1} - 10 \times 7^m) + 120\lambda + 20 \\ &= (2 \times 7 \times 7^m - 10 \times 7^m) + 120\lambda + 24 - 4 \\ &= (14 - 10)\,7^m - 4 + 24\,(5\lambda + 1) \\ &= 4\,(7^m - 1) + 24\,(5\lambda + 1) \\ &= 4 \times 6\mu + 24\,(5\lambda + 1) \quad \left[\because 7^m - 1 \text{ is a multiple of 6 for all } m \in N \therefore 7^m - 1 = 6\mu,\ \mu \in N\right] \\ &= 24\,(\mu + 5\lambda + 1), \text{ which is divisible by 24.} \end{aligned}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

EXAMPLE 22 *Prove that :*

(i) $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{n}\right) = (n+1)$ *for all* $n \in N$.

(ii) $\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right)\cdots\left(1 + \frac{2n+1}{n^2}\right) = (n+1)^2$ *for all* $n \in N$.

SOLUTION (i) Let $P(n)$ be the statement given by

$$P(n): \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{n}\right) = n + 1$$

Step I We have, $P(1): \left(1 + \frac{1}{1}\right) = (1 + 1)$

$\because$ $\left(1 + \frac{1}{1}\right) = 2 = (1 + 1)$. Therefore, $P(1)$ is true.

Step II Let $P(m)$ be true. Then,

$$\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{m}\right) = m + 1 \quad \text{....(i)}$$

$$\Rightarrow \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{m}\right)\left(1 + \frac{1}{m+1}\right) = (m+1)\left(1 + \frac{1}{m+1}\right) = \frac{(m+1)(m+2)}{m+1}$$

$$= m + 2$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

(ii) Let $P(n)$ be the statement given by

$$P(n): \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2$$

Step I We have, $P(1): \left(1+\frac{3}{1}\right)=(1+1)^2$

$\because$ $1+\frac{3}{1}=1+3=4=(1+1)^2$. Therefore, $P(1)$ is true.

Step II Let $P(m)$ be true. Then,

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2m+1}{m^2}\right)=(m+1)^2 \quad \ldots(i)$$

We shall now prove that $P(m+1)$ is true.

i.e. $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2m+1}{m^2}\right)\left(1+\frac{2(m+1)+1}{(m+1)^2}\right)=\left\{(m+1)+1\right\}^2$

Multiplying both sides of (i) by $1+\frac{2(m+1)+1}{(m+1)^2}$ i.e. $1+\frac{2m+3}{(m+1)^2}$, we obtain

$$\Rightarrow \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2m+1}{m^2}\right)\left(1+\frac{2m+3}{(m+1)^2}\right)=(m+1)^2\left(1+\frac{2m+3}{(m+1)^2}\right)$$

$$= (m+1)^2\left\{\frac{(m+1)^2+2m+3)}{(m+1)^2}\right\}$$

$$=(m^2+4m+4)$$

$$=(m+2)^2=\{(m+1)+1\}^2$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

EXAMPLE 23 *Prove by induction that* $4+8+12+\ldots+4n=2n(n+1)$ *for all* $n \in N$.

SOLUTION Let $P(n)$ be the statement given by $P(n): 4+8+12+\ldots+4n = 2n(n+1)$

Step I $P(1): 4 = 2\times1\times(1+1)$, which is true. Therefore, $P(1)$ is true.

Step II Let $P(m)$ be true. Then, $4+8+12+\ldots+4m = 2m(m+1)$ ...(i)

We shall now show that $P(m+1)$ is true i.e. $4+8+\ldots+4m+4(m+1) = 2(m+1)\{(m+1)+1\}$.

Now, $4+8+\ldots+4m+4(m+1) = 2m(m+1)+4(m+1)$ [Using (i)]

$$= (m+1)(2m+4)$$

$$= 2(m+1)(m+2) = 2(m+1)\{(m+1)+1\}$$

$\therefore$ $P(m+1)$ is true.

Thus $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true

Hence, by induction $P(n)$ is true for all $n \in N$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 24 *For all positive integer n, prove that* $\frac{n^7}{7}+\frac{n^5}{5}+\frac{2}{3}n^3-\frac{n}{105}$ *is an integer*

SOLUTION Let $P(n)$ be the statement given by $P(n): \frac{n^7}{7}+\frac{n^5}{5}+2\frac{n^3}{3}-\frac{n}{105}$ is an integer

Step I $P(1): \frac{1}{7}+\frac{1}{5}+\frac{2}{3}-\frac{1}{105}$ is an integer.

Since $\frac{1}{7}+\frac{1}{5}+\frac{2}{3}-\frac{1}{105}=\frac{15+21+70-1}{105}=1$, which is an integer. So, $P(1)$ is true.

Step II Let $P(m)$ be true. Then, $\frac{m^7}{7}+\frac{m^5}{5}+\frac{2m^3}{3}-\frac{m}{105}$ is an integer

Let $\frac{m^7}{7}+\frac{m^5}{5}+\frac{2m^3}{3}-\frac{m}{105}=\lambda, \lambda \in Z$ ...(i)

We shall now show that $P(m+1)$ is true for which we have to show that

$\frac{(m+1)^7}{7}+\frac{(m+1)^5}{5}+\frac{2(m+1)^3}{3}-\frac{(m+1)}{105}$ is an integer.

Now, $\frac{(m+1)^7}{7}+\frac{(m+1)^5}{5}+\frac{2(m+1)^3}{3}-\frac{(m+1)}{105}$

$$= \frac{1}{7}(m^7+7m^6+21m^5+35m^4+35m^3+21m^2+7m+1)$$

$$+\frac{1}{5}(m^5+5m^4+10m^3+10m^2+5m+1)+\frac{2}{3}(m^3+3m^2+3m+1)-\frac{m}{105}-\frac{1}{105}$$

$$=\left\{\frac{m^7}{7}+\frac{m^5}{5}+2\frac{m^3}{3}-\frac{m}{105}\right\}+m^6+3m^5+6m^4+7m^3+7m^2+4m+1$$

$$=\lambda+m^6+3m^5+6m^4+7m^3+7m^2+4m+1 \quad \text{[Using (i)]}$$

= an integer

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

i.e. $\frac{n^7}{7}+\frac{n^5}{5}+2\frac{n^3}{3}-\frac{n}{105}$ is an integer.

EXAMPLE 25 *Prove by the principle of mathematical induction that* $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ *is a natural number for all* $n \in N$. [NCERT EXEMPLAR]

SOLUTION Let $P(n)$ be the statement given by $P(n): \frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ is a natural number

Step I $P(1): \frac{1}{5}+\frac{1}{3}+\frac{7}{15}$ is a natural number.

$\because$ $\frac{1}{5}+\frac{1}{3}+\frac{7}{15}=\frac{15}{15}=1$, which is a natural number. So, $P(1)$ is true.

Step II *Let* $P(m)$ *be true. Then,*

$\frac{m^5}{5}+\frac{m^3}{3}+\frac{7m}{15}$ is a natural number. Let $\frac{m^5}{5}+\frac{m^3}{3}+\frac{7m}{15}=\lambda$ ...(i)

We shall now show that $P(m+1)$ is true, for which it is sufficient to prove that

$\frac{(m+1)^5}{5}+\frac{(m+1)^3}{3}+\frac{7(m+1)}{15}$ is a natural number.

Now, $\frac{(m+1)^5}{5}+\frac{(m+1)^3}{3}+\frac{7(m+1)}{15}$

$$=\frac{1}{5}(m^5+5m^4+10m^3+10m^2+5m+1)+\frac{1}{3}(m^3+3m^2+3m+1)+\frac{7}{15}m+\frac{7}{15}$$

$$= \left(\frac{m^5}{5} + \frac{m^3}{3} + \frac{7}{15}m\right) + (m^4 + 2m^3 + 3m^2 + 2m) + \frac{1}{5} + \frac{1}{3} + \frac{7}{15}$$

$$= \lambda + m^4 + 2m^3 + 3m^2 + 2m + 1 \qquad \text{[Using (i)]}$$

$$= \text{an integer}$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

i.e. $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7}{15}n$ is a natural number for all $n \in N$.

EXAMPLE 26 *Prove by the principle of mathematical induction that 3^{2n} when divided by 8, the remainder is always 1, for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by

$P(n)$: 3^{2n} when divided by 8, the remainder is 1 or, $P(n)$: $3^{2n} = 8\lambda + 1$ for some $\lambda \in N$

Step I $P(1): 3^2 = 8\lambda + 1$ for some $\lambda \in N$.

Clearly, $3^2 = 8 \times 1 + 1 = 8\lambda + 1$, where $\lambda = 1$. So, $P(1)$ is true

Step II Let $P(m)$ be true. Then, $3^{2m} = 8\lambda + 1$ for some $\lambda \in N$ ...(i)

We shall now show that $P(m+1)$ is true for which we have to show that $3^{2(m+1)}$ when divided by 8, the remainder is 1 i.e. $3^{2(m+1)} = 8\mu + 1$ for some $\mu \in N$.

Now, $3^{2(m+1)} = 3^{2m} \times 3^2 = (8\lambda + 1) \times 9$ [Using (i)]

$$= 72\lambda + 9 = 72\lambda + 8 + 1 = 8(9\lambda + 1) + 1 = 8\mu + 1, \quad \text{where } \mu = 9\lambda + 1 \in N$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$ i.e. 3^{2n} when divided by 8 the remainder is always 1.

EXAMPLE 27 *Prove by the principle of mathematical induction that $n < 2^n$ for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by $P(n)$: $n < 2^n$.

Step I $P(1)$: $1 < 2^1$.

Clearly, $1 < 2^1$. So, $P(1)$ is true

Step II Let $P(m)$ be true. Then, $m < 2^m$

We shall now show that $P(m+1)$ is true for which we will have to prove that $(m+1) < 2^{m+1}$.

Now,

$P(m)$ is true $\Rightarrow m < 2^m \Rightarrow 2m < 2.2^m \Rightarrow 2m < 2^{m+1} \Rightarrow (m+m) < 2^{m+1}$

$\Rightarrow$ $m + 1 \le m + m < 2^{m+1}$ $[\because 1 \le m \ \therefore m + 1 \le m + m]$

$\Rightarrow$ $(m+1) < 2^{m+1} \Rightarrow P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for all $n \in N$ i.e. $n < 2^n$ for all $n \in N$.

EXAMPLE 28 *Prove by induction the inequality $(1+x)^n \ge 1 + nx$ whenever x is positive and n is a positive integer.*

SOLUTION Let $P(n)$ be the statement given by $P(n) : (1+x)^n \geq 1 + nx$

Step I $P(1) : (1+x)^1 \geq 1 + 1(x)$

Clearly, $(1+x)^1 \geq 1 + 1(x)$. So, $P(1)$ is true

Step II Let $P(m)$ be true. Then, $(1+x)^m \geq 1 + mx$ ...(i)

We shall now prove that $P(m+1)$ is true whenever $P(m)$ is true. For this we have to show that $(1+x)^{m+1} \geq 1 + (m+1)x$.

Now, $P(m)$ is true

$\Rightarrow$ $(1+x)^m \geq 1 + mx$

$\Rightarrow$ $(1+x)(1+x)^m \geq (1+x)(1+mx)$ [Multiplying both sides by $(1+x)$]

$\Rightarrow$ $(1+x)^{m+1} \geq 1 + (m+1)x + mx^2$

$\Rightarrow$ $(1+x)^{m+1} \geq 1 + (m+1)x + mx^2 \geq 1 + (m+1)x$ [$\because mx^2 \geq 0$]

$\Rightarrow$ $(1+x)^{m+1} \geq 1 + (m+1)x \Rightarrow P(m+1)$ is true

Hence, by the principle of induction, $P(n)$ is true for all $n \in N$ i.e. $(1+x)^n \geq 1 + nx$ for all $n \in N$.

EXAMPLE 29 *Prove by induction that $(2n+7) < (n+3)^2$ for all natural numbers n. Using this, prove by induction that $(n+3)^2 \leq 2^{n+3}$ for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by $P(n) : (2n+7) < (n+3)^2$.

Step I $P(1) : (2 \times 1 + 7) < (1+3)^2$

$\because$ $(2 \times 1 + 7) = 9 < (1+3)^2$. Therefore, $P(1)$ is true.

Step II Let $P(m)$ be true. Then, $2m + 7 < (m+3)^2$. ... (i)

We shall now show that $P(m+1)$ is true whenever $P(m)$ is true. For this we have to show that $2(m+1) + 7 < (m+1+3)^2$.

Now,

$P(m)$ is true

$\Rightarrow$ $2m + 7 < (m+3)^2$

$\Rightarrow$ $2m + 7 + 2 < (m+3)^2 + 2$

$\Rightarrow$ $2(m+1) + 7 < m^2 + 6m + 11$

$\Rightarrow$ $2(m+1) + 7 < m^2 + 6m + 11 < m^2 + 8m + 16$

$\Rightarrow$ $2(m+1) + 7 < (m+4)^2 \Rightarrow \{2(m+1)+7\} < \{(m+1)+3\}^2 \Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Now, let $P'(n)$ be the statement given by $P'(n) : (n+3)^2 \leq 2^{n+3}$

Step I $P'(1) : (1+3)^2 \leq 2^{1+3}$

We find that $(1+3)^2 = 16 \leq 2^{1+3}$. So, $P'(1)$ is true

Step II Let $P'(m)$ be true. Then, $(m+3)^2 \leq 2^{m+3}$.

We shall now show that $P'(m+1)$ is true whenever $P'(m)$ is true. For this we have to show that $\{(m+1)+3\}^2 \leq 2^{(m+1)+3}$.

Now, $P'(m)$ is true

$\Rightarrow \quad (m+3)^2 \leq 2^{m+3}$

$\Rightarrow \quad (m+3)^2 + (2m+7) \leq 2^{m+3} + (2m+7)$

$\Rightarrow \quad (m+4)^2 \leq 2^{m+3} + (m+3)^2 \quad [\because\ 2m+7 < (m+3)^2 \therefore\ 2^{m+3} + (2m+7) < 2^{m+3} + (m+3)^2]$

$\Rightarrow \quad (m+4)^2 \leq 2^{m+3} + 2^{m+3} \quad [\because\ (m+3)^2 \leq 2^{m+3} \Rightarrow (m+3)^2 + 2^{m+3} \leq 2^{m+3} + 2^{m+3}]$

$\Rightarrow \quad (m+4)^2 \leq 2.2^{m+3} \Rightarrow (m+4)^2 \leq 2^{m+4} \Rightarrow \{(m+1)+3\}^2 \leq 2^{(m+1)+3} \Rightarrow P'(m+1)$ is true

Hence, by the principle of mathematical induction, $P'(n)$ is true for all $n \in N$ i.e. $(n+3)^2 \leq 2^{n+3}$ for all $n \in N$.

EXAMPLE 30 *Prove that:* $1^2 + 2^2 + 3^2 + \ldots + n^2 > \frac{n^3}{3}$ *for all* $n \in N$.

SOLUTION Let $P(n)$ be the statement given by $P(n): 1^2 + 2^2 + 3^2 + \ldots + n^2 > \frac{n^3}{3}$

Step I $\quad P(1): 1^2 > \frac{1^3}{3}$

We find that $1^2 = 1 > \frac{1}{3} = \frac{1^3}{3}$. So, $P(1)$ is true.

Step II $\quad$ Let $P(n)$ be true for $n = m$. Then, $1^2 + 2^2 + 3^2 + \ldots + m^2 > \frac{m^3}{3}$ ...(i)

We shall now prove that $P(m+1)$ is true. i.e. $1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{(m+1)^3}{3}$

Now, $\quad P(m)$ is true

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 > \frac{m^3}{3}$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{m^3}{3} + (m+1)^2$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{1}{3}(m^3 + 3m^2 + 6m + 3)$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{1}{3}\left\{(m^3 + 3m^2 + 3m + 1) + (3m+2)\right\}$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{1}{3}\left\{(m+1)^3 + (3m+2)\right\} > \frac{(m+1)^3}{3}$

$\Rightarrow \quad P(m+1)$ is true.

Thus, $\quad P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

EXAMPLE 31 *Prove that:* $1 + 2 + 3 + \ldots + n < \frac{(2n+1)^2}{8}$ *for all* $n \in N$.

SOLUTION Let $P(n)$ be the statement given by $P(n): 1 + 2 + 3 + \ldots + n < \frac{(2n+1)^2}{8}$.

Step I $\quad$ We have, $P(1): 1 < \frac{(2 \times 1 + 1)^2}{8}$.

Clearly, $1 < \frac{(2 \times 1 + 1)^2}{8} = \frac{9}{8}$. So, $P(1)$ is true.

Step II Let $P(m)$ be true. Then, $1 + 2 + 3 + \ldots + m < \frac{(2m+1)^2}{8}$ ...(i)

We shall now show that $P(m+1)$ is true.

i.e., $1 + 2 + 3 + \ldots + m + (m+1) < \frac{\{2(m+1)+1\}^2}{8}$

Now,

$P(m)$ is true

$$\Rightarrow \quad 1 + 2 + 3 + \ldots + m < \frac{(2m+1)^2}{8}$$

$$\Rightarrow \quad 1 + 2 + 3 + \ldots + m + (m+1) < \frac{(2m+1)^2}{8} + (m+1)$$

$$\Rightarrow \quad 1 + 2 + 3 + \ldots + m + (m+1) < \frac{(2m+1)^2 + 8(m+1)}{8}$$

$$\Rightarrow \quad 1 + 2 + 3 + \ldots + m + (m+1) < \frac{(4m^2 + 12m + 9)}{8}$$

$$\Rightarrow \quad 1 + 2 + 3 + \ldots + m + (m+1) < \frac{(2m+3)^2}{8} = \frac{\{2(m+1)+1\}^2}{8}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 32 *Prove by the principle of mathematical induction that for all $n \in N$,*

$$\sin \theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin\left(\frac{n+1}{2}\right)\theta \, \sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}}$$ [NCERT EXEMPLAR]

SOLUTION Let $P(n)$ be the statement given by

$$P(n) : \sin \theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin\left(\frac{n+1}{2}\right)\theta \, \sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}}$$

Step I We have, $P(1) : \sin \theta = \frac{\sin\left(\frac{1+1}{2}\right)\theta \, \sin\left(\frac{1 \times \theta}{2}\right)}{\sin \frac{\theta}{2}}$

Clearly, $\sin \theta = \frac{\sin\left(\frac{1+1}{2}\right)\theta \, . \sin\left(\frac{1 \times \theta}{2}\right)}{\sin \frac{\theta}{2}}$. So, $P(1)$ is true.

Step II Let $P(m)$ be true. Then,

$$\sin \theta + \sin 2\theta + \ldots + \sin m\theta = \frac{\sin\left(\frac{m+1}{2}\right)\theta \, \sin \frac{m\theta}{2}}{\sin \frac{\theta}{2}} \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true.

i.e. $$\sin\theta + \sin 2\theta + \ldots + \sin m\theta + \sin(m+1)\theta = \frac{\sin\left\{\frac{(m+1)+1}{2}\right\}\theta \,\sin\left(\frac{m+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

Now,

$$\sin\theta + \sin 2\theta + \ldots + \sin m\theta + \sin(m+1)\theta$$

$$= \frac{\sin\left(\frac{m+1}{2}\right)\theta \,\sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}} + \sin(m+1)\theta \qquad \text{[Using (i)]}$$

$$= \frac{\sin\left(\frac{m+1}{2}\right)\theta \,\sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}} + 2\sin\left(\frac{m+1}{2}\right)\theta \,\cos\left(\frac{m+1}{2}\right)\theta$$

$$= \sin\left(\frac{m+1}{2}\right)\theta\left\{\frac{\sin\left(\frac{m\theta}{2}\right)}{\sin\frac{\theta}{2}} + 2\cos\left(\frac{m+1}{2}\right)\theta\right\}$$

$$= \sin\left(\frac{m+1}{2}\right)\theta\left\{\frac{\sin\left(\frac{m\theta}{2}\right) + 2\sin\frac{\theta}{2}\cos\left(\frac{m+1}{2}\right)\theta}{\sin\frac{\theta}{2}}\right\}$$

$$= \sin\left(\frac{m+1}{2}\right)\theta\left\{\frac{\sin\left(\frac{m\theta}{2}\right) + \sin\left(\frac{m+2}{2}\right)\theta - \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}}\right\}$$

$$= \frac{\sin\left(\frac{m+1}{2}\right)\theta \,\sin\left(\frac{m+2}{2}\right)\theta}{\sin\frac{\theta}{2}} = \frac{\sin\left\{\frac{(m+1)+1}{2}\right\}\theta \,\sin\left(\frac{m+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 33 *Using principle of mathematical induction, prove that*

$$\cos\alpha\cos 2\alpha\cos 4\alpha \ldots \cos(2^{n-1}\alpha) = \frac{\sin 2^n\alpha}{2^n\sin\alpha} \text{ for all } n \in N. \qquad \text{[NCERT EXEMPLAR]}$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n) : \cos\alpha\cos 2\alpha\cos 4\alpha \ldots \cos(2^{n-1}\alpha) = \frac{\sin(2^n\alpha)}{2^n\sin\alpha}$$

Step I $\quad P(1) : \cos\alpha = \frac{\sin(2^1\alpha)}{2^1\sin\alpha}$

Clearly, $\frac{\sin(2^1\alpha)}{2^1\sin\alpha} = \frac{\sin 2\alpha}{2\sin\alpha} = \frac{2\sin\alpha\cos\alpha}{2\sin\alpha} = \cos\alpha$. So, $P(1)$ is true.

Step II Let $P(m)$ be true. Then,

$$\cos\alpha \cos 2\alpha \cos 4\alpha \dots \cos(2^{m-1}\alpha) = \frac{\sin(2^m\alpha)}{2^m \sin\alpha} \qquad \text{...(i)}$$

We shall now show that $P(m+1)$ is true. For this we have to show that

$$\cos\alpha \cos 2\alpha \cos 2^2\alpha \dots \cos(2^{m-1}\alpha)\cos(2^m\alpha) = \frac{\sin(2^{(m+1)}\alpha)}{2^{m+1}\sin\alpha}$$

Now, $\cos\alpha \cos 2\alpha \cos 2^2\alpha \dots \cos(2^{m-1}\alpha)\cos(2^m\alpha)$

$$= \{\cos\alpha \cos 2\alpha \cos 2^2\alpha \dots \cos(2^{m-1}\alpha)\}\cos(2^m\alpha)$$

$$= \frac{\sin(2^m\alpha)}{2^m\sin\alpha}\times\cos(2^m\alpha) \qquad \text{[Using (i)]}$$

$$= \frac{2\sin(2^m\alpha)\cos(2^m\alpha)}{2^{m+1}\sin\alpha} = \frac{\sin(2\cdot2^m\alpha)}{2^{m+1}\sin\alpha} = \frac{\sin(2^{m+1}\alpha)}{2^{m+1}\sin\alpha}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Type II PROBLEMS BASED UPON SECOND PRINCIPLE OF MATHEMATICAL INDUCTION

EXAMPLE 34 *Let $U_1 = 1$, $U_2 = 1$ and $U_{n+2} = U_{n+1} + U_n$ for $n \geq 1$. Use mathematical induction to show that:*

$$U_n = \frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n\right\} \quad \textit{for all } n \geq 1.$$

SOLUTION Let $P(n)$ be the statement given by $P(n): U_n = \frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n\right\}$

We find that:

$$U_1 = \frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^1 - \left(\frac{1-\sqrt5}{2}\right)^1\right\} = 1$$

and, $$U_2 = \frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^2 - \left(\frac{1-\sqrt5}{2}\right)^2\right\} = \frac{1}{\sqrt5}\left\{\left(\frac{1+5+2\sqrt5}{4}\right) - \left(\frac{1+5-2\sqrt5}{4}\right)\right\} = 1$$

So, $P(1)$ and $P(2)$ are true.

Let $P(n)$ be true for all $n \leq m$. i.e. $U_n = \frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n\right\}$ for all $n \leq m$ ...(i)

We shall now show that $P(n)$ is true for $n = m+1$.

i.e. $$U_{m+1} = \frac{1}{\sqrt5}\left\{\left(\frac{1+\sqrt5}{2}\right)^{m+1} - \left(\frac{1-\sqrt5}{2}\right)^{m+1}\right\}$$

We have,

$$U_{n+2} = U_{n+1} + U_n \quad \text{for } n \geq 1$$

$$\Rightarrow U_{m+1} = U_m + U_{m-1} \quad \text{for } m \geq 2 \qquad \text{[On replacing } n \text{ by } (m-1)]$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right\} + \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\right\} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left[\left\{\left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\right\} - \left\{\left(\frac{1-\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\right\}\right]$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{1-\sqrt{5}}{2}+1\right)\right\}$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{3-\sqrt{5}}{2}\right)\right\}$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{6+2\sqrt{5}}{4}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{6-2\sqrt{5}}{4}\right)\right\}$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{1-\sqrt{5}}{2}\right)^2\right\}$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m+1}\right\}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(n)$ is true for all $n \leq m \Rightarrow P(n)$ is true for all $n \leq m+1$.

Hence, $P(n)$ is true for all $n \in N$.

EXERCISE 11.2

BASIC

Prove the following by the principle of mathematical induction: (1-42)

1. $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$ i.e, the sum of the first n natural numbers is $\dfrac{n(n+1)}{2}$.
2. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
3. $1 + 3 + 3^2 + \ldots + 3^{n-1} = \dfrac{3^n - 1}{2}$
4. $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$
5. $1 + 3 + 5 + \ldots + (2n-1) = n^2$ i.e., the sum of first n odd natural numbers is n^2.
6. $\dfrac{1}{2.5} + \dfrac{1}{5.8} + \dfrac{1}{8.11} + \ldots + \dfrac{1}{(3n-1)(3n+2)} = \dfrac{n}{6n+4}$
7. $\dfrac{1}{1.4} + \dfrac{1}{4.7} + \dfrac{1}{7.10} + \ldots + \dfrac{1}{(3n-2)(3n+1)} = \dfrac{n}{3n+1}$

8. $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$

9. $\frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots+\frac{1}{(4n-1)(4n+3)}=\frac{n}{3(4n+3)}$

10. $1.2+2.2^2+3.2^3+\ldots+n.2^n=(n-1)\,2^{n+1}+2$

11. $2+5+8+11+\ldots+(3n-1)=\frac{1}{2}n(3n+1)$

12. $1.3+2.4+3.5+\ldots+n.(n+2)=\frac{1}{6}n(n+1)(2n+7)$

13. $1.3+3.5+5.7+\ldots+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$

14. $1.2+2.3+3.4+\ldots+n(n+1)=\frac{n(n+1)(n+2)}{3}$

15. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

16. $1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{1}{3}n(4n^2-1)$

17. $a+ar+ar^2+\ldots+ar^{n-1}=a\left(\frac{r^n-1}{r-1}\right), r\neq 1$

18. $a+(a+d)+(a+2d)+\ldots+(a+(n-1)d)=\frac{n}{2}[2a+(n-1)d]$

BASED ON LOTS

19. $5^{2n}-1$ is divisible by 24 for all $n\in N$

20. $3^{2n}+7$ is divisible by 8 for all $n\in N$

21. $5^{2n+2}-24n-25$ is divisible by 576 for all $n\in N$

22. $3^{2n+2}-8n-9$ is divisible by 8 for all $n\in N$

23. $(ab)^n=a^n b^n$ for all $n\in N$

24. $n(n+1)(n+5)$ is a multiple of 3 for all $n\in N$

25. $7^{2n}+2^{3n-3}\,.\,3^{n-1}$ is divisible by 25 for all $n\in N$

26. $2.7^n+3.5^n-5$ is divisible by 24 for all $n\in N$

27. $11^{n+2}+12^{2n+1}$ is divisible by 133 for all $n\in N$

28. $1\times 1!+2\times 2!+3\times 3!+\ldots+n\times n!=(n+1)!-1$ for all $N\in N$. [NCERT EXEMPLAR]

29. n^3-7n+3 is divisible by 3 for all $n\in N$.

30. $1+2+2^2+\ldots+2^n=2^{n+1}-1$ for all $n\in N$.

BASED ON HOTS

31. $7+77+777+\ldots+\underbrace{777\ldots\ldots\ldots7}_{n-\text{digits}}=\frac{7}{81}(10^{n+1}-9n-10)$ for all $n\in N$

32. $\frac{n^7}{7}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{n^2}{2}-\frac{37}{210}n$ is a positive integer for all $n \in N$

33. $\frac{n^{11}}{11}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{62}{165}n$ is a positive integer for all $n \in N$.

34. $\frac{1}{2}\tan\left(\frac{x}{2}\right)+\frac{1}{4}\tan\left(\frac{x}{4}\right)+\ldots+\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right)=\frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)-\cot x$ for all $n \in N$ and $0<x<\frac{\pi}{2}$

35. $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ for all natural numbers, $n \geq 2$.

36. $\frac{(2n)!}{2^{2n}(n!)^2} \leq \frac{1}{\sqrt{3n+1}}$ for all $n \in N$.

37. $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{n^2}<2-\frac{1}{n}$ for all $n>2, n \in N$.

38. $x^{2n-1}+y^{2n-1}$ is divisible by $x+y$ for all $n \in N$.

39. $\sin x+\sin 3x+\ldots+\sin(2n-1)x=\frac{\sin^2 nx}{\sin x}$ for all $n \in N$.

40. $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos(\alpha+(n-1)\beta)=\frac{\cos\left\{\alpha+\left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$ for all $n \in N$. [NCERT EXEMPLAR]

41. $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>\frac{13}{24}$ for all natural numbers $n>1$. [NCERT EXEMPLAR]

42. Using principle of mathematical induction prove that

$\sqrt{n}<\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}$ for all natural numbers $n \geq 2$. [NCERT EXEMPLAR]

43. Given $a_1=\frac{1}{2}\left(a_0+\frac{A}{a_0}\right)$, $a_2=\frac{1}{2}\left(a_1+\frac{A}{a_1}\right)$ and $a_{n+1}=\frac{1}{2}\left(a_n+\frac{A}{a_n}\right)$ for $n \geq 2$, where $a>0$, $A>0$.

Prove that $\frac{a_n-\sqrt{A}}{a_n+\sqrt{A}}=\left(\frac{a_1-\sqrt{A}}{a_1+\sqrt{A}}\right)2^{n-1}$.

44. Let $P(n)$ be the statement: $2^n \geq 3n$. If $P(r)$ is true, show that $P(r+1)$ is true. Do you conclude that $P(n)$ is true for all $n \in N$?

45. Show by the Principle of Mathematical induction that the sum S_n of the n terms of the series $1^2+2\times2^2+3^2+2\times4^2+5^2+2\times6^2+7^2+\ldots$ is given by

$$S_n=\begin{cases}\frac{n(n+1)^2}{2}, & \text{if } n \text{ is even}\\ \frac{n^2(n+1)}{2}, & \text{if } n \text{ is odd}\end{cases}$$ [NCERT EXEMPLAR]

46. Prove that the number of subsets of a set containing n distinct elements is 2^n for all $n \in N$. [NCERT EXEMPLAR]

47. A sequence $a_1, a_2, a_3, \ldots\ldots$ is defined by letting $a_1 = 3$ and $a_k = 7\ a_{k-1}$ for all natural numbers $k \geq 2$. Show that $a_n = 3.7^{n-1}$ for all $n \in N$. [NCERT EXEMPLAR]

48. A sequence $x_1, x_2, x_3, \ldots$ is defined by letting $x_1 = 2$ and $x_k = \dfrac{x_{k-1}}{n}$ for all natural numbers k, $k \geq 2$. Show that $x_n = \dfrac{2}{n!}$ for all $n \in N$. [NCERT EXEMPLAR]

49. A sequence $x_0, x_1, x_2, x_3, \ldots$ is defined by letting $x_0 = 5$ and $x_k = 4 + x_{k-1}$ for all natural number k. Show that $x_n = 5 + 4n$ for all $n \in N$ using mathematical induction. [NCERT EXEMPLAR]

50. The distributive law from algebra states that for all real numbers c, a_1 and a_2, we have

$$c\,(a_1 + a_2) = c\,a_1 + c\,a_2$$

Use this law and mathematical induction to prove that, for al natural numbers, $n \geq 2$, if $c, a_1, a_2, \ldots, a_n$ are any real numbers, then $c\,(a_1 + a_2 + \ldots + a_n) = c\,a_1 + c\,a_2 + \ldots + c\,a_n$.

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If $P(n)$: "$2 \times 4^{2n+1} + 3^{3n+1}$ is divisible by λ for all $n \in N$" is true, then the value of λ is ....
2. If $P(n): 2n < n!, n \in N$, then $P(n)$ is true for all $n \geq$ ....
3. If $P(n): 2^n < n!, n \in N$, then $P(n)$ is true for all $n >$ ....
4. For each $n \in N, 10^{2n-1} + 1$ is divisible by ....
5. If $P(n): n! > 2^{n-1}, n \in N$, then $P(n)$ is true for all $n >$ ....
6. If $P(n): n^3 - n$ is divisible by 6, $n \in N$, then $P(n)$ is true for all $n \geq$ ....
7. If $P(n): n^2 < 2^n, n \in N$, then $P(n)$ is true for all $n \geq$ ....
8. If $P(n): \sqrt{n} < \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \ldots + \dfrac{1}{\sqrt{n}}, n \in N$, then $P(n)$ is true for all $n \geq$ ....

ANSWERS

1. 11	2. 4	3. 3	4. 11
5. 2	6. 2	7. 5	8. 2

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per the requirement of the question.

1. State the first principle of mathematical induction.
2. Write the set of value of n for which the statement $P(n)$: $2n < n!$ is true.
3. State the second principle of mathematical induction.
4. If $P(n)$: $2 \times 4^{2n+1} + 3^{3n+1}$ is divisible by λ for all $n \in N$ is true, then find the value of λ.

ANSWERS

2. $\{n \in N : n \geq 4\}$ 4. 11

MULTIPLE CHOICES QUESTIONS (MCQs)

Make the correct alternative in each of the following.

1. If $x^n - 1$ is divisible by $x - \lambda$, then the least positive integral value of λ is
   (a) 1 (b) 2 (c) 3 (d) 4
2. For all $n \in N$, $3 \times 5^{2n+1} + 2^{3n+1}$ is divisible by
   (a) 19 (b) 17 (c) 23 (d) 25
3. If $10^n + 3 \times 4^{n+2} + \lambda$ is divisible by 9 for all $n \in N$, then the least positive integral value of λ is
   (a) 5 (b) 3 (c) 7 (d) 1
4. Let $P(n)$: $2^n < (1 \times 2 \times 3 \times \ldots \times n)$. Then the smallest positive integer for which $P(n)$ is true is
   (a) 1 (b) 2 (c) 3 (d) 4
5. A student was asked to prove a statement $P(n)$ by induction. He proved $P(k+1)$ is true whenever $P(k)$ is true for all $k > 5 \in N$ and also $P(5)$ is true. On the basis of this he could conclude that $P(n)$ is true.
   (a) for all $n \in N$ (b) for all $n > 5$ (c) for all $n \geq 5$ (d) for all $n < 5$
6. If $P(n) : 49^n + 16^n + \lambda$ is divisible by 64 for $n \in N$ is true, then the least negative integral value of λ is
   (a) -3 (b) -2 (c) -1 (d) -4

ANSWERS

1. (a) 2. (b) 3. (a) 4. (d) 5. (c) 6. (c)

SUMMARY

1. A sentence or description which can be judged to be true or false is called a statement. Statements involving mathematical relations are called mathematical statements.
2. Let $P(n)$ be a statement involving the natural number n such that
   (i) $P(1)$ is true.
   and, (ii) $P(m+1)$ is true, whenever $P(m)$ is true.
   Then, $P(n)$ is true for all $n \in N$.
   This is called first principle of mathematical induction.
3. Let $P(n)$ be a statement involving the natural number n such that
   (i) $P(1)$ is true
   and, (ii) $P(m+1)$ is true, whenever $P(n)$ is true for all $n \leq m$.
   Then, $P(n)$ is true for all $n \in N$.
   This is called second principle of mathematical induction.

CHAPTER 12

COMPLEX NUMBERS

12.1 INTRODUCTION

If a, b are natural numbers such that $a > b$, then the equation $x + a = b$ is not solvable in N, the set of natural numbers i.e. there is no natural number satisfying the equation $x + a = b$. So, the set of natural numbers is extended to form the set I of integers in which every equation of the form $x + a = b$; $a, b \in N$ is solvable. But, equations of the form $xa = b$, where $a, b \in I$, $a \neq 0$ are not solvable in I also. Therefore, the set I of integers is extended to obtain the set Q of all rational numbers in which every equation of the form $xa = b$, $a \neq 0$, $a, b \in I$ is uniquely solvable. The equations of the form $x^2 = 2$, $x^2 = 3$ etc. are not solvable in Q because there is no rational number whose square is 2. Such numbers are known as irrational numbers. The set Q of all rational numbers is extended to obtain the set R which includes both rational and irrational numbers. This set is known as the set of real numbers. The equations of the form $x^2 + 1 = 0$, $x^2 + 4 = 0$ etc. are not solvable in R i.e. there is no real number whose square is a negative real number. *Euler was the first mathematician to introduce the symbol i (iota) for the square root of -1 i.e. a solution of $x^2 + 1 = 0$ with the property $i^2 = -1$. He also called this symbol as the imaginary unit.*

12.2 INTEGRAL POWERS OF IOTA (i)

Positive integral powers of i: We have, $i = \sqrt{-1}$

$$\therefore \quad i^2 = -1,\ i^3 = i^2 \times i = -i,\ i^4 = (i^2)^2 = (-1)^2 = 1$$

In order to compute i^n for $n > 4$, we divide n by 4 and obtain the remainder r. Let m be the quotient when n is divided by 4. Then,

$$n = 4m + r, \text{ where } 0 \leq r < 4 \Rightarrow i^n = i^{4m+r} = (i^4)^m\, i^r = i^r$$

Thus, the value of i^n for $n > 4$ is i^r, where r is the remainder when n is divided by 4.

Negative integral powers of i: By the law of indices, we have

$$i^{-1} = \frac{1}{i} = \frac{i^3}{i^4} = i^3 = -i,\ i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,\ i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = i,\ i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

If $n > 4$, then $i^{-n} = \dfrac{1}{i^n} = \dfrac{1}{i^r}$, where r is the remainder when n is divided by 4

NOTE $\;$ *i^0 is defined as 1.*

The above discussion suggests the following algorithm to find integral exponents of i.

ALGORITHM

To find the value of i^n for $n \in Z$, we may follow the following steps.

Step I $\quad$ *If $n = 0$, then write $i^n = 1$.*

Step II *If $n > 0$, then*

$$i^n = \begin{cases} i, & \text{if } n = 1 \\ -1, & \text{if } n = 2 \\ -i, & \text{if } n = 3 \\ 1, & \text{if } n = 4 \\ i^r, & \text{if } n > 4, \text{ where } r \text{ is the remainder when } n \text{ is divided by } 4 \end{cases}$$

Step III *If $n < 0$, then $n = -m$, where $m > 0$.*

$$\therefore \quad i^n = \begin{cases} i^{-1} = \dfrac{1}{i} = -i, & \text{if } n = -1 \\ i^{-2} = \dfrac{1}{i^2} = \dfrac{1}{-1} = -1, & \text{if } n = -2 \\ i^{-3} = \dfrac{1}{i^3} = \dfrac{i}{i^4} = i, & \text{if } n = -3 \\ i^{-4} = \dfrac{1}{i^4} = \dfrac{1}{1} = 1, & \text{if } n = -4 \\ i^{-m} = \dfrac{1}{i^m} = \dfrac{1}{i^r}, & \text{where } r \text{ is the remainder when } m \text{ is divided by 4, if } n < -4. \end{cases}$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate the following:*

(i) i^{135} (ii) i^{19} (iii) i^{-999} (iv) $(-\sqrt{-1})^{4n+3}, n \in N$

SOLUTION (i) 135 leaves remainder as 3 when it is divided by 4. Therefore, $i^{135} = i^3 = -i$

(ii) The remainder is 3 when 19 is divided by 4. Therefore, $i^{19} = i^3 = -i$.

(iii) We have, $i^{-999} = 1/i^{999}$. On dividing 999 by 4, we obtain 3 as the remainder. Therefore, $i^{999} = i^3$. Hence, $i^{-999} = \dfrac{1}{i^{999}} = \dfrac{1}{i^3} = \dfrac{i}{i^4} = \dfrac{i}{1} = i$.

(iv) We have, $(-\sqrt{-1})^{4n+3} = (-i)^{4n+3} = (-i)^{4n}(-i)^3 = \{(-i)^4\}^n(-i^3) = 1 \times -i^3 = i$

EXAMPLE 2 *Show that:*

(i) $\left\{i^{19} + \left(\dfrac{1}{i}\right)^{25}\right\}^2 = -4$ (ii) $\left\{i^{17} - \left(\dfrac{1}{i}\right)^{34}\right\}^2 = 2i$

(iii) $\left\{i^{18} + \left(\dfrac{1}{i}\right)^{24}\right\}^3 = 0$ (iv) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$, *for all $n \in N$.*

SOLUTION (i) $\left\{i^{19} + \left(\dfrac{1}{i}\right)^{25}\right\}^2 = \left\{i^{19} + \dfrac{1}{i^{25}}\right\}^2 = \left\{i^3 + \dfrac{1}{i}\right\}^2 = \left\{-i + \dfrac{i^3}{i^4}\right\}^2$

$$= [-i + i^3]^2 = (-i - i)^2 = 4i^2 = -4.$$

(ii) $\left\{i^{17} - \left(\dfrac{1}{i}\right)^{34}\right\}^2 = \left\{i^{17} - \dfrac{1}{i^{34}}\right\}^2 = \left\{i - \dfrac{1}{i^2}\right\}^2 = \left\{i - \dfrac{1}{(-1)}\right\}^2 = (i+1)^2$

$$= i^2 + 2i + 1 = -1 + 2i + 1 = 2i$$

(iii) $\left\{i^{18}+\left(\frac{1}{i}\right)^{24}\right\}^{3}=\left\{i^{18}+\frac{1}{i^{24}}\right\}^{3}=\left(i^{2}+\frac{1}{1}\right)^{3}=(-1+1)^{3}=0$

(iv) $i^{n}+i^{n+1}+i^{n+2}+i^{n+3}=i^{n}+i^{n}\times i+i^{n}\times i^{2}+i^{n}\times i^{3}$

$=i^{n}(1+i+i^{2}+i^{3})=i^{n}(1+i-1-i)=i^{n}(0)=0$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Evaluate* $\sum_{n=1}^{13}(i^{n}+i^{n+1})$, *where* $n\in N$. [NCERT EXEMPLAR]

SOLUTION $\sum_{n=1}^{13}(i^{n}+i^{n+1})=\sum_{n=1}^{13}(i+1)i^{n}=(i+1)\sum_{n=1}^{13}i^{n}=(i+1)(i+i^{2}+i^{3}+\ldots+i^{13})$

$=(i+1)\times i\left(\frac{i^{13}-1}{i-1}\right)=(i^{2}+i)\left(\frac{i-1}{i-1}\right)=(-1+i)$ $[\because i^{13}=i]$

EXAMPLE 4 *Evaluate* $1+i^{2}+i^{4}+i^{6}+\ldots+i^{2n}$.

SOLUTION Let $S=1+i^{2}+i^{4}+i^{6}+\ldots+i^{2n}$. Then

$S=1+i^{2}+(i^{2})^{2}+(i^{2})^{3}+\ldots+(i^{2})^{n}$

$$\Rightarrow\quad S=1\left\{\frac{1-(i^{2})^{n+1}}{1-i^{2}}\right\}=\frac{1-(i^{2})^{n+1}}{1+1}=\frac{1}{2}\left\{1-(-1)^{n+1}\right\}=\begin{cases}\frac{1}{2}(1-1)=0, \text{ if } n \text{ is odd}\\ \frac{1}{2}(1+1)=1, \text{ if } n \text{ is even}\end{cases}$$

EXAMPLE 5 *For a positive integer n, find the value of* $(1-i)^{n}\left(1-\frac{1}{i}\right)^{n}$. [NCERT EXEMPLAR]

SOLUTION $(1-i)^{n}\left(1-\frac{1}{i}\right)^{n}=(1-i)^{n}(1+i)^{n}$ $\left[\because \frac{1}{i}=-i\right]$

$=\{(1-i)(1+i)\}^{n}=(1-i^{2})^{n}=(1+1)^{n}=2^{n}$

EXERCISE 12.1

BASIC

1. Evaluate the following:

(i) i^{457} (ii) i^{528} (iii) $\frac{1}{i^{58}}$ (iv) $i^{37}+\frac{1}{i^{67}}$

(v) $\left(i^{41}+\frac{1}{i^{257}}\right)^{9}$ (vi) $(i^{77}+i^{70}+i^{87}+i^{414})^{3}$ (vii) $i^{30}+i^{40}+i^{60}$

(viii) $i^{49}+i^{68}+i^{89}+i^{110}$

2. Show that $1+i^{10}+i^{20}+i^{30}$ is a real number.

3. Find the values of the following expressions:

(i) $i^{49}+i^{68}+i^{89}+i^{110}$ (ii) $i^{30}+i^{80}+i^{120}$

(iii) $i + i^2 + i^3 + i^4$

(iv) $i^5 + i^{10} + i^{15}$

(v) $\dfrac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$

(vi) $1 + i^2 + i^4 + i^6 + i^8 + \ldots + i^{20}$

(vii) $(1+i)^6 + (1-i)^3$ [NCERT EXEMPLAR]

ANSWERS

1. (i) i (ii) 1 (iii) -1 (iv) $2i$ (v) 0 (vi) -8 (vii) 1 (viii) $2i$
3. (i) $2i$ (ii) 1 (iii) 0 (iv) -1 (v) -1 (vi) 1 (vii) $-2-10i$

12.3 IMAGINARY QUANTITIES

The square root of a negative real number is called an imaginary quantity or an imaginary number.

For example, $\sqrt{-3}, \sqrt{-4}, \sqrt{-9/4}$ etc. are imaginary quantities.

THEOREM *If a, b are positive real numbers, then* $\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$.

PROOF We have,

$$\sqrt{-a} = \sqrt{-1 \times a} = \sqrt{-1} \times \sqrt{a} = i\sqrt{a} \text{ and, } \sqrt{-b} = \sqrt{-1 \times b} = \sqrt{-1} \times \sqrt{b} = i\sqrt{b}$$

$$\therefore \quad \sqrt{-a} \times \sqrt{-b} = (i\sqrt{a})(i\sqrt{b}) = i^2 (\sqrt{a} \times \sqrt{b}) = -1(\sqrt{ab}) = -\sqrt{ab}$$

NOTE 1 *For any two real numbers* $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ *is true only when at least one of a and b is either positive or zero. In other words,* $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ *is not valid if a and b both are negative.*

NOTE 2 *For any positive real number a, we have* $\sqrt{-a} = \sqrt{-1 \times a}\ \sqrt{-1} \times \sqrt{a} = i\sqrt{a}$.

ILLUSTRATION 1 *Compute the following:*

(i) $\sqrt{-144}$ (ii) $\sqrt{-4} \times \sqrt{\dfrac{-9}{4}}$ (iii) $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9}$

SOLUTION (i) $\sqrt{-144} = \sqrt{-1 \times 144} = \sqrt{-1} \times \sqrt{144} = 12i$

(ii) $\sqrt{-4} \times \sqrt{\dfrac{-9}{4}} = (2i)\left(\dfrac{3i}{2}\right) = 3i^2 = -3$

(iii) $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9} = 5i + 6i + 6i = 17i$

ILLUSTRATION 2 *A student writes the formula* $\sqrt{ab} = \sqrt{a}\sqrt{b}$. *Then he substitutes* $a = -1$ *and* $b = -1$ *and finds* $1 = -1$. *Explain where is he wrong?*

SOLUTION Since a and b both are negative. Therefore, $\sqrt{ab}$ cannot be written as $\sqrt{a}\sqrt{b}$. In fact, for a and b both negative, we have $\sqrt{a}\sqrt{b} = -\sqrt{ab}$.

ILLUSTRATION 3 *Is the following computation correct? If not give the correct computation:*

"$[\sqrt{(-2)} \cdot \sqrt{(-3)}] = \sqrt{(-2) \cdot (-3)} = \sqrt{6}$"

SOLUTION The said computation is not correct, because -2 and -3 both are negative and $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is true when at least one of a and b is positive or zero. The correct computation is

$$(\sqrt{-2})(\sqrt{-3}) = (i\sqrt{2})(i\sqrt{3}) = i^2\sqrt{6} = -\sqrt{6}$$

12.4 COMPLEX NUMBERS

COMPLEX NUMBER *If a, b are two real numbers, then a number of the form a + ib is called a complex number.*

For example, $7+2i, -1+i, 3-2i, 0+2i, 1+0i$ etc. are complex numbers.

Real and imaginary parts of a complex number: If $z=a+ib$ is a complex number, then 'a' is called the real part of z and 'b' is known as the imaginary part of z. The real part of z is denoted by Re (z) and the imaginary part by Im (z).

If $z=3-4i$, then $\text{Re}(z)=3$ and $\text{Im}(z)=-4$.

Purely real and purely imaginary complex numbers: A complex number z is purely real if its imaginary part is zero i.e. Im (z) = 0 and purely imaginary if its real part is zero i.e. Re (z) = 0.

Set of complex numbers: The set of all complex numbers is denoted by C i.e. $C=\{a+ib : a, b \in R\}$.

Since a real number 'a' can be written as $a+0i$. Therefore, every real number is a complex number. Hence, $R \subset C$, where R is the set of all real numbers.

12.5 EQUALITY OF COMPLEX NUMBERS

DEFINITION *Two complex numbers $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$ are equal if $a_1=a_2$ and $b_1=b_2$.*

i.e. $Re(z_1)=Re(z_2)$ and $Im(z_1)=Im(z_2)$.

Thus, $z_1=z_2 \Leftrightarrow \text{Re}(z_1)=\text{Re}(z_2)$ and $\text{Im}(z_1)=\text{Im}(z_2)$.

ILLUSTRATION 1 *If $z_1=2-iy$ and $z_2=x+3i$ are equal, find x and y.*

SOLUTION We have,

$$z_1=z_2 \Rightarrow 2-iy=x+3i \Rightarrow 2=x \text{ and } -y=3 \Rightarrow x=2 \text{ and } y=-3.$$

ILLUSTRATION 2 *If $(a+b)-i(3a+2b)=5+2i$, find a and b.*

SOLUTION We have,

$$(a+b)-i(3a+2b)=5+2i \Rightarrow a+b=5 \text{ and } -(3a+2b)=2 \Rightarrow a=-12, b=17$$

12.6 ADDITION OF COMPLEX NUMBERS

DEFINITION *Let $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$ be two complex numbers. Then their sum z_1+z_2 is defined as the complex number $(a_1+a_2)+i(b_1+b_2)$.*

It follows from this definition that the sum z_1+z_2 is a complex number such that

$$\text{Re}(z_1+z_2) = \text{Re}(z_1)+\text{Re}(z_2) \text{ and, } \text{Im}(z_1+z_2)=\text{Im}(z_1)+\text{Im}(z_2)$$

For example, If $z_1=2+3i$ and $z_2=3-2i$, then $z_1+z_2=(2+3)+(3-2)i=5+i$

12.6.1 PROPERTIES OF ADDITION OF COMPLEX NUMBERS

(i) Addition is Commutative: For any two complex numbers z_1 and z_2

$$z_1+z_2=z_2+z_1$$

PROOF Let $z_1=a_1+ib_1, z_2=a_2+ib_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$$\begin{aligned} z_1+z_2 &= (a_1+a_2)+i(b_1+b_2) && \text{[By definition of addition]} \\ &= (a_2+a_1)+i(b_2+b_1) && \text{[By commutativity of addition of real numbers]} \\ &= z_2+z_1 && \text{[By definition of addition]} \end{aligned}$$

Thus, $z_1+z_2 = z_2+z_1$ for all $z_1, z_2 \in C$.

Hence, addition of complex number is commutative.

(ii) Addition is Associative: For any three complex numbers z_1, z_2, z_3

$$(z_1+z_2)+z_3=z_1+(z_2+z_3)$$

PROOF Let $z_1=a_1+ib_1, z_2=a_2+ib_2$ and $z_3=a_3+ib_3$, where a_1, a_2, a_3 and b_1, b_2, b_3 are real numbers. Then,

$$\begin{aligned} (z_1+z_2)+z_3 &= [(a_1+a_2)+i(b_1+b_2)]+(a_3+ib_3) && \text{[By definition of addition]} \\ &= [(a_1+a_2)+a_3]+i[(b_1+b_2)+b_3] && \text{[By definition of addition]} \\ &= [(a_1+(a_2+a_3)]+i[b_1+(b_2+b_3)] && \text{[By associativity of addition on } R] \end{aligned}$$

$= (a_1 + i b_1) + [(a_2 + a_3) + i (b_2 + b_3)]$ [By definition of addition]

$= z_1 + (z_2 + z_3)$ [By definition of addition]

Thus, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ for all $z_1, z_2, z_3 \in C$.

Hence, addition of complex numbers is associative.

(iii) Existence of Additive Identity: The complex number $0 = 0 + i\,0$ is the identity element for addition i.e. $z + 0 = z = 0 + z$ for all $z \in C$.

PROOF Let $z = a + i b$ be an arbitrary complex number. Then,

$$z + 0 = (a + i b) + (0 + i\,0) = (a + 0) + i(b + 0) = a + i b = z$$

and, $$0 + z = (0 + i\,0) + (a + i b) = (0 + a) + i(0 + b) = a + i b = z$$

Thus, $z + 0 = z = 0 + z$ for all $z \in C$

Hence, the complex number $0 = 0 + i\,0$ is the identity element for addition.

(iv) Existence of Additive Inverse: For any complex number $z = a + i b$, there exists $-z = (-a) + i(-b)$ such that $z + (-z) = 0 = (-z) + z$.

PROOF Let $z = a + i b$ be an arbitrary complex number. Then, $-z = (-a) + i(-b)$ is also a complex number such that

$$z + (-z) = (a + i b) + \{(-a) + i(-b)\} = \{a + (-a)\} + i\{b + (-b)\} = 0 + i0 = 0$$

and $$(-z) + z = \{(-a) + i(-b)\} + (a + ib) = \{(-a) + a\} + i\{(-b) + b\} = 0 + i0 = 0.$$

Thus, for each complex number $z = a + i b$, there exists a complex number $-z = (-a) + i(-b)$ such that $z + (-z) = 0 = (-z) + z$.

The complex number $-z$ is called the *additive inverse of z*.

12.7 SUBTRACTION OF COMPLEX NUMBERS

DEFINITION *Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$ be two complex numbers. Then the subtraction of z_2 from z_1 is denoted by $z_1 - z_2$ and is defined as the addition of z_1 and $-z_2$.*

Thus, $z_1 - z_2 = z_1 + (-z_2) = (a_1 + ib_1) + (-a_2 - i b_2) = (a_1 - a_2) + i(b_1 - b_2)$

For example, If $z_1 = -2 + 3i$ and $z_2 = 4 + 5i$, then

$$z_1 - z_2 = (-2 + 3i) + (-4 - 5i) = (-2 - 4) + i(3 - 5) = -6 - 2i$$

12.8 MULTIPLICATION OF COMPLEX NUMBERS

Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$ be two complex numbers. Then the multiplication of z_1 with z_2 is denoted by $z_1 z_2$ and is defined as the complex number $(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$.

Thus, $$z_1 z_2 = (a_1 + i b_1)(a_2 + i b_2)$$
$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$
$$= [\text{Re}(z_1)\,\text{Re}(z_2) - \text{Im}(z_1)\,\text{Im}(z_2)] + i[\text{Re}(z_1)\,\text{Im}(z_2) + \text{Re}(z_2)\,\text{Im}(z_1)]$$

For example, If $z_1 = 3 + 2i$ and $z_2 = 2 - 3i$, then

$$z_1 z_2 = (3 + 2i)(2 - 3i) = (3 \times 2 - 2 \times (-3)) + i(3 \times -3 + 2 \times 2) = 12 - 5i$$

NOTE *The product $z_1 z_2$ can also be obtained if we actually carry out the multiplication $(a_1 + i b_1)(a_2 + i b_2)$ as given below:*

$$(a_1 + i b_1)(a_2 + i b_2) = a_1 a_2 + i a_1 b_2 + i b_1 a_2 + i^2 b_1 b_2$$
$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \quad [\because i^2 = -1]$$

12.8.1 PROPERTIES OF MULTIPLICATION

(i) Multiplication is commutative: For any two complex numbers z_1 and z_2

$$z_1 z_2 = z_2 z_1$$

PROOF Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$, where a_1, a_2, and b_1, b_2 are real numbers. Then,

$$z_1 z_2 = (a_1 + i\,b_1)(a_2 + i\,b_2) = (a_1 a_2 - b_1 b_2) + i\,(a_1 b_2 + a_2 b_1)$$

and, $$z_2 z_1 = (a_2 + i\,b_2)(a_1 + i\,b_1) = (a_2 a_1 - b_2 b_1) + i\,(b_2 a_1 + b_1 a_2)$$

$$= (a_1 a_2 - b_1 b_2) + i\,(a_1 b_2 + a_2 b_1) \quad \text{[By commutativity of mult. of real numbers]}$$

$\therefore$ $z_1 z_2 = z_2 z_1$

Thus, $z_1 z_2 = z_2 z_1$ for all $z_1, z_2 \in C$.

Hence, the multiplication of complex numbers is commutative on C.

(ii) Multiplication is associative: For any three complex numbers z_1, z_2, z_3

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

PROOF Let $z_1 = a_1 + i\,b_1$, $z_2 = a_2 + i\,b_2$ and $z_3 = a_3 + i\,b_3$ be any three complex numbers. Then,

$$\begin{aligned}(z_1 z_2) z_3 &= \{(a_1 + i\,b_1)(a_2 + ib_2)\}(a_3 + ib_3)\\
&= \{(a_1 a_2 - b_1 b_2) + i\,(a_1 b_2 + a_2 b_1)\}(a_3 + i\,b_3)\\
&= \{(a_1 a_2 - b_1 b_2)\,a_3 - (a_1 b_2 + a_2 b_1)\,b_3\} + i\,\{(a_1 a_2 - b_1 b_2)b_3 + (a_1 b_2 + a_2 b_1)\,a_3\}\\
&= \{a_1 (a_2 a_3 - b_2 b_3) - b_1 (a_2 b_3 + a_3 b_2)\} + i\,\{b_1 (a_2 a_3 - b_2 b_3) + a_1 (a_3 b_2 + a_2 b_3)\}\\
&= (a_1 + ib_1)\,\{(a_2 a_3 - b_2 b_3) + i\,(a_2 b_3 + a_3 b_2)\}\\
&= z_1 (z_2 z_3)\end{aligned}$$

Thus, $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ for all $z_1, z_2, z_3 \in C$.

Hence, multiplication is associative on C.

(iii) Existence of identity element for multiplication: The complex number $1 = 1 + i0$ is the identity element for multiplication i.e. for every complex number z, $z \cdot 1 = z = 1 \cdot z$.

PROOF Let $z = a + i\,b$. Then,

$$z \cdot 1 = (a + ib)(1 + i0) = (a \times 1 - b \times 0) + i\,(a \times 0 + 1 \times b) = a + ib.$$

Similarly, we obtain $1 \cdot z = z$

Thus, $z \cdot 1 = z = 1 \cdot z$, for all $z \in C$.

Hence, $1 = 1 + 0i$ is the multiplicative identity in C.

(iv) Existence of multiplicative inverse: Corresponding to every non-zero complex number $z = a + ib$ there exists a complex number $z_1 = x + iy$ such that $z \cdot z_1 = 1 = z_1 \cdot z$.

PROOF Clearly,

$$z \cdot z_1 = 1$$

$\Rightarrow$ $(a + ib)(x + iy) = 1 + i0 \Rightarrow (ax - by) + i\,(ay + bx) = 1 + i0 \Rightarrow ax - by = 1$ and $ay + bx = 0$.

Solving these two equations, we obtain: $x = \dfrac{a}{a^2 + b^2}$, $y = -\dfrac{b}{a^2 + b^2}$ $\quad [\because a \neq 0, b \neq 0]$

Thus, every non-zero complex number $z = a + i\,b$ possesses multiplicative inverse given by

$$\left\{\frac{a}{a^2 + b^2}\right\} + i\left\{\frac{-b}{a^2 + b^2}\right\}$$

NOTE *The multiplicative inverse of z is denoted by z^{-1}* or, $\dfrac{1}{z}$

ILLUSTRATION *Find the multiplicative inverse of $z = 3 - 2\,i$.*

SOLUTION Using the above formula, we obtain

$$z^{-1} = \frac{3}{3^2+(-2)^2} + \frac{i(-(-2))}{3^2+(-2)^2} = \frac{3}{13} + \frac{2}{13}i$$

(v) Multiplication of complex numbers is distributive over addition of complex numbers : For any three complex numbers z_1, z_2, z_3

(i) $z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$ *(Left distributivity)*

(ii) $(z_2+z_3)z_1 = z_2 z_1 + z_3 z_1$ *(Right distributivity)*

PROOF Let $z_1 = a_1 + i b_1, z_2 = a_2 + i b_2$ and $z_3 = a_3 + i b_3$. Then,

$$\begin{aligned} z_1(z_2+z_3) &= (a_1+ib_1)\{(a_2+a_3)+i(b_2+b_3)\} \\ &= \{a_1(a_2+a_3) - b_1(b_2+b_3)\} + i\{a_1(b_2+b_3)) + b_1(a_2+a_3)\} \\ &= [(a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)] + [(a_1a_3 - b_1b_3) + i(a_1b_3 + a_3b_1)] \\ &= z_1z_2 + z_1z_3 \end{aligned}$$

Similarly, it can be established that $(z_2+z_3)z_1 = z_2z_1 + z_3z_1$.

12.9 DIVISION OF COMPLEX NUMBERS

The division of a complex number z_1 *by a non-zero complex number* z_2 *is defined as the multiplication of* z_1 *by the multiplicative inverse of* z_2 *and is denoted by* $\frac{z_1}{z_2}$.

Thus, $\frac{z_1}{z_2} = z_1 z_2^{-1} = z_1\left\{\frac{1}{z_2}\right\}$

Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$. Then,

$$\frac{z_1}{z_2} = (a_1+ib_1)\left\{\frac{a_2}{a_2^2+b_2^2} + i\frac{(-b_2)}{a_2^2+b_2^2}\right\} \qquad \left[\because z = a+ib \Rightarrow \frac{1}{z} = \frac{a}{a^2+b^2} + \frac{i(-b)}{a^2+b^2}\right]$$

$$\Rightarrow \frac{z_1}{z_2} = \left(\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right) + i\left(\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}\right) \qquad \text{[By definition of multiplication]}$$

For example, If $z_1 = 2+3i$ and $z_2 = 1+2i$, then

$$\frac{z_1}{z_2} = z_1 \times \frac{1}{z_2} = (2+3i)\times\frac{1}{1+2i} = (2+3i)\frac{1}{5} - \frac{2}{5}i = \left(\frac{2}{5}+\frac{6}{5}\right) + i\left(-\frac{4}{5}+\frac{3}{5}\right) = \frac{8}{5} - \frac{1}{5}i$$

12.10 CONJUGATE OF A COMPLEX NUMBER

DEFINITION *Let* $z = a + ib$ *be a complex number. Then the conjugate of z is denoted by* $\bar{z}$ *and is equal to* $a - ib$.

Thus, $z = a+ib \Rightarrow \bar{z} = a - ib$

It follows from this definition that the conjugate of a complex number is obtained by replacing i by $-i$. For example, if $z = 3+4i$, then $\bar{z} = 3-4i$.

12.10.1 PROPERTIES OF CONJUGATE

THEOREM *If* z, z_1, z_2 *are complex numbers, then*

(i) $\overline{(\bar{z})} = z$ *(ii)* $z + \bar{z} = 2\,\mathrm{Re}(z)$ *(iii)* $z - \bar{z} = 2i\,\mathrm{Im}(z)$

(iv) $z = \bar{z} \Leftrightarrow z$ *is purely real* *(v)* $z + \bar{z} = 0 \Rightarrow z$ *is purely imaginary*

(vi) $z\bar{z} = \{\mathrm{Re}(z)\}^2 + \{\mathrm{Im}(z)\}^2$ *(vii)* $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$ *(viii)* $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(ix) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ *(x)* $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$.

PROOF Let $z = a + i\,b, z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$.

(i) $z = a + i\,b \Rightarrow \bar{z} = a - i\,b \Rightarrow (\bar{\bar{z}}) = \overline{(a - i\,b)} = a + i\,b \Rightarrow (\bar{\bar{z}}) = z.$

(ii) $z + \bar{z} = (a + i\,b) + (a - i\,b) = 2a = 2\,\mathrm{Re}\,(z)$

(iii) $z - \bar{z} = (a + i\,b) - (a - i\,b) = 2\,ib = 2\,i\,\mathrm{Im}\,(z)$

(iv) $z = \bar{z} \Leftrightarrow a + i\,b = a - i\,b \Leftrightarrow 2\,ib = 0 \Leftrightarrow b = 0 \Leftrightarrow \mathrm{Im}(z) = 0 \Rightarrow z$ is purely real

(v) $z + \bar{z} = 0 \Leftrightarrow (a + i\,b) + (a - ib) = 0 \Leftrightarrow 2a = 0 \Leftrightarrow a = 0 \Leftrightarrow \mathrm{Re}\,(z) = 0 \Leftrightarrow z$ is purely imaginary

(vi) $z\,\bar{z} = (a + ib)\,(a - ib) = a^2 + b^2 = \{\mathrm{Re}\,(z)\}^2 + \{\mathrm{Im}\,(z)\}^2$

(vii) We have, $z_1 + z_2 = (a_1 + a_2) + i\,(b_1 + b_2)$

$\therefore \quad \overline{z_1 + z_2} = (a_1 + a_2) - i\,(b_1 + b_2) = (a_1 - ib_1) + (a_2 - ib_2) = \overline{(a_1 + ib_1)} + \overline{(a_2 + ib_2)} = \bar{z}_1 + \bar{z}_2.$

(viii) We have, $z_1 - z_2 = (a_1 - a_2) + i\,(b_1 - b_2)$

$\therefore \quad \overline{z_1 - z_2} = (a_1 - a_2) - i\,(b_1 - b_2) = (a_1 - ib_1) - (a_2 - ib_2) = \overline{(a_1 + ib_1)} - \overline{(a_2 + ib_2)} = \bar{z}_1 - \bar{z}_2$

(ix) We have, $z_1\,z_2 = (a_1 + ib_1)\,(a_2 + ib_2) = (a_1\,a_2 - b_1\,b_2) + i\,(a_1\,b_2 + a_2\,b_1)$

$\therefore \quad \overline{z_1\,z_2} = (a_1\,a_2 - b_1\,b_2) - i\,(a_1\,b_2 + a_2\,b_1) = (a_1 - ib_1)\,(a_2 - ib_2) = \bar{z}_1\,\bar{z}_2$

(x) We have, $\dfrac{z_1}{z_2} = \dfrac{a_1 + ib_1}{a_2 + ib_2} = \left(\dfrac{a_1\,a_2 + b_1\,b_2}{a_2^2 + b_2^2}\right) + i\left(\dfrac{a_2\,b_1 - a_1\,b_2}{a_2^2 + b_2^2}\right)$

$$\therefore \quad \overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{a_1\,a_2 + b_1\,b_2}{a_2^2 + b_2^2}\right) - i\left(\frac{a_2\,b_1 - a_1\,b_2}{a_2^2 + b_2^2}\right) \qquad \text{...(i)}$$

$$\text{Now,} \quad \frac{\bar{z}_1}{\bar{z}_2} = \bar{z}_1 \times \frac{1}{\bar{z}_2} = (a_1 - ib_1)\left(\frac{a_2}{a_2^2 + b_2^2} + i\,\frac{b_2}{a_2^2 + b_2^2}\right) = \left(\frac{a_1\,a_2 + b_1\,b_2}{a_2^2 + b_2^2}\right) - i\left(\frac{a_2\,b_1 - a_1\,b_2}{a_2^2 + b_2^2}\right) \quad \text{... (ii)}$$

From (i) and (ii), we obtain: $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}.$

12.11 MODULUS OF A COMPLEX NUMBER

DEFINITION *The modulus of a complex number $z = a + i\,b$ is denoted by $|z|$ and is defined as*

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\{\mathrm{Re}\,(z)\}^2 + \{\mathrm{Im}\,(z)\}^2}$$

Clearly, $|z| \geq 0$ for all $z \in C$.

For example, If $z_1 = 3 - 4i, z_2 = -5 + 2i$ and $z_3 = 1 + \sqrt{-3}$, then

$$|z_1| = \sqrt{3^2 + (-4)^2} = 5,\ |z_2| = \sqrt{(-5)^2 + 2^2} = \sqrt{29} \text{ and, } |z_3| = |1 + i\sqrt{3}| = \sqrt{1^2 + (\sqrt{3})^2} = 2.$$

REMARK *In the set C of all complex numbers, the order relation is not defined. As such $z_1 > z_2$ or, $z_1 < z_2$ has no meaning but $|z_1| > |z_2|$ or, $|z_1| < |z_2|$ has got its meaning as $|z_1|$ and $|z_2|$ are real numbers.*

12.11.1 PROPERTIES OF MODULUS

THEOREM *If $z, z_1, z_2 \in C$, then*

(i) $|z| = 0 \Leftrightarrow z = 0$ i.e. $\mathrm{Re}\,(z) = \mathrm{Im}\,(z) = 0$

(ii) $|z| = |\bar{z}| = |-z|$

(iii) $-|z| \leq \mathrm{Re}\,(z) \leq |z|\,;\ -|z| \leq \mathrm{Im}\,(z) \leq |z|$

(iv) $z\,\bar{z} = |z|^2$

(v) $|z_1\,z_2| = |z_1|\,|z_2|$

(vi) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|};\ z_2 \neq 0$

(vii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\,(z_1\,\bar{z}_2)$

(viii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}\,(z_1\,\bar{z}_2)$

(ix) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

(x) $|a z_1 - b z_2|^2 + |b z_1 + a z_2|^2 = (a^2 + b^2)\left(|z_1|^2 + |z_2|^2\right)$, *where* $a, b \in R$.

[NCERT EXEMPLAR]

PROOF Let $z = a + ib$. Then,

(i) $|z| = 0 \Leftrightarrow \sqrt{a^2 + b^2} = 0 \Leftrightarrow a^2 + b^2 = 0 \Leftrightarrow a = 0$ and $b = 0 \Leftrightarrow \text{Re}(z) = \text{Im}(z) = 0$

(ii) Let $z = a + ib$. Then, $\bar{z} = a - ib$ and $-z = -a - ib$.

$\therefore$ $|z| = \sqrt{a^2 + b^2}, |\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$ and, $|-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2}$

Clearly, $|z| = |\bar{z}| = |-z|$

(iii) Let $z = a + ib$. Then, $|z| = \sqrt{a^2 + b^2}$.

Clearly, $-\sqrt{a^2 + b^2} \leq a \leq \sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2} \leq b \leq \sqrt{a^2 + b^2}$

$\Rightarrow$ $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$

(iv) Let $z = a + ib$. Then, $\bar{z} = a - ib$.

$\therefore$ $z\bar{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2 = \left\{\sqrt{a^2 + b^2}\right\}^2 = |z|^2$

(v) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

$$\Rightarrow \quad |z_1 z_2| = \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2} = \sqrt{a_1^2 a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2 + a_2^2 b_1^2}$$

$$= \sqrt{a_1^2 (a_2^2 + b_2^2) + b_1^2 (a_2^2 + b_2^2)} = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} = \sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}$$

$$= |z_1||z_2|$$

(vi) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$$\frac{z_1}{z_2} = z_1 \times \frac{1}{z_2} = (a_1 + ib_1)\left(\frac{a_2}{a_2^2 + b_2^2} + i\frac{(-b_2)}{a_2^2 + b_2^2}\right) = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) + i\left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right)^2 + \left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)^2} = \sqrt{\frac{(a_1 a_2 + b_1 b_2)^2 + (a_2 b_1 - a_1 b_2)^2}{(a_2^2 + b_2^2)^2}}$$

$$= \sqrt{\frac{a_1^2 a_2^2 + b_1^2 b_2^2 + a_2^2 b_1^2 + a_1^2 b_2^2}{(a_2^2 + b_2^2)^2}} = \sqrt{\frac{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}{(a_2^2 + b_2^2)^2}} = \sqrt{\frac{a_1^2 + b_1^2}{a_2^2 + b_2^2}}$$

$$= \frac{\sqrt{a_1^2 + b_1^2}}{\sqrt{a_2^2 + b_2^2}} = \frac{|z_1|}{|z_2|}$$

(vii) Clearly,

$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$ $[\because z\bar{z} = |z|^2]$

$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$ $[\because \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2]$

$= z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_1 \bar{z}_2 + z_2 \bar{z}_1$ [By distributivity of multiplication]

$= |z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + \overline{(z_1 \bar{z}_2)}$ $[\because \overline{(z_1 \bar{z}_2)} = \bar{z}_1 \overline{(\bar{z}_2)} = \bar{z}_1 z_2 = z_2 \bar{z}_1]$

$= |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1 \bar{z}_2)$ $[\because z + \bar{z} = 2\,\text{Re}(z)]$

(viii) Clearly,

$$|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)} \qquad [\because z\bar{z} = |z|^2]$$

$$= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \qquad [\because \overline{z_1 + z_2} = \bar{z}_1 - \bar{z}_2]$$

$$= z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_2 - z_2\bar{z}_1 \qquad \text{[By distributivity of multiplication]}$$

$$= |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \overline{(z_1\bar{z}_2)} \qquad [\because \overline{(z_1\bar{z}_2)} = \bar{z}_1(\bar{\bar{z}}_2) = \bar{z}_1 z_2]$$

$$= |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\bar{z}_2)$$

(ix) Using (vii) and (viii), we get

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\bar{z}_2)$$

$$= 2(|z_1|^2 + |z_2|^2)$$

(x) $$|az_1 - bz_2|^2 = (az_1 - bz_2)\overline{(az_1 - bz_2)}$$

$$= (az_1 - bz_2)(a\bar{z}_1 - b\bar{z}_2)$$

$$= a^2 z_1\bar{z}_1 - (az_1)(b\bar{z}_2) - (bz_2)(a\bar{z}_1) + b^2 z_2\bar{z}_2$$

$$= a^2|z_1|^2 - ab(z_1\bar{z}_2 + \bar{z}_1 z_2) + b^2|z_2|^2$$

$$= a^2|z_1|^2 - ab(z_1\bar{z}_2 + \overline{(z_1\bar{z}_2)}) + b^2|z_2|^2$$

$$= a^2|z_1|^2 - ab\{2\,\mathrm{Re}(z_1\bar{z}_2)\} + b^2|z_2|^2 \qquad [\because z_1\bar{z}_2 + \overline{(z_1\bar{z}_2)} = 2\,\mathrm{Re}(z_1\bar{z}_2)]$$

$$= a^2|z_1|^2 - 2ab\,\mathrm{Re}(z_1\bar{z}_2) + b^2|z_2|^2$$

Similarly, we obtain

$$|bz_1 + az_2|^2 = b^2|z_1|^2 + a^2|z_2|^2 + 2ab\,\mathrm{Re}(z_1\bar{z}_2)$$

$\therefore$ $$|az_1 - bz_2|^2 + |bz_1 + az_2|^2$$

$$= a^2|z_1|^2 - 2ab\,\mathrm{Re}(z_1\bar{z}_2) + b^2|z_2|^2 + b^2|z_1|^2 + a^2|z_2|^2 + 2ab\,\mathrm{Re}(z_1\bar{z}_2)$$

$$= |z_1|^2(a^2 + b^2) + |z_2|^2(b^2 + a^2)$$

$$= (a^2 + b^2)\left(|z_1|^2 + |z_2|^2\right)$$

12.12 RECIPROCAL OF A COMPLEX NUMBER

Let $z = a + ib$ be a non-zero complex number. Then,

$$\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} \qquad \left[\begin{array}{r}\text{Multiplying numerator and denominator}\\ \text{by conjugate of denominator}\end{array}\right]$$

$$\Rightarrow \quad \frac{1}{z} = \frac{a-ib}{a^2 - i^2 b^2} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} + \frac{i(-b)}{a^2+b^2}$$

Clearly, $\frac{1}{z}$ is equal to the multiplicative inverse of z. Also, $\frac{1}{z} = \frac{a-ib}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$

Thus, the multiplicative inverse of a non-zero complex number z is same as its reciprocal and is given by

$$\frac{\mathrm{Re}(z)}{|z|^2} + i\frac{(-\mathrm{Im}(z))}{|z|^2} = \frac{\bar{z}}{|z|^2}$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **EXPRESSING A COMPLEX NUMBER IN THE STANDARD FORM a + ib**

In order to express a complex number in the standard form, we may follow the following algorithm.

ALGORITHM

Step I *Write the complex number in the form* $\frac{a+ib}{c+id}$ *by using fundamental operations of addition, subtraction and multiplication.*

Step II *Multiply the numerator and denominator by the conjugate of the denominator.*

EXAMPLE 1 *Express the following in the form a + ib :*

(i) $(-5i)\left(\frac{1}{8}i\right)$ [NCERT] (ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3$ [NCERT]

(iii) $(5i)\left(-\frac{3}{5}i\right)$ [NCERT] (iv) i^9+i^{19}

(v) i^{-39} [NCERT] (vi) $(1-i)^4$ [NCERT]

SOLUTION (i) $(-5i)\left(\frac{1}{8}i\right)=-\frac{5}{8}i^2=-\frac{5}{8}\times-1=\frac{5}{8}=\frac{5}{8}+0\,i$

(ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3=-2i^2\times-\frac{1}{512}i^3=\frac{1}{256}\times i^2\times i^3=\frac{1}{256}i^5=\frac{i}{256}=0+\frac{1}{256}i$

(iii) $(5i)\left(-\frac{3}{5}i\right)=-3i^2=-3\times-1=3=3+0\,i$

(iv) $i^9+i^{19}=(i^4)^2\,i+(i^4)^4\,i^3=i+i^3=i-i=0=0+0\,i$

(v) $i^{-39}=(i^4)^{-10}\,i=i=0+1.i$

(vi) $(1-i)^4=\left\{(1-i)^2\right\}^2=(1-2i+i^2)^2=(1-2i-1)^2=(-2i)^2=4i^2=-4=-4+0i$

EXAMPLE 2 *Express each of the following in the form a + ib:*

(i) $3(7+7i)+i(7+7i)$ [NCERT] (ii) $(1-i)-(-1+6i)$ [NCERT]

(iii) $\left(\frac{1}{5}+\frac{2}{5}i\right)-\left(4+\frac{5}{2}i\right)$ [NCERT] (iv) $\left\{\left(\frac{1}{3}+\frac{7}{3}i\right)+\left(4+\frac{1}{3}i\right)\right\}-\left(-\frac{4}{3}+i\right)$ [NCERT]

SOLUTION (i) $3(7+7i)+i(7+7i)=21+21\,i+7i+7i^2=21+21\,i+7i-7=14+28\,i$

(ii) $(1-i)-(-1+6i)=1-i+1-6i=2-7i$

(iii) $\left(\frac{1}{5}+\frac{2}{5}\right)i-\left(4+\frac{5}{2}i\right)=\left(\frac{1}{5}-4\right)+\frac{2i}{5}-\frac{5i}{2}=-\frac{19}{5}-\frac{21}{10}i$

(iv)
$$\left\{\left(\frac{1}{3}+\frac{7}{3}i\right)+\left(4+\frac{1}{3}i\right)\right\}-\left(-\frac{4}{3}+i\right)=\left\{\left(\frac{1}{3}+4\right)+i\left(\frac{7}{3}+\frac{1}{3}\right)\right\}-\left(-\frac{4}{3}+i\right)$$
$$=\left(\frac{13}{3}+\frac{8}{3}i\right)+\frac{4}{3}-i$$
$$=\left(\frac{13}{3}+\frac{4}{3}\right)+\left(\frac{8}{3}-1\right)i=\frac{17}{3}+\frac{5}{3}i$$

EXAMPLE 3 *Express each of the following in the form a + ib:*

(i) $\left(\frac{1}{3}+3i\right)^3$ [NCERT] (ii) $\left(-2-\frac{1}{3}i\right)^3$ [NCERT]

(iii) $(5-3i)^3$ [NCERT] (iv) $(-\sqrt{3}+\sqrt{-2})(2\sqrt{3}-i)$

SOLUTION (i) $\left(\frac{1}{3}+3i\right)^3=\left(\frac{1}{3}\right)^3+(3i)^3+3\times\frac{1}{3}\times 3i\left(\frac{1}{3}+3i\right) = \frac{1}{27}+27i^3+3i\left(\frac{1}{3}+3i\right)$

$$= \frac{1}{27}+27i^3+i+9i^2=\frac{1}{27}-27i+i-9=-\frac{242}{27}-26i$$

(ii) $\left(-2-\frac{1}{3}i\right)^3=(-2)^3+\left(-\frac{1}{3}i\right)^3+3\times -2\times -\frac{1}{3}i\left(-2-\frac{1}{3}i\right)= -8-\frac{1}{27}i^3+2i\left(-2-\frac{1}{3}i\right)$

$$= -8+\frac{1}{27}i-4i-\frac{2}{3}i^2= -8+\frac{1}{27}i-4i+\frac{2}{3}=-\frac{22}{3}-\frac{107}{27}i$$

(iii) $(5-3i)^3=5^3+(-3i)^3+3\times 25\times -3i+3\times 5\times(-3i)^2=125+27i-225i-135=-10-198\,i$

(iv) $(-\sqrt{3}+\sqrt{-2})(2\sqrt{3}-i) = (-\sqrt{3}+i\sqrt{2})(2\sqrt{3}-i)=-6+\sqrt{3}i+2\sqrt{6}i-\sqrt{2}\,i^2$

$$=-6+(\sqrt{3}+2\sqrt{6})\,i+\sqrt{2}=(\sqrt{2}-6)+(\sqrt{3}+2\sqrt{6})\,i$$

EXAMPLE 4 *Express each one of the following in the standard form a + ib.*

(i) $\frac{1}{3-4i}$ (ii) $\frac{5+4i}{4+5i}$ (iii) $\frac{(1+i)^2}{3-i}$

(iv) $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ (v) $\frac{1}{-2+\sqrt{-3}}$ (vi) $\left(\frac{1}{1-2i}+\frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$

(vii) $\frac{1}{1-\cos\theta+2i\sin\theta}$ (viii) $\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}\,i)-(\sqrt{3}-i\sqrt{2})}$ [NCERT]

SOLUTION (i) $\frac{1}{3-4i}=\frac{1}{3-4i}\times\frac{3+4i}{3+4i}=\frac{3+4i}{9-16i^2}=\frac{3+4i}{9+16}=\frac{3}{25}+\frac{4}{25}i$

(ii) $\frac{5+4i}{4+5i}=\frac{5+4i}{4+5i}\times\frac{4-5i}{4-5i}=\frac{(20+20)+i(16-25)}{16-25i^2}=\frac{40-9i}{41}=\frac{40}{41}-\frac{9}{41}i$

(iii) $\frac{(1+i)^2}{3-i}=\frac{1+2i+i^2}{3-i}=\frac{2i}{3-i}=\frac{2i}{3-i}\cdot\frac{3+i}{3+i}=\frac{6i+2i^2}{9-i^2}=\frac{-2+6i}{10}=-\frac{1}{5}+\frac{3}{5}i$

(iv) $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}=\frac{(6+6)+i(-4+9)}{(2+2)+i(4-1)}=\frac{12+5i}{4+3i}=\frac{12+5i}{4+3i}\times\frac{4-3i}{4-3i}$

$$=\frac{(48+15)+i(-36+20)}{16-9i^2}=\frac{63}{25}-\frac{16}{25}i$$

(v) $\frac{1}{-2+\sqrt{-3}}=\frac{1}{-2+i\sqrt{3}}=\frac{1}{-2+i\sqrt{3}}\times\frac{-2-i\sqrt{3}}{-2-i\sqrt{3}}=\frac{-2-i\sqrt{3}}{4-3i^2}=-\frac{2}{7}-\frac{\sqrt{3}}{7}i$

(vi) $\left(\frac{1}{1-2i}+\frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)=\frac{1+i+3-6i}{(1+2)+i(-2+1)}\times\frac{3+4i}{2-4i}=\frac{4-5i}{3-i}\times\frac{3+4i}{2-4i}=\frac{(12+20)+i(16-15)}{(6-4)+i(-2-12)}$

$$=\frac{32+i}{2-14i}=\frac{32+i}{2-14i}\times\frac{2+14i}{2+14i}=\frac{(64-14)+i(2+448)}{4-196i^2}=\frac{50+450i}{200}=\frac{1}{4}+\frac{9}{4}i$$

(vii) $\dfrac{1}{1-\cos\theta+2i\sin\theta} = \dfrac{1}{1-\cos\theta+2i\sin\theta}\times\dfrac{1-\cos\theta-2i\sin\theta}{1-\cos\theta-2i\sin\theta}$

$$= \frac{1-\cos\theta-2i\sin\theta}{(1-\cos\theta)^2-4i^2\sin^2\theta} = \frac{1-\cos\theta-2i\sin\theta}{(1-\cos\theta)^2+4\sin^2\theta}$$

$$= \frac{1-\cos\theta-2i\sin\theta}{1-2\cos\theta+\cos^2\theta+4\sin^2\theta} = \frac{1-\cos\theta-2i\sin\theta}{2-2\cos\theta+3\sin^2\theta}$$

$$= \left(\frac{1-\cos\theta}{2-2\cos\theta+3\sin^2\theta}\right)+i\left(\frac{-2\sin\theta}{2-2\cos\theta+3\sin^2\theta}\right)$$

(viii) $\dfrac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}\,i)-(\sqrt{3}-i\sqrt{2})} = \dfrac{(9-\sqrt{5}\times-\sqrt{5})+i(3\times-\sqrt{5}+3\sqrt{5})}{\sqrt{3}+\sqrt{2}\,i-\sqrt{3}+\sqrt{2}\,i}$

$$= \frac{(9+5)+i\times 0}{2\sqrt{2}\,i} = \frac{14}{2\sqrt{2}\,i} = \frac{7}{\sqrt{2}i} = \frac{-7}{\sqrt{2}}i = 0-\frac{7}{\sqrt{2}}i$$

EXAMPLE 5 *Prove that the following complex numbers are purely real:*

(i) $\left(\dfrac{2+3i}{3+4i}\right)\left(\dfrac{2-3i}{3-4i}\right)$ (ii) $\left(\dfrac{3+2i}{2-3i}\right)+\left(\dfrac{3-2i}{2+3i}\right)$

SOLUTION (i) $\left(\dfrac{2+3i}{3+4i}\right)\left(\dfrac{2-3i}{3-4i}\right) = \dfrac{(2+3i)(2-3i)}{(3+4i)(3-4i)} = \dfrac{4-9i^2}{9-16i^2} = \dfrac{13}{25}$, which is purely real.

(ii) $\left(\dfrac{3+2i}{2-3i}\right)+\left(\dfrac{3-2i}{2+3i}\right) = \dfrac{3+2i}{2-3i}\times\dfrac{2+3i}{2+3i}+\dfrac{3-2i}{2+3i}\times\dfrac{2-3i}{2-3i} = \dfrac{(3+2i)(2+3i)}{4-9i^2}+\dfrac{(3-2i)(2-3i)}{4-9i^2}$

$$= \frac{13i}{13}-\frac{13i}{13} = 0,\ \text{which is purely real.}$$

EXAMPLE 6 *Express* $(1-2i)^{-3}$ *in the standard form* $a+ib$.

SOLUTION $(1-2i)^{-3} = \dfrac{1}{(1-2i)^3} = \dfrac{1}{1-8i^3-6i+12i^2} = \dfrac{1}{1+8i-6i-12} = \dfrac{1}{-11+2i}$

$$= \frac{1}{-11+2i}\times\frac{-11-2i}{-11-2i} = \frac{-11-2i}{(-11)^2-(2i)^2} = \frac{-11-2i}{125} = \frac{-11}{125}-\frac{2}{125}i$$

EXAMPLE 7 *Perform the suitable operations to express the result in the form* $a+ib$.

(i) $\dfrac{2-\sqrt{-25}}{1-\sqrt{-16}}$ (ii) $\dfrac{3-\sqrt{-16}}{1-\sqrt{-9}}$

SOLUTION We have,

(i) $\dfrac{2-\sqrt{-25}}{1-\sqrt{-16}} = \dfrac{2-5i}{1-4i} = \dfrac{2-5i}{1-4i}\times\dfrac{1+4i}{1+4i} = \dfrac{(2+20)+i(8-5)}{1-16i^2} = \dfrac{22+3i}{17} = \dfrac{22}{17}+\dfrac{3}{17}i$

(ii) $\dfrac{3-\sqrt{-16}}{1-\sqrt{-9}} = \dfrac{3-4i}{1-3i} = \dfrac{3-4i}{1-3i}\times\dfrac{1+3i}{1+3i} = \dfrac{(3+12)+i(-4+9)}{1-9i^2} = \dfrac{15}{10}+\dfrac{5}{10}i = \dfrac{3}{2}+\dfrac{1}{2}i$

EXAMPLE 8 *If* z_1, z_2 *are* $1-i, -2+4i$, *respectively, find* $\text{Im}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$.

SOLUTION $\dfrac{z_1 z_2}{\bar{z}_1} = \dfrac{(1-i)(-2+4i)}{(\overline{1-i})} = \dfrac{(-2+4)+i(2+4)}{1+i} = \dfrac{2+6i}{1+i}$

$$= \frac{2+6i}{1+i} \times \frac{1-i}{1-i} = \frac{(2+6)+i(6-2)}{1+1} = 4+2i$$

$$\therefore \quad \mathrm{Im}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = 2$$

Type II ON EQUALITY OF COMPLEX NUMBERS

Recall that two complex numbers z_1 and z_2 are equal iff $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

EXAMPLE 9 *Find the real values of x and y, if*

(i) $(3x-7) + 2iy = -5y + (5+x)i$ (ii) $(1-i)x + (1+i)y = 1-3i$

(iii) $(x+iy)(2-3i) = 4+i$ (iv) $\frac{x-1}{3+i} + \frac{y-1}{3-i} = i$

SOLUTION (i) We have

$$(3x-7) + 2iy = -5y + (5+x)i$$

$$\Rightarrow \quad 3x-7 = -5y \text{ and } 2y = 5+x \Rightarrow 3x+5y = 7 \text{ and } x-2y = -5 \Rightarrow x = -1, y = 2.$$

(ii) We have,

$$(1-i)x + (1+i)y = 1-3i$$

$$\Rightarrow \quad (x+y) + i(-x+y) = 1-3i \Rightarrow x+y = 1 \text{ and } -x+y = -3 \Rightarrow x = 2, y = -1$$

(iii) We have,

$$(x+iy)(2-3i) = 4+i$$

$$\Rightarrow \quad (2x+3y) + i(-3x+2y) = 4+i \Rightarrow 2x+3y = 4 \text{ and } -3x+2y = 1 \Rightarrow x = \frac{5}{13}, y = \frac{14}{13}$$

(iv) We have,

$$\frac{x-1}{3+i} + \frac{y-1}{3-i} = i$$

$$\Rightarrow \quad \frac{(x-1)(3-i)+(y-1)(3+i)}{(3+i)(3-i)} = i \Rightarrow \frac{(3x+3y-6)+i(y-x)}{9-i^2} = i$$

$$\Rightarrow \quad \left(\frac{3x+3y-6}{10}\right) + i\left(\frac{y-x}{10}\right) = 0+i \Rightarrow \frac{3x+3y-6}{10} = 0 \text{ and } \frac{y-x}{10} = 1$$

$$\Rightarrow \quad x+y-2 = 0 \text{ and } y-x = 10 \Rightarrow x = -4, \ y = 6.$$

EXAMPLE 10 *Find real values of x and y for which the following equalities hold:*

(i) $(1+i)y^2 + (6+i) = (2+i)x$ (ii) $(x^4+2xi) - (3x^2+iy) = (3-5i) + (1+2iy)$

SOLUTION (i) We have,

$$(1+i)y^2 + (6+i) = (2+i)x \Rightarrow (y^2+6) + i(y^2+1) = 2x + ix$$

$$\Rightarrow \quad y^2 + 6 = 2x \quad \text{...(i)} \quad \text{and,} \quad y^2 + 1 = x \quad \text{...(ii)}$$

From (i) and (ii), we get

$$y^2 + 6 = 2(y^2+1) \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

Substituting $y = \pm 2$ in (ii), we get $x = 5$. Thus, $x = 5$ and $y = 2$ or, $x = 5$ and $y = -2$

(ii) We have,

$$(x^4+2xi) - (3x^2+iy) = (3-5i) + (1+2iy)$$

$$\Rightarrow \quad (x^4 - 3x^2) + i(2x-y) = 4 + i(2y-5)$$

$$\Rightarrow \quad x^4 - 3x^2 = 4 \text{ and, } 2x - y = 2y - 5 \qquad \text{[On equating real and imaginary parts]}$$

$$\Rightarrow \quad x^4 - 3x^2 - 4 = 0, \ 2x - 3y + 5 = 0$$

Now, $x^4 - 3x^2 - 4 = 0 \Rightarrow (x^2-4)(x^2+1) = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$

Putting $x = \pm 2$ in $2x - 3y + 5 = 0$, we get

$y = 3$ when $x = 2$ and $y = 1/3$ when $x = -2$

Thus, $x = 2$ and $y = 3$ or, $x = -2$ and $y = 1/3$.

EXAMPLE 11 *If $a + ib = \frac{c+i}{c-i}$, where c is real, prove that: $a^2 + b^2 = 1$ and $\frac{b}{a} = \frac{2c}{c^2-1}$.*

SOLUTION We have,

$$a + ib = \frac{c+i}{c-i} \Rightarrow a + ib = \frac{(c+i)(c+i)}{(c-i)(c+i)} \Rightarrow a + ib = \frac{(c+i)^2}{c^2-i^2} \Rightarrow a + ib = \frac{c^2+2ic+i^2}{c^2-i^2}$$

$$\Rightarrow \quad a + ib = \frac{c^2-1}{c^2+1} + \frac{i\,2c}{c^2+1} \Rightarrow a = \frac{c^2-1}{c^2+1} \text{ and } b = \frac{2c}{c^2+1}$$

$$\Rightarrow \quad a^2 + b^2 = \left(\frac{c^2-1}{c^2+1}\right)^2 + \frac{4c^2}{(c^2+1)^2} \text{ and, } \frac{b}{a} = \left(\frac{2c}{c^2+1}\right) \div \left(\frac{c^2-1}{c^2+1}\right)$$

$$\Rightarrow \quad a^2 + b^2 = \frac{(c^2+1)^2}{(c^2+1)^2} = 1 \text{ and, } \frac{b}{a} = \frac{2c}{c^2-1}$$

EXAMPLE 12 *If $(x+iy)^{1/3} = a + ib$, $x, y, a\, b \in R$. Show that*

(i) $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$ (ii) $\frac{x}{a} - \frac{y}{b} = -2(a^2 + b^2)$ [NCERT EXEMPLAR]

SOLUTION We have,

$$(x+iy)^{1/3} = a + ib$$

$$\Rightarrow \quad (x+iy) = (a+ib)^3 \qquad \text{[On cubing both sides]}$$

$$\Rightarrow \quad x + iy = a^3 + 3a^2\, i\, b + 3a\, i^2\, b^2 + i^3\, b^3$$

$$\Rightarrow \quad x + i\, y = (a^3 - 3ab^2) + i(3a^2\, b - b^3)$$

$$\Rightarrow \quad x = a^3 - 3ab^2 \text{ and } y = 3a^2\, b - b^3$$

$$\Rightarrow \quad \frac{x}{a} = a^2 - 3b^2 \text{ and } \frac{y}{b} = 3a^2 - b^2$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = a^2 - 3b^2 + 3a^2 - b^2 = 4(a^2 - b^2) \text{ and } \frac{x}{a} - \frac{y}{b} = (a^2 - 3b^2) - (3a^2 - b^2) = -2(a^2 + b^2)$$

Type III ON CONJUGATE OF A COMPLEX NUMBER

EXAMPLE 13 *Multiply $3 - 2i$ by its conjugate.*

SOLUTION The conjugate of $3 - 2i$ is $3 + 2i$.

$\therefore$ Required product $= (3-2i)(3+2i) = 9 - 4i^2 = 9 + 4 = 13$

ALITER Let $z = 3 - 2i$. Then, $\bar{z} = 3 + 2i$. Therefore, $z\bar{z} = |z|^2 \Rightarrow z\bar{z} = 3^2 + (-2)^2 = 13$

EXAMPLE 14 *Find the conjugate of $\frac{1}{3+4i}$.*

SOLUTION Let $z = \frac{1}{3+4i}$. Then, $z = \frac{1}{3+4i} = \frac{1}{3+4i} \times \frac{3-4i}{3-4i} = \frac{3-4i}{9+16} = \frac{3}{25} - \frac{4}{25}i$

$$\therefore \quad \bar{z} = \frac{3}{25} + \frac{4}{25}i.$$

EXAMPLE 15 *Express the following complex numbers in the standard form. Also, find their conjugate:*

(i) $\dfrac{1-i}{1+i}$ (ii) $\dfrac{(1+i)^2}{3-i}$

(iii) $\dfrac{(2+3i)^2}{2-i}$ (iv) $\dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$ [NCERT EXEMPLAR]

SOLUTION (i) We have,

$$z = \frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{(1-i)^2}{1^2-i^2} = \frac{1-2i+i^2}{1-i^2} = \frac{1-2i-1}{1+1} = 0-i$$

$\therefore \quad \bar{z} = 0+i$

(ii) We have,

$$z = \frac{(1+i)^2}{3-i} = \frac{1+2i+i^2}{3-i} \times \frac{3+i}{3+i} = \frac{2i}{3-i} \times \frac{3+i}{3+i} = \frac{6i+2i^2}{9-i^2} = \frac{6i-2}{10} = -\frac{1}{5}+\frac{3}{5}i$$

$\therefore \quad \bar{z} = -\frac{1}{5}-\frac{3}{5}i$

(iii) We have,

$$z = \frac{(2+3i)^2}{2-i} = \frac{4+12i+9i^2}{2-i} = \frac{4+12i-9}{2-i} \times \frac{2+i}{2+i} = \frac{-5+12i}{2-i} \times \frac{2+i}{2+i} = \frac{-22+19i}{4-i^2} = -\frac{22}{5}+\frac{19}{5}i$$

$\therefore \quad \bar{z} = -\frac{22}{5}-\frac{19}{5}i$

(iv) Let $\dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$. Then,

$$z = \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}} \times \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}+\sqrt{5-12i}} = \frac{\left(\sqrt{5+12i}+\sqrt{5-12i}\right)^2}{(5+12i)-(5-12i)}$$

$$\Rightarrow \quad z = \frac{5+12i+5-12i+2\sqrt{5+12i}\,\sqrt{5-12i}}{5+12i-5+12i} = \frac{10+2\sqrt{25+144}}{24i} = \frac{3}{2i} = -\frac{3}{2}i = 0-\frac{3}{2}i$$

$\therefore \quad \bar{z} = 0+\frac{3}{2}i$

EXAMPLE 16 *Find real values of x and y for which the complex numbers $-3+ix^2y$ and x^2+y+4i are conjugate of each other.*

SOLUTION Since $-3+ix^2y$ and x^2+y+4i are complex conjugates.

$\therefore \quad -3+ix^2y = \overline{x^2+y+4i}$

$\Rightarrow \quad -3+ix^2y = x^2+y-4i$

$\Rightarrow \quad -3 = x^2+y \quad$ ...(i) $\quad$ and, $\quad x^2y = -4 \quad$ ...(ii)

$\Rightarrow \quad -3 = x^2-\dfrac{4}{x^2} \quad$ [Putting $y=-4/x^2$ from (ii) in (i)]

$\Rightarrow \quad x^4+3x^2-4 = 0$

$\Rightarrow \quad (x^2+4)(x^2-1) = 0 \Rightarrow x^2-1 = 0 \Rightarrow x = \pm 1 \quad$ [$\because$ $x^2+4 \neq 0$ for any real x]

From (ii), $y=-4$, when $x=\pm 1$. Hence, $x = 1, y = -4$ or, $x = -1, y = -4$

EXAMPLE 17 *Find the real numbers x and y, if $(x-iy)(3+5i)$ is the conjugate of $-6-24i$.* [NCERT]

SOLUTION We have, $(x-iy)(3+5i)=(3x+5y)+i(5x-3y)$

It is given that $(x-iy)(3+5i)$ is the conjugate of $-6-24i$.

$\therefore \quad (x-iy)(3+5i)=\overline{-6-24i}$

$\Rightarrow \quad (3x+5y)+i(5x-3y)=-6+24i \Rightarrow 3x+5y=-6$ and $5x-3y=24$

Solving these equations, we get $x=3,\ y=-3$.

Type IV **ON FINDING THE MULTIPLICATIVE INVERSE OR RECIPROCAL OF A NON-ZERO COMPLEX NUMBER**

EXAMPLE 18 *Find the multiplicative inverse of the following complex numbers:*

(i) $3+2i$ [NCERT] (ii) $(2+\sqrt{3}\,i)^2$

SOLUTION (i) Let $z=3+2i$. Then,

$$\frac{1}{z}=\frac{1}{3+2i}=\frac{3-2i}{(3+2i)(3-2i)}=\frac{3-2i}{9-4i^2}=\frac{3}{13}-\frac{2}{13}i$$

ALITER Let $z=3+2i$. Then, $\frac{1}{z}=\frac{\bar{z}}{|z|^2}=\frac{3-2i}{9+4}=\frac{3}{13}-\frac{2i}{13}$.

(ii) Let $z=(2+\sqrt{3}i)^2$. Then,

$$z=4+3i^2+4\sqrt{3}i=4-3+4\sqrt{3}i=1+4\sqrt{3}i$$

$$\therefore \quad \frac{1}{z}=\frac{1}{4+\sqrt{3}i}=\frac{1-4\sqrt{3}i}{(1+4\sqrt{3}i)(1-4\sqrt{3}i)}=\frac{1-4\sqrt{3}i}{1+48}=\frac{1}{49}-\frac{4\sqrt{3}i}{49}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type V **PROBLEMS BASED UPON CONJUGATE AND MODULUS OF A COMPLEX NUMBER**

EXAMPLE 19 *If* $\frac{a+ib}{c+id}=x+iy$, *prove that* $\frac{a-ib}{c-id}=x-iy$ *and* $\frac{a^2+b^2}{c^2+d^2}=x^2+y^2$.

SOLUTION We have,

$$\frac{a+ib}{c+id}=x+iy$$

$$\Rightarrow \quad \overline{\left(\frac{a+ib}{c+id}\right)}=\overline{x+iy} \qquad \text{[Taking Conjugate of both sides]}$$

$$\Rightarrow \quad \frac{\overline{a+ib}}{\overline{c+id}}=\overline{x+iy} \qquad \left[\because \overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}\right]$$

$$\Rightarrow \quad \frac{a-ib}{c-id}=x-iy$$

Thus, we have $\frac{a+ib}{c+id}=x+iy$ and $\frac{a-ib}{c-id}=x-iy$

$$\Rightarrow \quad \frac{a+ib}{c+id}\times\frac{a-ib}{c-id}=(x+iy)(x-iy) \Rightarrow \frac{(a+ib)(a-ib)}{(c+id)(c-id)}=(x+iy)(x-iy)$$

$$\Rightarrow \quad \frac{a^2+b^2}{c^2+d^2}=x^2+y^2 \qquad [\text{Using: } z\bar{z}=|z|^2]$$

EXAMPLE 20 *If* $\frac{(a+i)^2}{(2a-i)}=p+iq$, *show that:* $p^2+q^2=\frac{(a^2+1)^2}{(4a^2+1)}$.

SOLUTION We have,

$$\frac{(a+i)^2}{(2a-i)} = (p+iq) \quad ...(i)$$

$$\Rightarrow \quad \overline{\left\{\frac{(a+i)^2}{(2a-i)}\right\}} = \overline{(p+iq)} \quad \text{[Taking conjugate of both sides]}$$

$$\Rightarrow \quad \frac{\overline{(a+i)}^2}{\overline{(2a-i)}} = \overline{(p+iq)} \Rightarrow \frac{(a-i)^2}{(2a+i)} = p-iq \quad ...(ii)$$

Multiplying (i) and (ii), we obtain:

$$\frac{(a+i)^2}{(2a-i)} \times \frac{(a-i)^2}{(2a+i)} = (p+iq)(p-iq)$$

$$\Rightarrow \quad \frac{\{(a+i)(a-i)\}^2}{(2a-i)(2a+i)} = (p+iq)(p-iq) \Rightarrow \frac{(a^2+1)^2}{4a^2+1} = p^2+q^2 \quad [\text{Using : } z\bar{z} = |z|^2]$$

EXAMPLE 21 *If $a+ib = \dfrac{(x+i)^2}{2x^2+1}$, prove that $a^2+b^2 = \dfrac{(x^2+1)^2}{(2x^2+1)^2}$.* [NCERT]

SOLUTION We have,

$$a+ib = \frac{(x+i)^2}{2x^2+1} \quad ...(i)$$

$$\Rightarrow \quad \overline{a+ib} = \overline{\left\{\frac{(x+i)^2}{2x^2+1}\right\}} \quad \text{[Taking conjugate of both sides]}$$

$$\Rightarrow \quad \overline{a+ib} = \frac{\overline{(x+i)}^2}{(2x^2+1)} \Rightarrow a-ib = \frac{(x-i)^2}{2x^2+1} \quad ...(ii)$$

Multiplying (i) and (ii), we get

$$(a+ib)(a-ib) = \frac{(x+i)^2(x-i)^2}{(2x^2+1)(2x^2+1)}$$

$$\Rightarrow \quad a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2} \quad \left[\because (x+i)(x-i) = x^2 - i^2 = x^2+1\right]$$

EXAMPLE 22 *If $x+iy = \sqrt{\dfrac{a+ib}{c+id}}$, prove that: $(x^2+y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}$.* [NCERT]

SOLUTION We have,

$$x+iy = \sqrt{\frac{a+ib}{c+id}} \Rightarrow x-iy = \sqrt{\frac{a-ib}{c-id}} \quad \text{[Taking conjugate of both sides]}$$

$$\therefore \quad (x+iy)(x-iy) = \sqrt{\frac{a+ib}{c+id}} \times \sqrt{\frac{a-ib}{c-id}} = \sqrt{\frac{a+ib}{c+id} \times \frac{a-ib}{c-id}}$$

$$\Rightarrow \quad x^2+y^2 = \sqrt{\frac{a^2+b^2}{c^2+d^2}} \Rightarrow (x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$$

EXAMPLE 23 *Find the least positive value of n, if $\left(\dfrac{1+i}{1-i}\right)^n = 1$.* [NCERT]

SOLUTION We have,

$$\frac{1+i}{1-i}=\frac{1+i}{1-i}\times\frac{1+i}{1+i}=\frac{(1+i)^2}{1-i^2}=\frac{1+2i+i^2}{1-i^2}=\frac{1+2i-1}{1+1}=i$$

$\therefore \quad \left(\frac{1+i}{1-i}\right)^n=1 \Rightarrow i^n=1 \Rightarrow n$ is a multiple of $4 \Rightarrow$ The smallest positive value of n is 4.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 24 *Find real θ such that $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is purely real.* [NCERT]

SOLUTION Clearly,

$$\frac{3+2i\sin\theta}{1-2i\sin\theta}=\frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$$

$$=\frac{(3-4\sin^2\theta)+i(6\sin\theta+2\sin\theta)}{1+4\sin^2\theta}$$

$$=\frac{(3-4\sin^2\theta)+i(6\sin\theta+2\sin\theta)}{1+4\sin^2\theta}=\frac{3-4\sin^2\theta}{1+4\sin^2\theta}+\frac{i\,8\sin\theta}{1+4\sin^2\theta}$$

It is given that $\frac{3+2\sin\theta}{1-2i\sin\theta}$ is purely real. Therefore, its imaginary part is zero.

i.e. $\quad \frac{8\sin\theta}{1+4\sin^2\theta}=0 \Rightarrow \sin\theta=0 \Rightarrow \theta=n\pi, n\in Z$

EXAMPLE 25 *The sum and product of two complex numbers are real if and only if they are conjugate of each other.*

SOLUTION First, let the two complex numbers be conjugate of each other. Let complex numbers be $z_1=a+ib$ and $z_2=a-ib$. Then,

$z_1+z_2=(a+ib)+(a-ib)=2a$, which is real.

And, $z_1 z_2=(a+ib)(a-ib)=a^2-i^2b^2=a^2+b^2$, which is also real.

Thus, if z_1 and z_2 are conjugate of each other. Then, Their sum z_1+z_2 and product z_1z_2 both are real.

Conversely, let z_1 and z_2 be two complex numbers such that their sum z_1+z_2 and product $z_1 z_2$ both are real. Then, we have to prove that z_1 and z_2 are conjugate of each other.

Let $\quad z_1=a_1+ib_1$ and $z_2=a_2+ib_2$. Then,

$z_1+z_2=(a_1+a_2)+i(b_1+b_2)$ and $z_1z_2=(a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1)$

Now, $\quad z_1+z_2$ and z_1z_2 are real

$\Rightarrow \quad (a_1+a_2)+i(b_1+b_2)$ and, $(a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1)$ are real

$\Rightarrow \quad b_1+b_2=0$ and $a_1b_2+a_2b_1=0$ $\quad$ [$\because$ z is real $\Leftrightarrow$ Im $(z)=0$]

$\Rightarrow \quad b_2=-b_1$ and $a_1b_2+a_2b_1=0$

$\Rightarrow \quad b_2=-b_1$ and $-a_1b_1+a_2b_1=0$

$\Rightarrow \quad b_2=-b_1$ and $(a_2-a_1)b_1=0 \Rightarrow b_2=-b_1$ and $a_2-a_1=0 \Rightarrow b_2=-b_1$ and $a_2=a_1$

$\therefore \quad z_2=a_2+ib_2=a_1-ib_1 \Rightarrow z_2=\bar{z}_1 \Rightarrow z_1$ and z_2 are conjugate of each other

EXAMPLE 26 *If $(1+i)(1+2i)(1+3i)\ldots(1+ni)=(x+iy)$, show that: $2.5.10\ldots(1+n^2)=x^2+y^2$.*

SOLUTION We have,

$(1+i)(1+2i)(1+3i)\ \ldots\ (1+ni)=x+iy$

$\Rightarrow \quad |(1+i)(1+2i)\ldots(1+ni)|=|x+iy|$ $\quad$ [Taking modulus of both sides]

$\Rightarrow \quad |1+i|\ |1+2i| \ldots |1+ni| = |x+iy| \qquad [\because |z_1 z_2 \ldots z_n| = |z_1||z_2| \ldots |z_n|]$

$\Rightarrow \quad \sqrt{1+1}\ \sqrt{1+4} \ldots \sqrt{1+n^2} = \sqrt{x^2+y^2}$

$\Rightarrow \quad 2.5.10 \ldots (1+n^2) = (x^2+y^2)$ [On squaring both sides]

EXAMPLE 27 *If* $(a+ib)(c+id)(e+if)(g+ih) = A+iB$, *prove that*
$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$

SOLUTION We have,

$(a+ib)(c+id)(e+if)(g+ih) = A+iB$

$\Rightarrow \quad |(a+ib)(c+id)(e+if)(g+ih)| = |A+iB|$ [Taking modulus of both sides]

$\Rightarrow \quad |a+ib|\ |c+id|\ |e+if|\ |g+ih| = |A+iB|$ [Using: $|z_1 z_2 \ldots z_n| = |z_1|\ |z_2| \ldots |z_n|$]

$\Rightarrow \quad \sqrt{a^2+b^2}\ \sqrt{c^2+d^2}\ \sqrt{c^2+f^2}\ \sqrt{g^2+h^2} = \sqrt{A^2+B^2}$

$\Rightarrow \quad (a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$ [On squaring both sides]

EXAMPLE 28 *If* z_1, z_2 *are complex numbers such that* $\frac{2z_1}{3z_2}$ *is purely imaginary number, then find* $\left|\frac{z_1-z_2}{z_1+z_2}\right|$.

SOLUTION It is given that $\frac{2z_1}{3z_2}$ is purely imaginary. Therefore,

$$\frac{2z_1}{3z_2} = \lambda i \text{ for some } \lambda \in R \Rightarrow \frac{z_1}{z_2} = \frac{3\lambda}{2} i$$

Now, $$\left|\frac{z_1-z_2}{z_1+z_2}\right| = \left|\frac{\frac{z_1}{z_2}-1}{\frac{z_1}{z_2}+1}\right| = \left|\frac{\frac{3}{2}\lambda i - 1}{\frac{3}{2}\lambda i + 1}\right| = \frac{|-2+3\lambda i|}{|2+3\lambda i|} = \frac{\sqrt{4+9\lambda^2}}{\sqrt{4+9\lambda^2}} = 1 \qquad \left[\because \frac{z_1}{z_2} = \frac{3\lambda}{2} i\right]$$

Type VI ON FINDING THE VALUE OF A POLYNOMIAL FOR A GIVEN COMPLEX VALUE OF THE VARIABLE

EXAMPLE 29 *If* $x = -5+2\sqrt{-4}$, *find the value of* $x^4+9x^3+35x^2-x+4$.

SOLUTION We have, $x = -5+2\sqrt{-4}$

$\Rightarrow \quad x+5 = 4i \Rightarrow (x+5)^2 = 16i^2 \Rightarrow x^2+10x+25 = -16 \Rightarrow x^2+10x+41 = 0$

$\therefore \quad x^4+9x^3+35x^2-x+4 = x^2(x^2+10x+41) - x(x^2+10x+41) + 4(x^2+10x+41) - 160$

$= x^2(0) - x(0) + 4(0) - 160 = -160 \qquad [\because x^2+10x+41 = 0]$

Thus, the value of the given polynomial for $x = -5+2\sqrt{-4}$ is -160.

EXAMPLE 30 *Find the value of* x^3+7x^2-x+16, *when* $x = 1+2i$.

SOLUTION We have,

$x = 1+2i \Rightarrow x-1 = 2i \Rightarrow (x-1)^2 = 4i^2 \Rightarrow x^2-2x+1 = -4 \Rightarrow x^2-2x+5 = 0$

$\therefore \quad x^3+7x^2-x+16 = x(x^2-2x+5) + 9(x^2-2x+5) + (12x-29)$

$= x(0) + 9(0) + 12x - 29 \qquad [\because x^2-2x+5 = 0]$

$= 12(1+2i) - 29 = -17+24i \qquad [\because x = 1+2i]$

Hence, the value of the given polynomial when $x = 1+2i$ is $-17+24i$.

EXAMPLE 31 *Find the value of* $2x^4+5x^3+7x^2-x+41$, *when* $x=-2-\sqrt{3}i$. [NCERT EXEMPLAR]

SOLUTION We have, $x=-2-\sqrt{3}i$

$\Rightarrow \quad x+2=-\sqrt{3}i \Rightarrow (x+2)^2=3i^2 \Rightarrow x^2+4x+4=-3 \Rightarrow x^2+4x+7=0.$

$\therefore \quad 2x^4+5x^3+7x^2-x+41$

$=2x^2(x^2+4x+7)-3x(x^2+4x+7)+5(x^2+4x+7)+6$

$=2x^2\times 0-3x\times 0+5\times 0+6=6 \qquad [\because x^2+4x+7=0]$

EXAMPLE 32 *If* $z=2-3i$, *show that* $z^2-4z+13=0$ *and hence find the value of* $4z^3-3z^2+169$.

SOLUTION We have,

$z=2-3i \Rightarrow z-2=-3i \Rightarrow (z-2)^2=(-3i)^2 \Rightarrow z^2-4z+4=9i^2 \Rightarrow z^2-4z+13=0$

$\therefore \quad 4z^3-3z^2+169=4z(z^2-4z+13)+13(z^2-4z+13)=4z(0)+13(0)=0 \qquad [\because z^2-4z+13=0]$

Type VII MISCELLANEOUS PROBLEMS

EXAMPLE 33 *Prove that:* $x^4+4=(x+1+i)(x+1-i)(x-1+i)(x-1-i)$.

SOLUTION We have, $(x+1+i)(x+1-i)(x-1+i)(x-1-i)$

$=\{(x+1)^2-i^2\}\{(x-1)^2-i^2\}=\{(x+1)^2+1\}\{(x-1)^2+1\}=\{x^2+2x+2\}\{x^2-2x+2\}$

$=\{x^2+2+2x\}\{x^2+2-2x\}=(x^2+2)^2-(2x)^2=x^4+4x^2+4-4x^2=x^4+4$

EXAMPLE 34 *If* $z\,(\neq 1)$ *is a complex number such that* $|z|=1$, *prove that* $\dfrac{z-1}{z+1}$ *is purely imaginary. What will be your conclusion if* $z=1$?

SOLUTION Let $z=x+iy$. Then, $|z|=1 \Rightarrow \sqrt{x^2+y^2}=1 \Rightarrow x^2+y^2=1$

$$\therefore \quad \frac{z-1}{z+1}=\frac{x+iy-1}{x+iy+1}=\frac{(x-1)+iy}{(x+1)+iy}=\frac{(x-1)+iy}{(x+1)+iy}\times\frac{(x+1)-iy}{(x+1)-iy}$$

$$=\frac{(x^2-1+y^2)+i(xy+y-xy+y)}{(x+1)^2+y^2}=\frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$

$$=\frac{2iy}{(x+1)^2+y^2}, \text{ which is purely imaginary} \qquad [\because x^2+y^2=1]$$

Now, $z=1 \Rightarrow x+iy=1+i.0 \Rightarrow x=1$ and $y=0$.

$$\therefore \quad \frac{z-1}{z+1}=\frac{x+iy-1}{x+iy+1}=\frac{1+i0-1}{1+i0+1}=0, \text{ which is purely real}$$

EXAMPLE 35 *If* $z=x+iy$ *and* $w=\dfrac{1-iz}{z-i}$, *show that* $|w|=1 \Rightarrow z$ *is purely real.*

SOLUTION We have, $|w|=1 \Rightarrow \left|\dfrac{1-iz}{z-i}\right|=1 \Rightarrow \dfrac{|1-iz|}{|z-i|}=1 \Rightarrow |1-iz|=|z-i|$

$\Rightarrow \quad |1-i(x+iy)|=|x+iy-i|$, where $z=x+iy$

$\Rightarrow \quad |1+y-ix|=|x+i(y-1)|$

$\Rightarrow \quad \sqrt{(1+y)^2+(-x)^2}=\sqrt{x^2+(y-1)^2}$

$\Rightarrow \quad (1+y)^2+x^2=x^2+(y-1)^2 \Rightarrow y=0 \Rightarrow z=x+i0=x$, which is purely real

EXAMPLE 36 *Show that a real value of x will satisfy the equation* $\frac{1-ix}{1+ix} = a - ib$ *if* $a^2 + b^2 = 1$, *where a, b are real.*

SOLUTION We have, $\frac{1-ix}{1+ix} = \frac{a-ib}{1}$

Applying componendo and dividendo, we obtain

$$\frac{(1-ix)+(1+ix)}{(1-ix)-(1+ix)} = \frac{a-ib+1}{a-ib-1} \Rightarrow \frac{2}{-2ix} = \frac{1+a-ib}{-(1-a+ib)} \Rightarrow ix = \frac{1-a+ib}{1+a-ib}$$

$$\Rightarrow \quad ix = \frac{(1-a+ib)}{(1+a-ib)} \times \frac{(1+a+ib)}{(1+a+ib)} = \frac{1-a^2-b^2+2ib}{(1+a)^2-i^2b^2} = \frac{1-a^2-b^2+2ib}{(1+a)^2+b^2}$$

$$\Rightarrow \quad ix = \frac{2ib}{(1+a)^2+b^2}, \text{ if } a^2+b^2 = 1 \Rightarrow x = \frac{2b}{(1+a)^2+b^2}, \text{ which is real}$$

EXAMPLE 37 *If* α and β *are different complex numbers with* $|\beta| = 1$, *find* $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$. [NCERT]

SOLUTION Clearly,

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2 = \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\overline{\left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)} = \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\frac{\bar{\beta}-\bar{\alpha}}{1-\alpha\bar{\beta}}\right) = \frac{(\beta-\alpha)(\bar{\beta}-\bar{\alpha})}{(1-\bar{\alpha}\beta)(1-\alpha\bar{\beta})}$$

$$= \frac{\beta\bar{\beta}-\beta\bar{\alpha}-\alpha\bar{\beta}+\alpha\bar{\alpha}}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+\bar{\alpha}\beta\alpha\bar{\beta}} = \frac{|\beta|^2-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+(\alpha\bar{\alpha})(\beta\bar{\beta})}$$

$$= \frac{|\alpha|^2-\alpha\bar{\beta}-\bar{\alpha}\beta+|\beta|^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2|\beta|^2} = \frac{|\alpha|^2-\alpha\bar{\beta}-\bar{\alpha}\beta+1}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2} = 1 \qquad [\because |\beta| = 1]$$

$$\therefore \quad \left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = 1.$$

EXAMPLE 38 *If* $|z_1| = |z_2| = \ldots = |z_n| = 1$, *prove that* $|z_1+z_2+z_3+\ldots+z_n| = \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right|$.

[NCERT EXEMPLAR]

SOLUTION We find that

$$|z_1+z_2+z_3+\ldots+z_n| = \left|\frac{z_1\bar{z}_1}{\bar{z}_1}+\frac{z_2\bar{z}_2}{\bar{z}_2}+\frac{z_3\bar{z}_3}{\bar{z}_3}+\ldots+\frac{z_n\bar{z}_n}{\bar{z}_n}\right| = \left|\frac{|z_1|^2}{\bar{z}_1}+\frac{|z_2|^2}{\bar{z}_2}+\frac{|z_3|^2}{\bar{z}_3}+\ldots+\frac{|z_n|^2}{\bar{z}_n}\right|$$

$$= \left|\frac{1}{\bar{z}_1}+\frac{1}{\bar{z}_2}+\frac{1}{\bar{z}_3}+\ldots+\frac{1}{\bar{z}_n}\right| \qquad [\because |z_1| = |z_2| = \ldots = |z_n| = 1]$$

$$= \left|\overline{\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right)}\right| = \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right| \qquad [\because |z| = |\bar{z}|]$$

EXAMPLE 39 *Find non-zero integral solutions of* $|1-i|^x = 2^x$. [NCERT]

SOLUTION We have, $|1-i|^x = 2^x$

$$\Rightarrow \quad (\sqrt{2})^x = 2^x \Rightarrow 2^{x/2} = 2^x \Rightarrow 2^{x/2} = 1 \Rightarrow 2^{x/2} = 2^0 \Rightarrow \frac{x}{2} = 0 \Rightarrow x = 0.$$

Hence, the given equation has no non-zero integral solution.

EXAMPLE 40 *Find all non-zero complex numbers z satisfying* $\bar{z} = i z^2$.

SOLUTION Let $z = x + iy$. Then,

$$\bar{z} = i z^2$$

$\Rightarrow x - iy = i(x^2 - y^2 + 2i\,xy) \Rightarrow x - iy = i(x^2 - y^2) - 2xy \Rightarrow (x + 2xy) - i(x^2 - y^2 + y) = 0$

$\Rightarrow x + 2xy = 0$ ...(i) and, $x^2 - y^2 + y = 0$ ...(ii)

Now,

$$x + 2xy = 0 \Rightarrow x(1 + 2y) = 0 \Rightarrow x = 0 \text{ or, } 1 + 2y = 0 \Rightarrow x = 0 \text{ or, } y = -\frac{1}{2}$$

Case I When $x = 0$: Putting $x = 0$ in (ii), we obtain

$$-y^2 + y = 0 \Rightarrow y(y - 1) = 0 \Rightarrow y = 0, y = 1$$

Thus, the pairs of values of x and y are: $x = 0, y = 0; x = 0, y = 1$

$\therefore \quad z = 0 + i\,0 = 0, \; z = 0 + 1\,i = i$

Case II When $y = -\frac{1}{2}$: Putting $y = -\frac{1}{2}$ in (ii), we get

$$x^2 - y^2 + y = 0 \Rightarrow x^2 - \frac{1}{4} - \frac{1}{2} = 0 \Rightarrow x^2 - \frac{3}{4} = 0 \Rightarrow x = \pm\frac{\sqrt{3}}{2}$$

Thus, the pairs of values of x and y are: $x = \frac{\sqrt{3}}{2}, y = \frac{-1}{2}$ and, $x = \frac{-\sqrt{3}}{2}, y = \frac{-1}{2}$

$\therefore \quad z = \frac{\sqrt{3}}{2} - \frac{1}{2}i, \; z = \frac{-\sqrt{3}}{2} - \frac{1}{2}i$

Hence, $z = 0, i, \frac{\sqrt{3}}{2} - \frac{1}{2}i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

EXAMPLE 41 *If* $iz^3 + z^2 - z + i = 0$, *then show that* $|z| = 1$.

SOLUTION We have, $iz^3 + z^2 - z + i = 0$

Dividing both sides by i, we obtain

$$z^3 - iz^2 + iz + 1 = 0 \Rightarrow z^2(z - i) + i(z - i) = 0 \Rightarrow (z - i)(z^2 + i) = 0 \Rightarrow z = i \text{ or, } z^2 = -i$$

Now, $z = i \Rightarrow |z| = |i| = 1$ and, $z^2 = -i \Rightarrow |z^2| = |-i| = 1 \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1.$

Hence, in either case, we obtain: $|z| = 1$.

EXAMPLE 42 *Solve the equation* $z^2 + |z| = 0$, *where z is a complex number.*

SOLUTION Let $z = x + iy$. Then,

$$z^2 + |z| = 0 \Rightarrow (x + iy)^2 + \sqrt{x^2 + y^2} = 0 \Rightarrow (x^2 - y^2) + \sqrt{x^2 + y^2} + 2i\,xy = 0$$

$\Rightarrow \quad x^2 - y^2 + \sqrt{x^2 + y^2} = 0$ ...(i) and, $2xy = 0$ ...(ii)

Now, $2xy = 0 \Rightarrow xy = 0 \Rightarrow x = 0$ or, $y = 0$

Case I *When* $y = 0$: Putting $y = 0$ in (i), we get

$$x^2 + \sqrt{x^2} = 0 \Rightarrow x^2 + |x| = 0$$

Clearly, $x^2 + |x| > 0$ for all $x > 0$. So, let $x < 0$. In this case, we have

$$x^2 + |x| = 0$$

$\Rightarrow \quad x^2 - x = 0$ $\quad [\because x < 0 \; \therefore |x| = -x]$

$\Rightarrow \quad x(x - 1) = 0 \Rightarrow x = 0, x = 1$

But, $x<0$. So, the equation $x^2+|x|=0$ has no solution for $x<0$.

Clearly, $x=0$ satisfies the equation $x^2+|x|=0$. Thus, we have $x=0,\ y=0$. Therefore, $z=0$

Case I *When* $x=0$: Putting $x=0$ in (i), we get

$$-y^2+\sqrt{y^2}=0 \Rightarrow -y^2+|y|=0$$

If $y>0$, then $|y|=y$.

$\therefore$ $$-y^2+|y|=0 \Rightarrow -y^2+y=0 \Rightarrow y=0, y=1 \Rightarrow y=1 \qquad [\because y>0]$$

If $y<0$, then $|y|=-y$.

$$-y^2+|y|=0 \Rightarrow -y^2-y=0 \Rightarrow y=0,-1 \Rightarrow y=-1 \qquad [\because y<0]$$

Thus, we obtain $x=0,\ y=1$ or, $x=0,\ y=-1$. Therefore, $z=0+i$ or, $z=0-i$.

Hence, $z=0, i$ and $-i$ are solutions of $z^2+|z|=0$.

EXAMPLE 43 *Solve the equation* $z^2=\bar{z}$. [NCERT EXEMPLAR]

SOLUTION Let $z=x+iy$. Then,

$$z^2=\bar{z}$$

$$\Rightarrow \quad (x+iy)^2=x-iy \Rightarrow x^2+2ixy+(iy)^2=x-iy \Rightarrow (x^2-y^2)+2ixy=x-iy$$

$$\Rightarrow \quad x^2-y^2=x \quad \ldots(i) \quad \text{and,} \quad 2xy=-y \quad \ldots(ii)$$

Now, $2xy=-y \Rightarrow (2x+1)\,y=0 \Rightarrow 2x+1=0$ or $y=0 \Rightarrow x=-\frac{1}{2}$ or $y=0$

Following cases arise :

Case I *When* $y=0$: Putting $y=0$ in (i), we obtain

$$x^2=x \Rightarrow x(x-1)=0 \Rightarrow x=0 \text{ or, } x=1$$

Thus, we obtain ($x=0$ and $y=0$) or ($x=1$ and $y=0$). Therefore, $z=0+i0=0$ or, $z=1+i0$.

Case II *When* $x=-\frac{1}{2}$: Putting $x=-\frac{1}{2}$ in (i), we obtain

$$\frac{1}{4}-y^2=-\frac{1}{2} \Rightarrow y^2=\frac{3}{4} \Rightarrow y=\pm\frac{\sqrt{3}}{2}$$

Thus, we obtain $\left(x=-\frac{1}{2} \text{ and } y=\frac{\sqrt{3}}{2}\right)$ or $\left(x=-\frac{1}{2} \text{ and } y=-\frac{\sqrt{3}}{2}\right)$.

$\therefore$ $$z=-\frac{1}{2}+i\frac{\sqrt{3}}{2} \text{ or } z=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

Hence the values of z satisfying the given equation are

$$z=0+i0, z=1+i0, z=-\frac{1}{2}+i\frac{\sqrt{3}}{2} \text{ and } z=-\frac{1}{2}-\frac{i\sqrt{3}}{2}$$

EXAMPLE 44 *Solve the equation* $|z+1|=z+2(1+i)$. [NCERT EXEMPLAR]

SOLUTION Let $z=x+iy$. Then, $z+1=(x+1)+iy$ and, $|z+1|=\sqrt{(x+1)^2+y^2}$

Given that:

$$|z+1|=z+2(1+i)$$

$$\Rightarrow \quad \sqrt{(x+1)^2+y^2}=(x+iy)+2(1+i) \Rightarrow \sqrt{(x+1)^2+y^2}+0i=(x+2)+(y+2)i$$

$$\Rightarrow \quad \sqrt{(x+1)^2+y^2}=x+2 \text{ and } y+2=0 \Rightarrow (x+1)^2+y^2=(x+2)^2 \text{ and } y=-2$$

$\Rightarrow \quad y^2 = 2x+3$ and $y=-2 \Rightarrow 4 = 2x+3$ and $y=-2 \Rightarrow x = \frac{1}{2}$ and $y=-2$

Hence, $z = \frac{1}{2} - 2i$

EXAMPLE 45 *If $|z^2-1| = |z|^2+1$, then show that z lies on the imaginary axis.* [NCERT EXEMPLAR]

SOLUTION Let $z = x+iy$. Then, $z^2 = x^2-y^2+2ixy$ and $|z|^2 = x^2+y^2$.

$\therefore \quad |z^2-1| = |z|^2+1$

$\Rightarrow \quad \left|(x^2-y^2)+2i\,xy\right| = x^2+y^2+1$

$\Rightarrow \quad \sqrt{(x^2-y^2-1)^2+4x^2y^2} = x^2+y^2+1 \Rightarrow (x^2-y^2-1)^2+4x^2y^2 = (x^2+y^2+1)^2$

$\Rightarrow \quad x^4+y^4+1-2x^2+2y^2-2x^2y^2+4x^2y^2 = x^4+y^4+1+2x^2y^2+2x^2+2y^2$

$\Rightarrow \quad 4x^2 = 0 \Rightarrow x = 0$

$\therefore \quad z = x+iy = 0+iy$, which is purely imaginary.

Thus, z is purely imaginary and hence it lies on y-axis.

EXAMPLE 46 *If the imaginary part of $\frac{2z+1}{iz+1}$ is –2, then show that the locus of the point representing z in the argand plane is a straight line.* [NCERT EXEMPLAR]

SOLUTION Let $z = x+iy$. Then,

$$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+i2y}{(1-y)+ix} = \frac{(2x+1)+i2y}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix}$$

$$= \frac{(2x+1-y)+i(2y-2y^2-2x^2-x)}{(1-y)^2+x^2} = \left\{\frac{2x+1-y}{x^2+(1-y)^2}\right\} + i\left\{\frac{2y-2y^2-2x^2-x}{x^2+(1-y)^2}\right\}$$

$\therefore \quad \text{Im}\left(\frac{2z+1}{iz+1}\right) = \frac{2y-2y^2-2x^2-x}{x^2+(1-y)^2}$. But, it is given that $\text{Im}\left(\frac{2z+1}{iz+1}\right) = -2$.

$\therefore \quad \frac{2y-2y^2-2x^2-x}{x^2+(1-y)^2} = -2 \Rightarrow 2y-2y^2-2x^2-x = -2x^2-2(1-y)^2 \Rightarrow x+2y-2=0,$

which is a straight line. Hence, the locus of z is a starlight line.

EXAMPLE 47 *If the real part of $\frac{\bar{z}+2}{\bar{z}-1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.* [NCERT EXEMPLAR]

SOLUTION Let $z = x+iy$. Then, $\bar{z} = x-iy$

$$\therefore \quad \frac{\bar{z}+2}{\bar{z}-1} = \frac{x-iy+2}{x-iy-1} = \frac{(x+2)-iy}{(x-1)-iy} = \frac{(x+2)-iy}{(x-1)-iy} \times \frac{(x-1)+iy}{(x-1)+iy}$$

$$= \frac{(x^2+y^2+x-2)+3iy}{(x-1)^2+y^2} = \left\{\frac{x^2+y^2+x-2}{(x-1)^2+y^2}\right\} + i\left\{\frac{3y}{(x-1)^2+y^2}\right\}$$

It is given that the real part of $\frac{\bar{z}+2}{\bar{z}-1}$ is 4.

$\therefore \quad \frac{x^2+y^2+x-2}{(x-1)^2+y^2} = 4 \Rightarrow 3x^2+3y^2-9x+6=0 \Rightarrow x^2+y^2-3x+2=0,$ which represents a circle.

EXAMPLE 48 *If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + a = 0$, where $a \in R$, represents a circle.* [NCERT EXEMPLAR]

SOLUTION We have, $z = x + iy \Rightarrow \bar{z} = x - iy$

$\therefore \; z\bar{z} + 2(z + \bar{z}) + a = 0$

$\Rightarrow (x + iy)(x - iy) + 2(x + iy + x - iy) + a = 0$

$\Rightarrow x^2 + y^2 + 4x + a = 0 \Rightarrow (x+2)^2 + (y-0)^2 = (\sqrt{4-a})^2$, which represents a circle for all $a \le 4$.

EXAMPLE 49 *Show that $\left|\dfrac{z-2}{z-3}\right| = 2$ represents a circle. Find its centre and radius.* [NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then,

$$\left|\frac{z-2}{z-3}\right| = 2 \Rightarrow \left|\frac{(x-2)+iy}{(x-3)+iy}\right| = 2 \Rightarrow |(x-2)+iy| = 2|(x-3)+iy|$$

$$\Rightarrow \sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2}$$

$$\Rightarrow (x-2)^2 + y^2 = 4\left\{(x-3)^2 + y^2\right\}$$

$$\Rightarrow 3x^2 + 3y^2 - 20x + 32 = 0$$

$$\Rightarrow x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

$\Rightarrow \left(x - \dfrac{10}{3}\right)^2 + (y-0)^2 = \left(\dfrac{2}{3}\right)^2$, which represents a circle with centre at $\left(\dfrac{10}{3}, 0\right)$ and radius $\dfrac{2}{3}$

EXAMPLE 50 *Find a complex number z satisfying the equation $z + \sqrt{2}|z+1| + i = 0$.* [NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then,

$$z + \sqrt{2}|z+1| + i = 0$$

$\Rightarrow x + iy + \sqrt{2}|(x+1) + iy| + i = 0$ $\quad\Rightarrow x + \sqrt{2}\sqrt{(x+1)^2 + y^2} + (y+1)i = 0$

$\Rightarrow x + \sqrt{2(x+1)^2 + 2y^2} = 0$ and $(y+1) = 0$ $\quad\Rightarrow x + \sqrt{2(x+1)^2 + 2y^2} = 0$ and $y = -1$

$\Rightarrow x + \sqrt{2(x+1)^2 + 2} = 0$ and $y = -1$ $\quad\Rightarrow \sqrt{2(x+1)^2 + 2} = -x$ and $y = -1$

$\Rightarrow 2(x+1)^2 + 2 = x^2$ and $y = -1$ $\quad\Rightarrow x^2 + 4x + 4 = 0$ and $y = -1$

$\Rightarrow (x+2)^2 = 0$ and $y = -1$ $\quad\Rightarrow x = -2$ and $y = -1$

Hence, $z = x + iy = -2 - i$.

EXAMPLE 51 *Let z_1 and z_2 be two complex numbers such that*

$|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = k\left(1 - |z_1|^2\right)\left(1 - |z_2|^2\right)$. *Find the value of k.* [NCERT EXEMPLAR]

SOLUTION We find that

$$\begin{aligned}|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 &= (1 - \bar{z}_1 z_2)\overline{(1 - \bar{z}_1 z_2)} - (z_1 - z_2)\overline{(z_1 - z_2)} \\ &= (1 - \bar{z}_1 z_2)(1 - z_1 \bar{z}_2) - (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= (1 - \bar{z}_1 z_2 - z_1 \bar{z}_2 + \bar{z}_1 z_2 z_1 \bar{z}_2) - (z_1 \bar{z}_1 - z_1 \bar{z}_2 - \bar{z}_1 z_2 + z_2 \bar{z}_2) \\ &= 1 - \bar{z}_1 z_2 - z_1 \bar{z}_2 + (z_1 \bar{z}_1)(z_2 \bar{z}_2) - \left(|z_1|^2 - z_1 \bar{z}_2 - \bar{z}_1 z_2 + |z_2|^2\right)\end{aligned}$$

$$=1-\bar{z}_1 z_2 - z_1\bar{z}_2 + |z_1|^2|z_2|^2 - |z_1|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 - |z_2|^2$$

$$=1-|z_1|^2-|z_2|^2+|z_1|^2|z_2|^2 = (1-|z_1|^2)(1-|z_2|^2)$$

$\therefore \quad |1-\bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = k(1-|z|^2)(1-|z_2|^2)$

$\Rightarrow \quad \left(1-|z_1|^2\right)\left(1-|z_2|^2\right) = k\left(1-|z_1|^2\right)\left(1-|z_2|^2\right) \Rightarrow k=1.$

EXERCISE 12.2

BASIC

1. Express the following complex numbers in the standard form $a + ib$:

(i) $(1+i)(1+2i)$ (ii) $\frac{3+2i}{-2+i}$ (iii) $\frac{1}{(2+i)^2}$

(iv) $\frac{1-i}{1+i}$ (v) $\frac{(2+i)^3}{2+3i}$ (vi) $\frac{(1+i)(1+\sqrt{3}i)}{1-i}$

(vii) $\frac{2+3i}{4+5i}$ (viii) $\frac{(1-i)^3}{1-i^3}$ (ix) $(1+2i)^{-3}$

(x) $\frac{3-4i}{(4-2i)(1+i)}$ (xi) $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ [NCERT]

(xii) $\frac{5+\sqrt{2}i}{1-\sqrt{2}i}$ [NCERT]

2. Find the real values of x and y, if

(i) $(x+iy)(2-3i) = 4+i$ (ii) $(3x-2iy)(2+i)^2 = 10(1+i)$

(iii) $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$ (iv) $(1+i)(x+iy) = 2-5i$

3. Find the conjugates of the following complex numbers:

(i) $4-5i$ (ii) $\frac{1}{3+5i}$ (iii) $\frac{1}{1+i}$ (iv) $\frac{(3-i)^2}{2+i}$

(v) $\frac{(1+i)(2+i)}{3+i}$ (vi) $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ [NCERT]

4. Find the multiplicative inverse of the following complex numbers:

(i) $1-i$ (ii) $(1+i\sqrt{3})^2$ (iii) $4-3i$ (iv) $\sqrt{5}+3i$

5. If $z_1 = 2-i$, $z_2 = 1+i$, find $\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right|$. [NCERT]

6. If $z_1 = 2-i$, $z_2 = -2+i$, find (i) $\mathrm{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$ (ii) $\mathrm{Im}\left(\frac{1}{z_1\bar{z}_1}\right)$ [NCERT]

7. Find the modulus of the complex number $z = \frac{1+i}{1-i} - \frac{1-i}{1+i}$ [NCERT]

BASED ON LOTS

8. If $x+iy = \frac{a+ib}{a-ib}$, prove that $x^2+y^2=1$ [NCERT]

9. Find the least positive integral value of n for which $\left(\frac{1+i}{1-i}\right)^n$ is real.

10. Find the real values of θ for which the complex number $\frac{1+i\cos\theta}{1-2i\cos\theta}$ is purely real.

11. Find the smallest positive integer value of n for which $\frac{(1+i)^n}{(1-i)^{n-2}}$ is a real number.

12. If $\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=x+iy$, find (x, y) [NCERT EXEMPLAR]

13. If $\frac{(1+i)^2}{2-i}=x+iy$, find $x+y$. [NCERT EXEMPLAR]

14. If $\left(\frac{1-i}{1+i}\right)^{100}=a+ib$, find (a, b). [NCERT EXEMPLAR]

15. If $a=\cos\theta+i\sin\theta$, find the value of $\frac{1+a}{1-a}$. [NCERT EXEMPLAR]

BASED ON HOTS

16. Evaluate the following:
(i) $2x^3+2x^2-7x+72$, when $x=\frac{3-5i}{2}$ (ii) $x^4-4x^3+4x^2+8x+44$, when $x=3+2i$
(iii) $x^4+4x^3+6x^2+4x+9$, when $x=-1+i\sqrt{2}$ (iv) $x^6+x^4+x^2+1$, when $x=\frac{1+i}{\sqrt{2}}$.

17. Find the number of solutions of $z^2+|z|^2=0$. [NCERT EXEMPLAR]
18. If $(1+i)z=(1-i)\bar{z}$, then show that $z=-i\bar{z}$. [NCERT EXEMPLAR]
19. Solve the system of equations $\mathrm{Re}(z^2)=0$, $|z|=2$. [NCERT EXEMPLAR]
20. If $\frac{z-1}{z+1}$ is purely imaginary number $(z\neq-1)$, find the value of $|z|$. [NCERT EXEMPLAR]
21. If z_1 is a complex number other than -1 such that $|z_1|=1$ and $z_2=\frac{z_1-1}{z_1+1}$, then show that the real part of z_2 is zero. [NCERT EXEMPLAR]
22. If $|z+1|=z+2(1+i)$, find z. [NCERT EXEMPLAR]
23. Solve the equation $|z|=z+1+2i$. [NCERT EXEMPLAR]
24. What is the smallest positive integer n for which $(1+i)^{2n}=(1-i)^{2n}$?
25. If z_1, z_2, z_3 are complex numbers such that $|z_1|=|z_2|=|z_3|=\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=1$, then find the value of $|z_1+z_2+z_3|$. [NCERT EXEMPLAR]

ANSWERS

1. (i) $-1+3i$ (ii) $-\frac{4}{5}-\frac{7}{5}i$ (iii) $\frac{3}{25}-\frac{4}{25}i$ (iv) $-i$
(v) $\frac{37}{13}+\frac{16}{13}i$ (vi) $-\sqrt{3}+i$ (vii) $\frac{23}{41}+\frac{2}{41}i$ (viii) $-2+0i$
(ix) $\frac{-11}{125}+\frac{2i}{125}$ (x) $\frac{1}{4}-\frac{3}{4}i$ (xi) $\frac{307}{442}+i\frac{599}{442}$ (xii) $1+2\sqrt{2}i$

2. (i) $x=\frac{5}{13}$, $y=\frac{14}{13}$ (ii) $x=\frac{14}{15}$, $y=\frac{1}{5}$ (iii) $x=3$, $y=-1$ (iv) $x=-\frac{3}{2}$, $y=-\frac{7}{2}$

3. (i) $4+5i$ (ii) $\frac{1}{34}(3+5i)$ (iii) $\frac{1}{2}+\frac{1}{2}i$ (iv) $2+4i$

(v) $\frac{3}{5}-\frac{4}{5}i$ (vi) $\frac{63}{25}+\frac{16}{25}i$

4. (i) $\frac{1}{2}+\frac{1}{2}i$ (ii) $-\frac{1}{8}-i\frac{\sqrt{3}}{8}$ (iii) $\frac{4}{25}+\frac{3}{25}i$ (iv) $\frac{\sqrt{5}}{14}-\frac{3i}{14}$

5. $\frac{4}{\sqrt{2}}$ 6. (i) $-\frac{2}{5}$ (ii) 0 7. 2 9. $n=2$

10. $\theta=2n\pi\pm\frac{\pi}{2}, n\in Z$ 11. 1 12. $(0,-2)$ 13. $\frac{2}{5}$ 14. $(1,0)$

15. $i\cot\frac{\theta}{2}$ 16. (i) 4 (ii) 5 (iii) 12 (iv) 0

17. Infinitely many solutions of the form $z=0+iy,\ y\in R$. 19. $\sqrt{2}(1\pm i), \sqrt{2}(-1\pm i)$

20. 1 22. $\frac{1}{2}-2i$ 23. $\frac{3}{2}-2i$ 24. $n=2$ 25. 1

HINTS TO SELECTED PROBLEMS

12. $\frac{1+i}{1-i}=\frac{(1+i)^2}{(1-i)(1+i)}=\frac{1+2i-1}{2}=i$ and $\frac{1-i}{1+i}=\frac{(1-i)^2}{(1-i)(1+i)}=\frac{1-2i-1}{2}=-i$

$\therefore \left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=x+iy$

$\Rightarrow i^3-(-i)^3=x+iy \Rightarrow i^3+i^3=x+iy \Rightarrow 2i^3=x+iy \Rightarrow 0-2i=x+iy \Rightarrow x=0, y=-2$

13. $\frac{(1+i)^2}{2-i}=x+iy \Rightarrow \frac{1+2i+i^2}{2-i}=x+iy \Rightarrow \frac{2i}{2-i}=x+iy$

$\Rightarrow \frac{2i(2+i)}{(2+i)(2-i)}=x+iy \Rightarrow \frac{4i+2i^2}{4-i^2}=x+iy \Rightarrow \frac{-2+4i}{5}=x+iy \Rightarrow -\frac{2}{5}+\frac{4}{5}i=x+iy$

$\Rightarrow x=-\frac{2}{5}, y=\frac{4}{5}$

14. $\left(\frac{1-i}{1+i}\right)^{100}=a+ib \Rightarrow (-i)^{100}=a+ib \Rightarrow 1+0i=a+ib \Rightarrow a=1, b=0$

15. We have, $a=\cos\theta+i\sin\theta$

$\therefore \frac{1+a}{1-a}=\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}=\left(\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}\right)\times\left(\frac{1-\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta}\right)$

$\Rightarrow \frac{1+a}{1-a}=\frac{(1-\cos^2\theta-\sin^2\theta)+2i\sin\theta}{(1-\cos\theta)^2+\sin^2\theta}=\frac{i\sin\theta}{1-\cos\theta}=\frac{2i\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}=i\cot\frac{\theta}{2}$

17. Let $z=x+iy$. Then, $z^2=x^2-y^2+2ixy$ and $|z|^2=x^2+y^2$.

$\therefore z^2+|z|^2=0$

$\Rightarrow x^2-y^2+2i\,xy+x^2+y^2=0 \Rightarrow 2x^2+2ixy=0 \Rightarrow 2x^2=0$ and $2xy=0 \Rightarrow x=0$ and $y\in R$

$\therefore z=0+iy$, where $y\in R$.

18. We have, $(1+i)z=(1-i)\bar{z}$

$$\Rightarrow \quad \frac{z}{\bar{z}} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{1-2i+i^2}{1+i^2} = -i \Rightarrow z = -i\bar{z}$$

19. Let $z = x + iy$. Then, $z^2 = x^2 - y^2 + 2ixy$ and $|z| = \sqrt{x^2 + y^2}$.

$\therefore$ $\text{Re}(z^2) = 0$ and $|z| = 2 \Rightarrow x^2 - y^2 = 0$ and $x^2 + y^2 = 4 \Rightarrow x^2 = y^2 = 2 \Rightarrow x = \pm\sqrt{2}, y = \pm\sqrt{2}$

$\therefore$ $z = \pm\sqrt{2} \pm \sqrt{2}i$

20. Let $z = x + iy$. Then,

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$

If $\frac{z-1}{z+1}$ is purely imaginary, then

$$\text{Re}\left(\frac{z-1}{z+1}\right) = 0 \Rightarrow \frac{x^2+y-1}{(x+1)^2+y^2} = 0 \Rightarrow x^2 + y^2 = 1 \Rightarrow |z| = 1$$

21. Let $z_1 = x + iy$. Then,

$$z_2 = \frac{z_1 - 1}{z_1 + 1} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2} \Rightarrow \text{Re}(z_2) = \frac{x^2+y^2-1}{(x+1)^2+y^2} = 0 \qquad [\because |z_1| = 1 \Rightarrow x^2 + y^2 = 1]$$

22. Let $z = x + iy$. Then, $z + 1 = (x+1) + iy$ and $z + 2(1+i) = (x+2) + i(y+2)$

$\therefore$ $|z+1| = z + 2(1+i)$

$\Rightarrow$ $\sqrt{(x+1)^2 + y^2} = (x+2) + i(y+2) \Rightarrow \sqrt{(x+1)^2+y^2} = x + 2$ and $y + 2 = 0$

$\Rightarrow$ $(x+1)^2 + y^2 = (x+2)^2$ and $y = -2 \Rightarrow y^2 = 2x + 3$ and $y = -2$

$\Rightarrow$ $4 = 2x + 3$ and $y = -2 \Rightarrow x = \frac{1}{2}$ and $y = -2 \Rightarrow z = x + iy = \frac{1}{2} - 2i$

23. Let $z = x + iy$. Then, $|z| = \sqrt{x^2 + y^2}$ and $z + 1 + 2i = (x+1) + i(y+2)$.

$\therefore$ $|z| = z + 1 + 2i \Rightarrow \sqrt{x^2+y^2} = (x+1) + i(y+2) \Rightarrow \sqrt{x^2+y^2} = x + 1$ and $y + 2 = 0$

$\Rightarrow$ $x^2 + y^2 = (x+1)^2$ and $y = -2 \Rightarrow x^2 + 4 = (x+1)^2$ and $y = -2 \Rightarrow x = \frac{3}{2}$ and $y = -2$

Hence, $z = x + iy = \frac{3}{2} - 2i$

24. $(1+i)^{2n} = (1-i)^{2n} \Rightarrow \{(1+i)^2\}^n = \{(1-i)^2\}^n \Rightarrow (1 + 2i + i^2)^n = (1 - 2i + i^2)^n$

$\Rightarrow$ $(2i)^n = (-2i)^n \Rightarrow i^n = (-1)^n i^n \Rightarrow (-1)^n = 1 \Rightarrow n$ is a multiple of 2.

12.13 SQUARE ROOTS OF A COMPLEX NUMBER

Let $a + ib$ be a complex number such that $\sqrt{a+ib} = x + iy$, where x and y are real numbers. Then,

$$\sqrt{a+ib} = x + iy \Rightarrow (a+ib) = (x+iy)^2 \Rightarrow a + ib = (x^2 - y^2) + 2ixy$$

On equating real and imaginary parts, we obtain

$$x^2 - y^2 = a \quad \text{...(i)} \quad \text{and,} \quad 2xy = b \quad \text{...(ii)}$$

Now, $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$

$\Rightarrow$ $(x^2 + y^2)^2 = a^2 + b^2 \Rightarrow (x^2 + y^2) = \sqrt{a^2 + b^2}$ $\qquad [\because x^2 + y^2 > 0]$ ...(iii)

Solving equations (i) and (ii), we get

$$x^2 = \frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\} \text{ and } y^2 = \frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} \text{ and } y = \pm\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}$$

If b is positive, then from equation (ii), we find that x and y are of the same sign.

$$\therefore \quad \sqrt{a+ib} = \pm\left[\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right]$$

If b is negative, then from equation (ii), we find that x and y are of different signs.

$$\therefore \quad \sqrt{a+ib} = \pm\left[\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} - i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\right]$$

REMARK *It is evident from the above discussion that for any complex number z, we have*

(i) $\sqrt{z} = \pm\left\{\sqrt{\frac{|z|+\text{Re}(z)}{2}} + i\sqrt{\frac{|z|-\text{Re}(z)}{2}}\right\}$, if $\text{Im}(z) > 0$

(ii) $\sqrt{z} = \pm\left\{\sqrt{\frac{|z|+\text{Re}(z)}{2}} - i\sqrt{\frac{|z|-\text{Re}(z)}{2}}\right\}$, if $\text{Im}(z) < 0$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the square roots of the following complex numbers:* (i) $7-24i$ (ii) $5+12i$

SOLUTION (i) Let $\sqrt{7-24i} = x+iy$. Then,

$$\sqrt{7-24i} = x+iy \Rightarrow 7-24i = (x+iy)^2 \Rightarrow 7-24i = (x^2-y^2)+2ixy$$

$$\Rightarrow \quad x^2-y^2 = 7 \quad \text{...(i)} \quad \text{and,} \quad 2xy = -24 \quad \text{...(ii)}$$

$$\therefore \quad (x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$

$$\Rightarrow \quad (x^2+y^2)^2 = 49+576 = 625 \Rightarrow x^2+y^2 = 25 \quad [\because x^2+y^2>0] \quad \text{...(iii)}$$

On solving (i) and (iii), we obtain:

$$x^2 = 16 \text{ and } y^2 = 9 \Rightarrow x = \pm 4 \text{ and } y = \pm 3$$

From (ii) we observe that $2xy$ is negative. So, x and y are of opposite signs.

$$\therefore \quad (x = 4 \text{ and } y = -3) \text{ or, } (x = -4 \text{ and } y = 3)$$

Hence, $\sqrt{7-24i} = x+iy = \pm(4-3i)$

ALITER Let $z = 7-24i$. Then, $\text{Re}(z) = 7$ and $|z| = \sqrt{49+576} = 25$. We find that $\text{Im}(z) < 0$.

$$\therefore \quad \sqrt{7-24i} = \pm\left\{\sqrt{\frac{|z|+\text{Re}(z)}{2}} - i\sqrt{\frac{|z|-\text{Re}(z)}{2}}\right\} = \pm\left\{\sqrt{\frac{25+7}{2}} - i\sqrt{\frac{25-7}{2}}\right\} = \pm(4-3i)$$

(ii) Let $\sqrt{5+12i} = x+iy$. Then,

$$\sqrt{5+12i} = x+iy \Rightarrow 5+12i = (x+iy)^2 \Rightarrow 5+12i = (x^2-y^2)+2ixy$$

$$\Rightarrow \quad x^2-y^2 = 5 \quad \text{...(i)} \quad \text{and,} \quad 2xy = 12 \quad \text{...(ii)}$$

$$\therefore \quad (x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$

$$\Rightarrow \quad (x^2+y^2)^2 = 5^2+12^2 = 169 \Rightarrow x^2+y^2 = 13 \quad [\because x^2+y^2>0] \quad \text{...(iii)}$$

On solving (i) and (iii), we get

$$x^2 = 9 \text{ and } y^2 = 4 \Rightarrow x = \pm 3 \text{ and } y = \pm 2$$

From (ii) we observe that $2xy$ is positive. So, x and y are of the same sign.

$\therefore \quad (x = 3 \text{ and } y = 2) \text{ or, } (x = -3 \text{ and } y = -2)$

Hence, $\sqrt{5+12i} = x + iy = \pm(3+2i)$.

ALITER Let $z = 5 + 12i$. Then, $\text{Re}(z) = 5$, and $|z| = \sqrt{25+144} = 13$. We find that $\text{Im}(z) > 0$.

$$\therefore \quad \sqrt{5+12i} = \pm\left\{\sqrt{\frac{|z|+\text{Re}(z)}{2}} + i\sqrt{\frac{|z|-\text{Re}(z)}{2}}\right\} = \pm\left\{\sqrt{\frac{13+5}{2}} + i\sqrt{\frac{13-5}{2}}\right\} = \pm(3+2i)$$

EXAMPLE 2 *Find the square roots of $-15-8i$.*

SOLUTION Let $\sqrt{-15-8i} = x + iy$. Then,

$$\sqrt{-15-8i} = x + iy \Rightarrow -15-8i = (x+iy)^2 \Rightarrow -15-8i = (x^2-y^2) + 2i\,xy$$

$$\Rightarrow \quad -15 = x^2 - y^2 \quad \text{...(i)} \quad \text{and,} \quad 2xy = -8 \quad \text{...(ii)}$$

$$\therefore \quad (x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$

$$\Rightarrow \quad (x^2+y^2)^2 = (-15)^2 + 64 = 289 \Rightarrow x^2 + y^2 = 17 \quad \text{...(iii)}$$

On solving (i) and (iii), we get

$$x^2 = 1 \text{ and } y^2 = 16 \Rightarrow x = \pm 1 \text{ and } y = \pm 4$$

From (ii), we observe that $2xy$ is negative. So, x and y are of opposite signs.

$\therefore \quad (x = 1 \text{ and } y = -4) \text{ or, } (x = -1 \text{ and } y = 4)$

Hence, $\sqrt{-15-8i} = x + iy = \pm(1-4i)$

EXAMPLE 3 *Find the square root of i.*

SOLUTION Let $\sqrt{i} = x + iy$. Then,

$$\sqrt{i} = x + iy \Rightarrow i = (x+iy)^2 \Rightarrow (x^2-y^2) + 2i\,xy = 0 + i$$

$$\Rightarrow \quad x^2 - y^2 = 0 \quad \text{...(i)} \quad \text{and,} \quad 2xy = 1 \quad \text{...(ii)}$$

$$\therefore \quad (x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$

$$\Rightarrow \quad (x^2+y^2)^2 = 0 + 1 = 1 \Rightarrow x^2 + y^2 = 1 \quad [\because x^2+y^2 > 0] \quad \text{...(iii)}$$

Solving (i) and (iii), we get

$$x^2 = 1/2 \text{ and } y^2 = 1/2 \Rightarrow x = \pm 1/\sqrt{2} \text{ and } y = \pm 1/\sqrt{2}$$

From equation (ii) we find that $2xy$ is positive. So, x and y are of same sign.

$$\therefore \quad \left(x = \frac{1}{\sqrt{2}} \text{ and } y = \frac{1}{\sqrt{2}}\right) \text{ or, } \left(x = -\frac{1}{\sqrt{2}} \text{ and } y = -\frac{1}{\sqrt{2}}\right)$$

Hence, $\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \pm\frac{1}{\sqrt{2}}(1+i)$

ALITER Let $z = i$. Then, $\text{Re}(z) = 0$ and $|z| = 1$. We find that $\text{Im}(z) > 0$.

$$\therefore \sqrt{i} = \pm\left\{\sqrt{\frac{|z|+\text{Re}(z)}{2}} + i\sqrt{\frac{|z|-\text{Re}(z)}{2}}\right\} = \pm\left\{\sqrt{\frac{1+0}{2}} + i\sqrt{\frac{1-0}{2}}\right\} = \pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2}}(1+i)$$

EXERCISE 12.3

BASIC

1. Find the square root of the following complex numbers:
(i) $-5+12i$ (ii) $-7-24i$ (iii) $1-i$ (iv) $-8-6i$
(v) $8-15i$ (vi) $-11-60\sqrt{-1}$ (vii) $1+4\sqrt{-3}$ (viii) $-i$

ANSWERS

1. (i) $\pm(2+3i)$ (ii) $\pm(3-4i)$ (iii) $\pm\left\{\left(\sqrt{\frac{\sqrt{2}+1}{2}}\right)-\left(\sqrt{\frac{\sqrt{2}-1}{2}}\right)i\right\}$ (iv) $\pm(1-3i)$
(v) $\pm\frac{1}{\sqrt{2}}(5-3i)$ (vi) $\pm(5-6i)$ (vii) $\pm(2+\sqrt{3}i)$ (viii) $\pm\frac{1}{\sqrt{2}}(1-i)$

12.14 REPRESENTATIONS OF A COMPLEX NUMBER

A complex number can be represented in the following forms:
(i) Geometrical form (ii) Vectorial form (iii) Trigonometrical form or, Polar form
In this section, we shall learn about these three representations of a complex number.

12.14.1 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

A complex number $z = x + iy$ can be represented by a point (x, y) on the plane which is known as the Argand plane. To represent $z = x + iy$ geometrically we take two mutually perpendicular straight lines $X'OX$ and $Y'OY$. Now plot a point whose x and y coordinates are respectively the real and imaginary parts of z. This point $P(x, y)$ represents the complex number $z = x + iy$.

If a complex number is purely real, then its imaginary part is zero. Therefore, a purely real number is represented by a point on x-axis. A purely imaginary complex number is represented by a point on y-axis. That is why x-axis is known as the *real axis* and y-axis, as the imaginary axis.

Conversely, *if* $P(x, y)$ is a point in the plane, then the point $P(x, y)$ represents a complex number $z = x + iy$. The complex number $z = x + iy$ is known as the *affix* of the point P.

Thus, there exists a one-one correspondence between the points of the plane and the members (elements) of the set C of all complex numbers, *i.e.*, for every complex number $z = x + iy$ there exists uniquely a point (x, y) on the plane and for every point (x, y) of the plane there exists uniquely a complex number $z = x + iy$.

The plane in which we represent a complex number geometrically is known as the **complex plane** or **Argand plane** or the **Gaussian plane**. The point P, plotted on the Argand plane, is called the Argand diagram.

The length of the line segment OP is called the *modulus* of z and is denoted by $|z|$.
From Fig. 12.1, we obtain

$$OP^2 = OM^2 + MP^2 \Rightarrow OP^2 = x^2 + y^2 \Rightarrow OP = \sqrt{x^2+y^2}$$

Thus, $|z| = \sqrt{x^2+y^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$

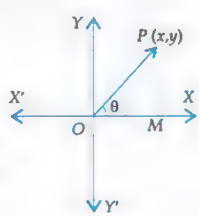


Fig.12.1

The angle θ which OP makes with positive direction of x-axis in anticlockwise sense is called the *argument or amplitude of* z and is denoted by arg(z) or amp(z).
From Fig. 12.1, we have

$$\tan\theta = \frac{PM}{OM} = \frac{y}{x} = \frac{\text{Im}(z)}{\text{Re}(z)} \Rightarrow \theta = \tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)$$

This angle θ has infinitely many values differing by multiples of 2π. The unique value of θ such that $-\pi < \theta \le \pi$ is called the *principal value of the amplitude or principal argument*. This formula for

determining the argument of $z = x + iy$ has severe drawback, because $z_1 = 1 + i\sqrt{3}$ and $z_2 = -1 - i\sqrt{3}$ are two distinct complex numbers represented by two distinct points in the Argand plane but their arguments seem to be $\tan^{-1}\sqrt{3} = \pi/3$ or $4\pi/3$ which is not correct. In fact the argument is the common solution of the simultaneous trigonometric equations

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and,} \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

Since the above system of equations has infinitely many solutions. Therefore, there can be infinitely many arguments of $z = x + iy$. The argument θ which satisfies the inequality $-\pi < \theta \le \pi$ is usually known as the *principal argument* of z. The argument of z depends upon the quadrant in which the point P lies as discussed below.

12.14.2 ARGUMENT OR AMPLITUDE OF A COMPLEX NUMBER FOR DIFFERENT SIGNS OF REAL AND IMAGINARY PARTS

(i) *Argument of $z = x + iy$ when $x > 0$ and $y > 0$:* Since x and y both are positive, therefore the point $P(x, y)$ representing $z = x + iy$ in the Argand plane lies in the first quadrant. Let θ be the argument of z and let α be the acute angle satisfying $\tan\alpha = |y/x|$. Then it is evident from Fig. 12.2 that $\theta = \alpha$.

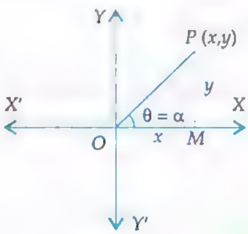


Fig.12.2

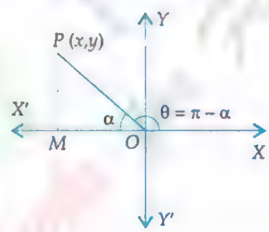


Fig.12.3

Thus, if x and y both are positive, then the argument of $z = x + iy$ is the acute angle given by $\tan\alpha = \dfrac{y}{x}$.

(ii) *Argument of $z = x + iy$ when $x < 0$ and $y > 0$:* In this case, the point $P(x, y)$ representing $z = x + iy$ in the Argand plane lies in the second quadrant. Let θ be the argument of z and let α be the acute angle satisfying $\tan\alpha = |y/x|$. Then it is evident from Fig. 12.3 that $\theta = \pi - \alpha$.

Thus, if $x < 0$ and $y > 0$, then the argument of $z = x + iy$ is $\pi - \alpha$, where α is the acute angle given by $\tan\alpha = \left|\dfrac{y}{x}\right|$.

(iii) *Argument of $z = x + iy$ when $x < 0$ and $y < 0$:* In this case, the point $P(x, y)$ representing $z = x + iy$ lies in the third quadrant. Let θ be the argument of z and α be the acute angle given by $\tan\alpha = |y/x|$. Then from Fig. 12.4, we obtain $\theta = -(\pi - \alpha) = \alpha - \pi$.

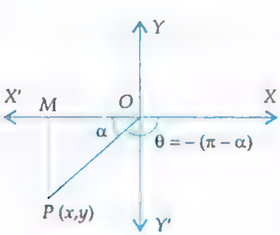


Fig.12.4

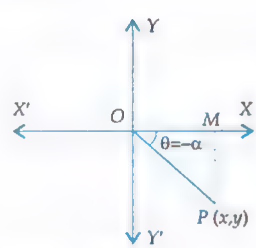


Fig.12.5

Thus, if $x < 0$ and $y < 0$ then the argument of $z = x + iy$ is $\alpha - \pi$ where α is the acute angle given by $\tan\alpha = |y/x|$.

(iv) *Argument of* $z = x + iy$ *when* $x > 0$ *and* $y < 0$: In this case, the point $P(x, y)$ representing $z = x + iy$ lies in the fourth quadrant. Let θ be the argument of z and let α be the acute angle given by $\tan\alpha = |y/x|$. Then from Fig. 12.5, we obtain $\theta = -\alpha$.

Thus, if $x > 0$ *and* $y < 0$, *then the argument of* $z = x + iy$ *is* $-\alpha$ *where* α *is the acute angle given by* $\tan\alpha = |y/x|$.

The above discussion suggests us the following algorithm for finding the argument of a complex number $z = x + iy$.

ALGORITHM

Step I *Find the acute angle* α *given by* $\tan\alpha = |y/x|$.

Step II *Determine quadrant in which the point* $P(x, y)$ *lies.*

If $P(x, y)$ *belongs to the first quadrant, then* $\arg(z) = \alpha$.

If $P(x, y)$ *belongs to the second quadrant, then* $\arg(z) = \pi - \alpha$.

If $P(x, y)$ *belongs to the third quadrant, the* $\arg(z) = -(\pi - \alpha)$ or $\pi + \alpha$.

If $P(x, y)$ *belongs to the fourth quadrant, then* $\arg(z) = -\alpha$ or $2\pi - \alpha$.

ILLUSTRATION 1 *Find the modulus and argument of each of the following complex numbers:*

(i) $1 + i\sqrt{3}$ [NCERT] (ii) $-2 + 2i\sqrt{3}$

(iii) $-\sqrt{3} - i$ (iv) $2\sqrt{3} - 2i$

SOLUTION (i) Let $z = 1 + i\sqrt{3}$ and let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$$

We observe that $\text{Re}(z) > 0$ and $\text{Im}(z) > 0$. So, the point representing z lies in the first quadrant.

$\therefore$ $\arg(z) = \alpha = \frac{\pi}{3}$. We find that $|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$.

(ii) Let $z = -2 + 2\sqrt{3}\,i$. Then, $|z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$. Let α be the angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then, $\tan\alpha = \left|\frac{2\sqrt{3}}{-2}\right| = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$.

We find that $\text{Re}(z) < 0$ and $\text{Im}(z) > 0$. So, the point representing z lies in the second quadrant.

$\therefore$ $\arg(z) = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

(iii) Let $z = -\sqrt{3} - i$. Then, $|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$. Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then, $\tan\alpha = \left|\frac{-1}{-\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$.

We find that $\text{Re}(z) < 0$ and $\text{Im}(z) < 0$. So, the point representing z lies in the third quadrant.

$\therefore$ $\arg(z) = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}$

(iv) Let $z = 2\sqrt{3} - 2i$. Then, $|z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = 4$. Let α be the acute angle given by $\tan\alpha \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then, $\tan\alpha = \left|\frac{-2}{2\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$.

We observe that $\text{Re}(z) > 0$ and $\text{Im}(z) < 0$. So, the point representing z lies in the fourth quadrant.

$\therefore$ $\arg(z) = -\alpha = -\pi/6$

ILLUSTRATION 2 *Find the modulus and argument of the following complex numbers:*

(i) $\frac{1+i}{1-i}$ [NCERT] (ii) $\frac{1}{1+i}$ [NCERT]

SOLUTION (i) Let $z = \frac{1+i}{1-i}$. Then,

$$z = \frac{1+i}{1-i} = \frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{1+2i+i^2}{1-i^2} = \frac{1+2i-1}{1+1} = i = 0 + i \Rightarrow |z| = \sqrt{0^2+1^2} = 1$$

Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then, $\tan\alpha = \frac{1}{0} = \infty \Rightarrow \alpha = \frac{\pi}{2}$.

We find that $\text{Re}(z) = 0$ and $\text{Im}(z) = 1 > 0$. So the point representing z lies on y-axis. Consequently, $\arg(z) = \alpha = \frac{\pi}{2}$. Hence, $|z| = 1$ and $\arg(z) = \frac{\pi}{2}$.

(ii) Let $z = \frac{1}{1+i}$. Then, $z = \frac{1}{1+i} = \frac{1}{1+i} \times \frac{1-i}{1-i} = \frac{1-i}{1-i^2} = \frac{1-i}{1+1} = \frac{1}{2} - \frac{1}{2}i$

$$\therefore \quad |z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then, $\tan\alpha = \left|\frac{-1/2}{1/2}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$.

We observe that $\text{Re}(z) = \frac{1}{2} > 0$ and $\text{Im}(z) = -\frac{1}{2} < 0$. So, the point representing z lies in the fourth quadrant. Therefore, $\arg(z) = -\alpha = -\frac{\pi}{4}$. Hence, $|z| = \frac{1}{\sqrt{2}}$ and $\arg(z) = -\frac{\pi}{4}$.

12.14.3 VECTORIAL REPRESENTATION OF A COMPLEX NUMBER

A complex number $z = x + iy$ can be represented by the position vector OP of point $P(x, y)$ in a two dimensional plane because a complex number depends on two things viz. (i) its modulus and (ii) its argument which are also the requirements of a vector on a plane.

In Fig. 12.6, the complex number $z = x + i\,y$ is represented by the vector $\vec{OP}$ and in such a case $|z|$ is the length OP and $\arg(z)$ is the angle which the directed line OP makes with the positive direction of x-axis.

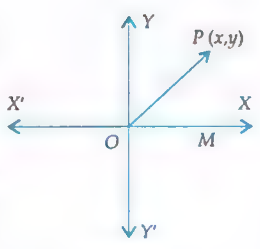


Fig.12.6

12.14.4 POLAR OR TRIGONOMETRICAL FORM OF A COMPLEX NUMBER

Let $z = x + i\,y$ be a complex number represented by a point $P(x, y)$ in the Argand plane. Then by the geometrical representation of $z = x + i\,y$, we obtain

$$OP = |z| \text{ and, } \angle POX = \theta = \arg(z)$$

In $\Delta\, POM$, we obtain

$$\cos\theta = \frac{OM}{OP} = \frac{x}{|z|} \Rightarrow x = |z|\cos\theta$$

and, $$\sin\theta = \frac{PM}{OP} = \frac{y}{|z|} \Rightarrow y = |z|\sin\theta$$

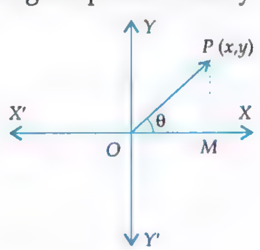


Fig.12.7

$$\therefore \quad z = x + i\,y$$

$\Rightarrow$ $z = |z| \cos\theta + i|z| \sin\theta = |z|(\cos\theta + i\sin\theta) = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg(z)$

This form of z is called a polar form of z. *If* we use the general value of the argument θ, then the polar form of z is given by

$$z = r\{\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)\}\text{, where } r = |z|,\ \theta = \arg(z) \text{ and } n \text{ is an integer.}$$

12.14.5 MULTIPLICATION OF A COMPLEX NUMBER BY IOTA

Let $z = x + iy$ be a complex number represented by a point $P(x, y)$ in the argand plane. Let $r(\cos\theta + i\sin\theta)$ be the polar form of z. Then, $r = |z|$ and $\arg(z) = \theta$.

Now, $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $iz = ir(\cos\theta + i\sin\theta) = r(-\sin\theta + i\cos\theta) = r\{\cos(\pi/2 + \theta) + \sin(\pi/2 + \theta)\}$

This means that iz is a complex number such that $|iz| = r = |z|$ and $\arg(iz) = \frac{\pi}{2} + \theta = \frac{\pi}{2} + \arg(z)$.

Thus, multiplication of a complex number by i results in rotating the vector joining the origin to point representing z through a right angle.

12.14.6 POLAR FORM OF A COMPLEX NUMBER FOR DIFFERENT SIGNS OF REAL AND IMAGINARY PARTS

Let $|z| = r$ and α be the acute angle given by $\tan\alpha = \left|\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right|$. Let θ be the argument of z.

Case I Polar form of $z = x + iy$ when $x > 0$ *and* $y > 0$: In this case, we have $\theta = \alpha$.
So, the polar form of $z = x + iy$ is $z = r(\cos\alpha + i\sin\alpha)$

Case II Polar form of $z = x\ iy$ when $x < 0$ and $y > 0$: In this case, we have $\theta = \pi - \alpha$.
So, the polar form of $z = x + iy$ is

$$z = r[\cos(\pi - \alpha) + i\sin(\pi - \alpha)] = r(-\cos\alpha + i\sin\alpha)$$

Case III Polar form of $z = x + iy$ *when* $x < 0$ and $y < 0$: In this case, we have $\theta = -(\pi - \alpha)$.
So, the polar form of z is given by

$$z = r[\cos(\pi - \alpha) + i\sin(-(\pi - \alpha))] = r(-\cos\alpha - i\sin\alpha)$$

Case IV Polar form of $z = x + iy$ when $x > 0$ and $y < 0$: In this case, we have $\theta = -\alpha$.
So, the polar form of z is

$$z = r[\cos(-\alpha) + i\sin(-\alpha)] = r(\cos\alpha - i\sin\alpha)$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Write the following complex numbers in the polar form:*

(i) $-3\sqrt{2} + 3\sqrt{2}\,i$ (ii) $1 + i$

(iii) $-1 - i$ [NCERT] (iv) $1 - i$ [NCERT EXEMPLAR]

SOLUTION (i) Let $z = -3\sqrt{2} + 3\sqrt{2}i$. Then, $r = |z| = \sqrt{(-3\sqrt{2})^2 + (3\sqrt{2})^2} = 6$.

Let α be the acute angle given by $\tan\alpha = \left|\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right|$. Then, $\tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$.

The point representing z lies in the second quadrant. So, the argument θ of z is given by

$$\theta = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Hence, the polar form of $z = -3\sqrt{2} + 3\sqrt{2}\,i$ is

$$z = r(\cos\theta + i\sin\theta) = 6\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

(ii) Let $z = 1 + i$. Then, $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then, $\tan\alpha = \frac{1}{1} = 1 \Rightarrow \alpha = \frac{\pi}{4}$.

We find that the point (1, 1) representing z lies in first quadrant. Therefore, the argument of z is given by $\theta = \alpha = \frac{\pi}{4}$. Hence, the polar form of $z = 1 + i$ is

$$z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(iii) Let $z = -1 - i$. Then, $r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$. Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then, $\tan\alpha = \left|\frac{-1}{-1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$.

Clearly, the point $(-1, -1)$ representing z lies in the third quadrant. Therefore, the argument of z is given by

$$\theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}.$$

Hence, the polar form of $z = -1 - i$ is

$$z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left\{\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right\} = \sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

(iv) Let $z = 1 - i$. Then, $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$.

Then, $\tan\alpha = \left|\frac{-1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$.

We find that the point $(1, -1)$ representing z lies in the fourth quadrant. Therefore, the argument of z is given by $\theta = -\alpha = -\frac{\pi}{4}$. Hence, the polar form of $z = 1 - i$ is

$$r(\cos\theta + i\sin\theta) = \sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right\} = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

EXAMPLE 2 *Find the modulus and principal argument of* $(1 + i)$ *and hence express it in the polar form.* [NCERT]

SOLUTION Let $z = 1 + i$. Then, $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then, $\tan\alpha = \left|\frac{1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$. Clearly, the point (1, 1) representing $z = 1 + i$ lies in first quadrant. Therefore, $\theta = \arg(z) = \frac{\pi}{4}$.

Hence, the polar form of $z = 1 + i$ is $z = |z|(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

EXAMPLE 3 *Find the modulus and principal argument of* $-2i$.

SOLUTION Let $z = -2i = 0 + (-2)i$. Then, $|z| = \sqrt{0 + (-2)^2} = 2$. We find that the point $(0, -2)$ representing $z = -2i$ lies on the negative side of imaginary axis. Therefore, principal argument of is $-\frac{\pi}{2}$.

EXAMPLE 4 *Find the modulus and principal argument of* -4. [NCERT]

SOLUTION Let $z = -4 + 0i$. Then, $|z| = \sqrt{(-4)^2 + 0} = 4$. We find that the point $(-4, 0)$ representing $z = -4 + 0i$ lies on the negative side of real axis. Therefore, principal argument of z is π.

EXAMPLE 5 *Express the following complex numbers in the polar form:*

(i) $\dfrac{1+i}{1-i}$ (ii) $\dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}$

SOLUTION (i) Let $z = \dfrac{1+i}{1-i}$. and, let $r(\cos\theta + i\sin\theta)$ be the polar form of z. Then, $r = |z|$ and $\theta = \arg(z)$.

Now, $$z = \frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1-2i+i^2}{1-i^2} = \frac{1+2i-1}{1+1} = i = 0 + 1\,i.$$

$\therefore$ $r = |z| = \sqrt{0+1} = 1$.

Clearly, the point $(0, 1)$ representing $z = 0 + i$ lies on positive direction of imaginary axis. Therefore, $\arg(z) = \dfrac{\pi}{2}$. Hence, the polar form of z is

$$z = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

(ii) Let $z = \dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}$ and, let $r(\cos\theta + i\sin\theta)$ be the polar form of z. Then, $r = |z|$ and $\theta = \arg(z)$.

Now, $$z = \frac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i} = \frac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}\cdot\frac{(5-\sqrt{3}\,i)}{(5-\sqrt{3}\,i)} = \frac{28+28\sqrt{3}\,i}{28} = 1 + i\sqrt{3}$$

$\therefore$ $r = |z| = \sqrt{1+3} = 2$.

Let α be acute angle given by $\tan\alpha = \dfrac{|\text{Im}(z)|}{|\text{Re}(z)|} = \dfrac{\sqrt{3}}{1} = \sqrt{3} \Rightarrow \alpha = \dfrac{\pi}{3}$.

Clearly, the point $(1, \sqrt{3})$ representing z lies in first quadrant. Therefore, $\theta = \arg(z) = \alpha = \dfrac{\pi}{3}$.

Hence, the polar form of z is $2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 6 *Put the complex number* $\dfrac{1+7i}{(2-i)^2}$ *in the form* $r(\cos\theta + i\sin\theta)$, *where* r *is a positive real number and* $-\pi < \theta \le \pi$. [NCERT]

SOLUTION Let $z = \dfrac{1+7i}{(2-i)^2}$. Then,

$$z = \frac{1+7i}{4-4i+i^2} = \frac{1+7i}{3-4i} = \frac{1+7i}{3-4i}\times\frac{3+4i}{3+4i} = \frac{-25+25i}{25} = -1+i$$

$\therefore$ $r = |z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

Let α be the acute angle given by $\tan\alpha = \left|\dfrac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then, $\tan\alpha = \left|-\dfrac{1}{1}\right| = 1 \Rightarrow \alpha = \dfrac{\pi}{4}$

Clearly, the point $(-1, 1)$ representing z lies in the second quadrant. Therefore,

$$\therefore \quad \theta = \arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Hence, z in the polar form is given by $z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

EXAMPLE 7 *Find the modulus and argument of the following complex numbers and convert them in polar form:*

(i) $\dfrac{1+2i}{1-3i}$ [NCERT] (ii) $\dfrac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ [NCERT] (iii) $\dfrac{1+3i}{1-2i}$ [NCERT]

SOLUTION (i) Let $z = \dfrac{1+2i}{1-3i}$. Then, $z = \dfrac{1+2i}{1-3i} \times \dfrac{1+3i}{1+3i} = \dfrac{(1-6)+i(2+3)}{1+9} = -\dfrac{1}{2} + \dfrac{1}{2}i$

$$\therefore \quad r = |z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

Let α be the acute angle given by $\tan\alpha = \left|\dfrac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then, $\tan\alpha = \left|\dfrac{-1/2}{1/2}\right| = 1 \Rightarrow \alpha = \dfrac{\pi}{4}$

We find that $\text{Re}(z) = -\dfrac{1}{2} < 0$ and $\text{Im}(z) = \dfrac{1}{2} > 0$. So, the point representing z lies in the second quadrant. Therefore, $\theta = \arg(z) = \pi - \alpha = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$. Hence, the polar form of z is

$$r(\cos\theta + i\sin\theta) = \frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).$$

(ii) Let $z = \dfrac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$. Then,

$$z = \frac{i-1}{\frac{1}{2} + i\frac{\sqrt{3}}{2}} = \frac{2(-1+i)}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{2\{(-1+\sqrt{3}) + i(1+\sqrt{3})\}}{1+3} = \left(\frac{\sqrt{3}-1}{2}\right) + i\left(\frac{\sqrt{3}+1}{2}\right)$$

$$\therefore \quad |z| = \sqrt{\left(\frac{\sqrt{3}-1}{2}\right)^2 + \left(\frac{\sqrt{3}+1}{2}\right)^2} = \sqrt{\frac{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2}{4}} = \sqrt{\frac{2(3+1)}{4}} = \sqrt{2}$$

Let α be the acute angle given by $\tan\alpha = \left|\dfrac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{\frac{\sqrt{3}+1}{2}}{\frac{\sqrt{3}-1}{2}}\right| = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \tan\frac{5\pi}{12} \Rightarrow \alpha = \frac{5\pi}{12}$$

We find that $\text{Re}(z) = \dfrac{\sqrt{3}-1}{2} > 0$ and, $\text{Im}(z) = \dfrac{\sqrt{3}+1}{2} > 0$. So, the point representing z lies in the first quadrant. Therefore, $\theta = \arg(z) = \alpha = \dfrac{5\pi}{12}$. Hence, the polar form of z is

$$r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right).$$

(iii) Let $z = \dfrac{1+3i}{1-2i}$. Then, $z = \dfrac{1+3i}{1-2i} = \dfrac{1+3i}{1-2i} \times \dfrac{1+2i}{1+2i} = \dfrac{(1-6)+i(3+2)}{1+4} = -1+i$

$\therefore \quad r = |z| = \sqrt{(-1^2 + 1^2} = \sqrt{2}.$

Let α be the acute angle given by $\tan \alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then, $\tan \alpha = \left|\frac{1}{-1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$.

We find that Re $(z) < 0$ and Im $(z) > 0$. So, the point representing z lies in the second quadrant.

$\therefore \quad \theta = \arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$

Hence, the polar form of z is $r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 8 *For any complex number z, prove that* $|\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}|z|$.

SOLUTION Let $z = r(\cos\theta + i\sin\theta)$. Then, $|z| = r$ and $\arg(z) = \theta$.

Now,

$$|\text{Re}(z)| + |\text{Im}(z)| = |r\cos\theta| + |r\sin\theta| = r\{|\cos\theta| + |\sin\theta|\} \qquad [\because r = |z| > 0]$$

$$\therefore \quad \{|\text{Re}(z)| + |\text{Im}(z)|\}^2 = r^2\{|\cos\theta| + |\sin\theta|\}^2 = r^2\{1 + |2\sin\theta\cos\theta|\}$$

$$= r^2\{1 + |\sin 2\theta|\} \le r^2(1+1) \qquad [\because |\sin 2\theta| \le 1]$$

$$\Rightarrow \quad |\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}\,r \Rightarrow |\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}|z|$$

EXAMPLE 9 *If z and w are two complex number such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then show that $\bar{z}w = -i$.* [NCERT EXEMPLAR]

SOLUTION Let $|z| = r$ and $\arg(z) = \theta$. Then, $z = r(\cos\theta + \sin\theta)$.

Now, $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$

$$\Rightarrow \quad |z||w| = 1 \text{ and } \arg(w) = \arg(z) - \frac{\pi}{2} \Rightarrow |w| = \frac{1}{r} \text{ and } \arg(w) = \theta - \frac{\pi}{2}$$

$$\therefore \quad w = |w|\{\cos(\arg w) + i\sin(\arg w)\}$$

$$\Rightarrow \quad w = \frac{1}{r}\left\{\cos\left(\theta - \frac{\pi}{2}\right) + i\sin\left(\theta - \frac{\pi}{2}\right)\right\} = \frac{1}{r}\{\sin\theta - i\cos\theta\} = -\frac{i}{r}(\cos\theta + i\sin\theta)$$

$$\therefore \quad \bar{z}w = r(\cos\theta - i\sin\theta) \times -\frac{i}{r}(\cos\theta + i\sin\theta) = -i(\cos^2\theta + \sin^2\theta) = -i.$$

EXAMPLE 10 *What is the locus of z, if amplitude of $(z - 2 - 3i)$ is $\frac{\pi}{4}$?*

SOLUTION Let $z = x + iy$. Then, $z - 2 - 3i = (x + iy) - 2 - 3i = (x - 2) + i(y - 3)$

Let θ be the amplitude of $(x - 2) + i(y - 3)$. Then, $\tan\theta = \frac{y-3}{x-2}$. It is given that the argument of $z - 2 - 3i$ is $\frac{\pi}{4}$.

$$\therefore \quad \tan\frac{\pi}{4} = \frac{y-3}{x-2} \Rightarrow 1 = \frac{y-3}{x-2} \Rightarrow x - y + 1 = 0, \text{ which is a straight line.}$$

Hence, the locus of z is a straight line.

EXAMPLE 11 *Show that the complex number z, satisfying $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ lies on a circle.*

[NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then,

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x^2-1+y^2)+2iy}{(x+1)^2+y^2}$$

$$= \left\{\frac{x^2+y^2-1}{(x+1)^2+y^2}\right\} + i\left\{\frac{2y}{(x+1)^2+y^2}\right\}$$

Let θ be the argument of $\frac{z-1}{z+1}$. Then, $\tan\theta = \frac{2y/(x+1)^2+y^2}{(x^2+y^2-1)/(x+1)^2+y^2} = \frac{2y}{x^2+y^2-1}$

But, it is given that $\arg\left(\frac{z-1}{z+1}\right)$ is $\frac{\pi}{4}$ i.e. $\theta = \frac{\pi}{4}$.

$\therefore \quad \tan\frac{\pi}{4} = \frac{2y}{x^2+y^2-1} \Rightarrow x^2+y^2-1 = 2y \Rightarrow x^2+y^2-2y-1=0 \Rightarrow (x-0)^2+(y-1)^2=(\sqrt{2})^2$,

which represents a circle.

EXAMPLE 12 *If $\arg(z-1) = \arg(z+3i)$, then find $(x-1):y$, where $z = x+iy$.* [NCERT EXEMPLAR]

SOLUTION We have, $z = x + iy$.

$\therefore \quad z-1 = (x-1)+iy$ and $z+3i = x+i(y+3)$

Let θ_1 and θ_2 be the arguments of $z-1$ and $z+3i$. Then, $\tan\theta_1 = \frac{y}{x-1}$ and $\tan\theta_2 = \frac{y+3}{x}$.

It is given that $\arg(z-1) = \arg(z+3i)$ i.e. $\theta_1 = \theta_2$.

$\therefore \quad \tan\theta_1 = \tan\theta_2$

$\Rightarrow \quad \frac{y}{x-1} = \frac{y+3}{x} \Rightarrow 3x-y-3=0 \Rightarrow 3(x-1)=y \Rightarrow \frac{x-1}{y} = \frac{1}{3} \Rightarrow (x-1):y = 1:3$

EXAMPLE 13 *If for complex numbers z_1 and z_2, $\arg(z_1)-\arg(z_2)=0$, then show that $|z_1-z_2| = ||z_1|-|z_2||$.* [NCERT EXEMPLAR]

SOLUTION Let $|z|=r_1$ and $|z_2|=r_2$. It is given that $\arg(z_1)-\arg(z_2)=0$

i.e. $\quad \arg(z_1) = \arg(z_2) = \theta$ (say)

$\therefore \quad z_1 = r_1(\cos\theta + i\sin\theta)$ and $z_2 = r_2(\cos\theta + i\sin\theta)$

$\Rightarrow \quad z_1 - z_2 = (r_1-r_2)\cos\theta + i(r_1-r_2)\sin\theta$

$\Rightarrow \quad |z_1-z_2|^2 = (r_1-r_2)^2\cos^2\theta + (r_1-r_2)^2\sin^2\theta = (r_1-r_2)^2(\cos^2\theta+\sin^2\theta) = (r_1-r_2)^2$

$\Rightarrow \quad |z_1-z_2| = |r_1-r_2| \Rightarrow |z_1-z_2| = ||z_1|-|z_2||$

EXAMPLE 14 *If z, z_1 and z_2 are complex numbers, prove that:*

(i) $\arg(\bar{z}) = -\arg(z)$. In general, $\arg(\bar{z}) = 2n\pi - \arg(z)$ (ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(iii) $\arg(z_1\bar{z_2}) = \arg(z_1) - \arg(z_2)$ (iv) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

SOLUTION (i) Let $z = r(\cos\theta + i\sin\theta)$ be the polar form of z. Then, $|z| = r$ and $\arg(\bar{z}) = \theta$.

Now, $\quad z = r(\cos\theta\, i + i\sin\theta)$

$\Rightarrow \quad \bar{z} = r(\cos\theta - i\sin\theta) = r\{\cos(-\theta) + i\sin(-\theta)\} \Rightarrow |\bar{z}| = r$ and $\arg(\bar{z}) = -\theta$

Since $\cos\theta$ and $\sin\theta$ are periodic functions with period 2π. Therefore, in general

$\arg(\bar{z}) = 2n\pi - \arg(z)$

(ii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers in their polar forms. Then,

$|z_1| = r_1, |z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

$\therefore \quad z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow \quad z_1 z_2 = r_1 r_2 \{(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1 \sin\theta_2)\}$

$\Rightarrow \quad z_1 z_2 = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

$\Rightarrow \quad |z_1 z_2| = r_1 r_2$ and, $\arg(z_1 z_2) = \theta_1 + \theta_2$

$\Rightarrow \quad |z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

REMARK *It follows from the above result that*

$|z_1 z_2 \ldots z_n| = |z_1||z_2|\ldots|z_n|$ *and,* $\arg(z_1 z_2 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n)$

Replacing $z_1, z_2, z_3, \ldots, z_n$ by z, *we obtain:* $|z^n| = |z|^n$ *and* $\arg(z^n) = n\arg(z)$

(iii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$\overline{z_2} = r_2(\cos\theta_2 - i\sin\theta_2) = r_2\{\cos(-\theta_2) + i\sin(-\theta_2)\}$

$\therefore \quad z_1\overline{z_2} = r_1 r_2[\cos\{\theta_1 + (-\theta_2)\} + i\sin\{\theta_1 + (-\theta_2)\}]$ [Using (ii)]

$\Rightarrow \quad z_1\overline{z_2} = r_1 r_2\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$

$\Rightarrow \quad \arg(z_1\overline{z_2}) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

(iv) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$|z_1| = r_1, |z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

$$\therefore \quad \frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\,\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\left\{\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right\}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\left\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right\} \Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

EXAMPLE 15 *Let* $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ *and* $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ *be two complex numbers. Then, prove that*

(i) $|z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2\cos(\theta_1 - \theta_2)$

or, $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(ii) $|z_1 - z_2|^2 = r_1^2 + r_2^2 - 2r_1 r_2\cos(\theta_1 - \theta_2)$

or, $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

SOLUTION We have, $z_1 = r_1\cos\theta_1 + i\sin\theta_1$ and, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore \quad |z_1| = r_1, |z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

(i) $\quad z_1 + z_2 = (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$

$\therefore \quad |z_1 + z_2|^2 = (r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2$

$\Rightarrow \quad |z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2\cos(\theta_1 - \theta_2)$

$\Rightarrow \quad |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(ii) $\quad z_1 - z_2 = (r_1\cos\theta_1 - r_2\cos\theta_2) + i(r_1\sin\theta_1 - r_2\sin\theta_2)$

$\therefore \quad |z_1 - z_2|^2 = (r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2$

$\Rightarrow \quad |z_1 - z_2|^2 = r_1^2 + r_2^2 - 2r_1 r_2\cos(\theta_1 - \theta_2)$

$$\Rightarrow \quad |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

EXAMPLE 16 *For any two complex numbers* z_1 and z_2, *prove that :*

(i) $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2} \Leftrightarrow \frac{z_1}{z_2}$ *is purely imaginary.*

(ii) $|z_1 + z_2| = |z_1||z_2| \Leftrightarrow \arg(z_1) = \arg(z_2) \Leftrightarrow \frac{z_1}{z_2}$ *is purely real.*

(iii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ *is purely imaginary*

SOLUTION Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then, $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$.

(i) We have,

$$|z_1 + z_2| = |z_1 - z_2|$$

$$\Leftrightarrow \quad |z_1 + z_2|^2 = |z_1 - z_2|^2$$

$$\Leftrightarrow \quad r_1^2 + r_2^2 + 2r_1 r_2\cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 - 2r_1 r_2\cos(\theta_1 - \theta_2)$$

$$\Leftrightarrow \quad 4r_1 r_2\cos(\theta_1 - \theta_2) = 0$$

$$\Leftrightarrow \quad \cos(\theta_1 - \theta_2) = 0$$

$$\Leftrightarrow \quad \theta_1 - \theta_2 = \frac{\pi}{2} \text{ i.e. } \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$$

$$\Leftrightarrow \quad \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} \qquad \left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)\right]$$

$\Leftrightarrow$ $\frac{z_1}{z_2}$ is purely imaginary.

(ii) We have,

$$|z_1 + z_2| = |z_1| + |z_2|$$

$$\Leftrightarrow \quad |z_1 + z_2|^2 = (r_1 + r_2)^2 \qquad [\because |z_1| = r_1 \text{ and } |z_2| = r_2]$$

$$\Leftrightarrow \quad r_1^2 + r_2^2 + 2r_1 r_2\cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 + 2r_1 r_2$$

$$\Leftrightarrow \quad \cos(\theta_1 - \theta_2) = 1$$

$$\Leftrightarrow \quad \theta_1 - \theta_2 = 0 \text{ i.e. } \arg(z_1) - \arg(z_2) = 0 \text{ or, } \arg(z_1) = \arg(z_2)$$

$$\Leftrightarrow \quad \arg\left(\frac{z_1}{z_2}\right) = 0 \qquad \left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)\right]$$

$\Leftrightarrow$ $\frac{z_1}{z_2}$ is purely real

(iii) We have,

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$$

$$\Leftrightarrow \quad r_1^2 + r_2^2 + 2r_1 r_2\cos(\theta_1 - \theta_2) = r_1^2 + r_2^2$$

$$\Leftrightarrow \quad 2r_1 r_2\cos(\theta_1 - \theta_2) = 0$$

$$\Leftrightarrow \quad \cos(\theta_1 - \theta_2) = 0$$

$$\Leftrightarrow \quad \theta_1 - \theta_2 = \frac{\pi}{2}$$

$$\Leftrightarrow \quad \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} \qquad \left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \theta_1 - \theta_2\right]$$

$\Leftrightarrow$ $\quad \frac{z_1}{z_2}$ is purely imaginary.

EXAMPLE 17 *For any two complex numbers z_1 and z_2, prove the following triangle inequalities:*

(i) $|z_1+z_2| \le |z_1|+|z_2|$ (ii) $|z_1-z_2| \le |z_1|+|z_2|$

(iii) $|z_1+z_2| \ge |z_1|-|z_2|$ (iv) $|z_1-z_2| \ge |z_1|-|z_2|$

SOLUTION (i) We have,

$|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.

$\because$ $\quad \cos(\theta_1-\theta_2) \le 1$

$\Rightarrow$ $\quad 2|z_1||z_2|\cos(\theta_1-\theta_2) \le 2|z_1||z_2|$ [Multiplying both sides by $2|z_1||z_2|$]

$\Rightarrow$ $\quad |z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2) \le |z_1|^2+|z_2|^2+2|z_1||z_2|$

$\Rightarrow$ $\quad |z_1+z_2|^2 \le \left(|z_1|+|z_2|\right)^2 \Rightarrow |z_1+z_2| \le |z_1|+|z_2|$

(ii) We have,

$|z_1-z_2|^2 = |z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2)$

$\because$ $\quad -1 \le \cos(\theta_1-\theta_2) \le 1$

$\therefore$ $\quad -1 \le -\cos(\theta_1-\theta_2) \le 1$

$\Rightarrow$ $\quad -\cos(\theta_1-\theta_2) \le 1$

$\Rightarrow$ $\quad -2|z_1||z_2|\cos(\theta_1-\theta_2) \le 2|z_1||z_2|$

$\Rightarrow$ $\quad |z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2) \le |z_1|^2+|z_2|^2+2|z_1||z_2|$

$\Rightarrow$ $\quad |z_1-z_2|^2 \le \left(|z_1|+|z_2|\right)^2 \Rightarrow |z_1-z_2| \le |z_1|+|z_2|$

(iii) We have,

$|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)$

$\because$ $\quad -1 \le \cos(\theta_1-\theta_2) \le 1$

$\Rightarrow$ $\quad \cos(\theta_1-\theta_2) \ge -1$

$\Rightarrow$ $\quad 2|z_1||z_2|\cos(\theta_1-\theta_2) \ge -2|z_1||z_2|$

$\Rightarrow$ $\quad |z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2) \ge |z_1|^2+|z_2|^2-2|z_1||z_2|$

$\Rightarrow$ $\quad |z_1+z_2|^2 \ge \left(|z_1|-|z_2|\right)^2 \Rightarrow |z_1+z_2| \ge |z_1|-|z_2|$

(iv) We have,

$|z_1-z_2|^2 = |z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2)$

$\therefore$ $\quad -1 \le \cos(\theta_1-\theta_2) \le 1$

$\Rightarrow$ $\quad \cos(\theta_1-\theta_2) \le 1$

$\Rightarrow$ $\quad -\cos(\theta_1-\theta_2) \ge -1$

$\Rightarrow$ $\quad -2|z_1||z_2|\cos(\theta_1-\theta_2) \ge -2|z_1||z_2|$

$\Rightarrow$ $\quad |z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2) \ge |z_1|^2+|z_2|^2-2|z_1||z_2|$

$\Rightarrow$ $\quad |z_1-z_2|^2 \ge \left(|z_1|-|z_2|\right)^2 \Rightarrow |z_1-z_2| \ge |z_1|-|z_2|$

EXAMPLE 18 *If* $z_r = \cos\left(\frac{\pi}{3^r}\right) + i \sin\left(\frac{\pi}{3^r}\right)$, $r = 1, 2, 3, \ldots$, *prove that* $z_1 z_2 z_3 \ldots z_\infty = i$.

SOLUTION We know that, if $z_1 = r_1 (\cos\theta_1 + i \sin\theta_1)$, $z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$, ... are complex numbers, then

$$z_1 z_2 z_3 \ldots z_n = r_1 r_2 r_3 \ldots r_n \{\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i \sin(\theta_1 + \theta_2 + \ldots + \theta_n)\}$$

Here, $z_r = \cos\frac{\pi}{3^r} + i \sin\frac{\pi}{3^r}$, $r = 1, 2, 3, \ldots$

$$\Rightarrow \quad |z_r| = \sqrt{\cos^2\frac{\pi}{3^r} + \sin^2\frac{\pi}{3^r}} = 1, r = 1, 2, 3, \ldots \text{ and, } \arg(z_r) = \frac{\pi}{3^r}, r = 1, 2, 3, \ldots$$

$$\therefore \quad z_1 z_2 z_3 \ldots z_n = \cos\left\{\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots \frac{\pi}{3^n}\right\} + i \sin\left\{\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots + \frac{\pi}{3^n}\right\}$$

$$= \cos\left\{\frac{\frac{\pi}{3}\left(1 - \frac{1}{3^n}\right)}{\left(1 - \frac{1}{3}\right)}\right\} + i \sin\left\{\frac{\frac{\pi}{3}\left(1 - \frac{1}{3^n}\right)}{\left(1 - \frac{1}{3}\right)}\right\} = \cos\left\{\frac{\pi}{2}\left(1 - \frac{1}{3^n}\right)\right\} + i \sin\left\{\frac{\pi}{2}\left(1 - \frac{1}{3^n}\right)\right\}$$

$$\text{Hence, } z_1 z_2 z_3 \ldots z_\infty = \lim_{n \to \infty} (z_1 z_2 z_3 \ldots z_n) = \lim_{n \to \infty} \left[\cos\left\{\frac{\pi}{2}\left(1 - \frac{1}{3^n}\right)\right\} + i \sin\left\{\frac{\pi}{2}\left(1 - \frac{1}{3^n}\right)\right\}\right]$$

$$= \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = i \qquad \left[\because \lim_{n \to \infty} \frac{1}{3^n} = 0\right]$$

EXAMPLE 19 *If* $x_n = \cos\frac{\pi}{2^n} + i \sin\frac{\pi}{2^n}$, *prove that* $x_1 x_2 x_3 \ldots x_\infty = -1$.

SOLUTION We find that

$$x_1 x_2 \ldots x_n = \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i \sin\frac{\pi}{2^2}\right)\left(\cos\frac{\pi}{2^2} + i \sin\frac{\pi}{2^3}\right) \ldots \left(\cos\frac{\pi}{2^n} + i \sin\frac{\pi}{2^n}\right)$$

$$= \cos\left\{\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots + \frac{\pi}{2^n}\right\} + i \sin\left\{\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots + \frac{\pi}{2^n}\right\}$$

$$= \cos\left\{\frac{\frac{\pi}{2}\left(1 - \frac{1}{2^n}\right)}{\left(1 - \frac{1}{2}\right)}\right\} + i \sin\left\{\frac{\frac{\pi}{2}\left(1 - \frac{1}{2^n}\right)}{\left(1 - \frac{1}{2}\right)}\right\} = \cos\left\{\pi\left(1 - \frac{1}{2^n}\right)\right\} + i \sin\left\{\pi\left(1 - \frac{1}{2^n}\right)\right\}$$

$$\therefore \quad x_1 x_2 x_3 \ldots x_\infty = \lim_{n \to \infty} (x_1 x_2 x_3 \ldots x_n) = \lim_{n \to \infty} \cos\left\{\pi\left(1 - \frac{1}{2^n}\right)\right\} + i \sin\left\{\pi\left(1 - \frac{1}{2^n}\right)\right\}$$

$$= \cos\pi + i \sin\pi = -1$$

EXAMPLE 20 *Let* z_1 *and* z_2 *be two complex numbers such that* $\bar{z}_1 + i\bar{z}_2 = 0$ *and* $\arg(z_1 z_2) = \pi$. *Then, find* $\arg(z_1)$. [NCERT EXEMPLAR]

SOLUTION It is given that

$$\bar{z}_1 + i\bar{z}_2 = 0 \Rightarrow \bar{z}_1 = -i\bar{z}_2 \Rightarrow \overline{\left(\bar{z}_1\right)} = \overline{\left(-i\bar{z}_2\right)} \qquad \text{[Taking conjugate of both sides]}$$

$\Rightarrow z_1 = i z_2$

$$\Rightarrow z_2 = -i z_1 \Rightarrow \arg(z_2) = \arg(-i z_1) \Rightarrow \arg(z_2) = \arg(-i) + \arg(z_1) \Rightarrow \arg(z_2) = -\frac{\pi}{2} + \arg(z_1) \quad \ldots\text{(i)}$$

It is also given that

$$\arg(z_1 z_2) = \pi \Rightarrow \arg(z_1) + \arg(z_2) = \pi \Rightarrow \arg(z_1) - \frac{\pi}{2} + \arg(z_1) = \pi \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 2\arg(z_1) = \frac{3\pi}{2} \Rightarrow \arg(z_1) = \frac{3\pi}{4}$$

EXAMPLE 21 *If z_1 and z_2 both satisfy $z + \bar{z} = 2|z-1|$ and arg $(z_1 - z_2) = \frac{\pi}{4}$, then find $\text{Im}(z_1 + z_2)$.* [NCERT EXEMPLAR]

SOLUTION Let $z_1 = x_1 + iy$ and $z_2 = x_2 + i\,y_2$. It is given that z_1 and z_2 satisfy $z + \bar{z} = 2|z-1|$.

$$\therefore \quad z_1 + \bar{z}_1 = 2|z_1 - 1| \quad \text{and} \quad z_2 + \bar{z}_2 = 2|z_2 - 1|$$

$$\Rightarrow \quad 2x_1 = 2|(x_1-1) + i\,y_1| \quad \text{and} \quad 2x_2 = 2|(x_2 - 1) + i\,y_2|$$

$$\Rightarrow \quad x_1 = \sqrt{(x_1-1)^2 + y_1^2} \quad \text{and} \quad x_2 = \sqrt{(x_2-1)^2 + y_2^2}$$

$$\Rightarrow \quad x_1^2 = (x_1-1)^2 + y_1^2 \quad \text{and} \quad x_2^2 = (x_2-1)^2 + y_2^2$$

$$\Rightarrow \quad 2x_1 = 1 + y_1^2 \quad \ldots\text{(i)} \quad \text{and} \quad 2x_2 = 1 + y_2^2 \qquad \ldots\text{(ii)}$$

$$\Rightarrow \quad 2(x_1 - x_2) = y_1^2 - y_2^2 \qquad \text{[Subtracting (ii) from (i)]}$$

$$\Rightarrow \quad 2\left(\frac{x_1 - x_2}{y_1 - y_2}\right) = y_1 + y_2 \qquad \ldots\text{(iii)}$$

Now, $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2 \Rightarrow z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

It is given that $\arg(z_1 - z_2) = \frac{\pi}{4}$. Therefore,

$$\tan\frac{\pi}{4} = \frac{y_1 - y_2}{x_1 - x_2} \Rightarrow 1 = \frac{y_1 - y_2}{x_1 - x_2} \Rightarrow \frac{x_1 - x_2}{y_1 - y_2} = 1 \qquad \ldots\text{(iv)}$$

From (iii) and (iv), we obtain: $2 = y_1 + y_2 \Rightarrow \text{Im}(z_1 + z_2) = 2$.

EXAMPLE 22 *If a complex number z lies in the interior or on the boundary of a circle of radius 3 units and centre $(-4, 0)$, find the greatest and least values of $|z+1|$.* [NCERT EXEMPLAR]

SOLUTION Let C $(-4, 0)$ be the centre of a circle of radius 3 units and $P(z)$ be a point in the interior or on the boundary of the circle. Then,

$$CP \le 3 \Rightarrow |z - (-4 + 0i)| \le 3 \Rightarrow |z + 4| \le 3 \qquad \ldots\text{(i)}$$

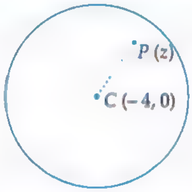


Fig. 12.8

Now,

$$|z+1| = |z+4-3| \le |z+4| + |-3| \qquad [\because |z_1 + z_2| \le |z_1| + |z_2|]$$

$$\Rightarrow \quad |z+1| \le 3 + 3 = 6 \qquad \text{[Using (i)]}$$

So, the greatest value of $|z+1|$ is 6.

We know that the modulus of any complex number is greater than or equal to zero. Therefore,

$$|z+1| \ge 0 \quad \text{for all } z.$$

So, the least value of $|z+1|$ is 0.

EXAMPLE 23 *Locate the points for which $3 < |z| < 4$.* [NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then, $|z| = \sqrt{x^2 + y^2}$.

Now

$$3 < |z| < 4$$

$\Leftrightarrow \quad 9 < |z|^2 < 16 \Leftrightarrow 9 < x^2 + y^2 < 16 \Leftrightarrow 9 < x^2 + y^2$ and $x^2 + y^2 < 16$

Clearly, $x^2 + y^2 > 9$ represents the exterior of the circle $x^2 + y^2 = 9$ and $x^2 + y^2 < 16$ represents the interior of the circle $x^2 + y^2 = 16$. Hence, $9 < x^2 + y^2 < 16$ represents the shaded portion between the circles $x^2 + y^2 = 9$ and $x^2 + y^2 = 16$ i.e. the circular annulus.

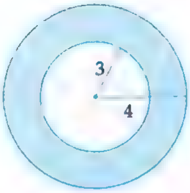


Fig. 12.9

EXERCISE 12.4

BASIC

1. Find the modulus and argument of the following complex numbers and hence express each of them in the polar form:
   (i) $1 + i$ (ii) $\sqrt{3} + i$ [NCERT] (iii) $1 - i$ [NCERT]
   (iv) $\frac{1-i}{1+i}$ (v) $\frac{1}{1+i}$ (vi) $\frac{1+2i}{1-3i}$
   (vii) $\sin 120° - i \cos 120°$ (viii) $\frac{-16}{1 + i\sqrt{3}}$ [NCERT]
2. Write $(i^{25})^3$ in polar form. [NCERT EXEMPLAR]

BASED ON HOTS

3. Express the following complex numbers in the form $r(\cos\theta + i \sin\theta)$:
   (i) $1 + i \tan\alpha$ (ii) $\tan\alpha - i$
   (iii) $1 - \sin\alpha + i\cos\alpha$ (iv) $\frac{1-i}{\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}}$
   [NCERT EXEMPLAR]
4. If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$. [NCERT EXEMPLAR]
5. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, prove that
   $$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = 0.$$ [NCERT EXEMPLAR]
6. Express $\sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)$ in polar form. [NCERT EXEMPLAR]

ANSWERS

1. (i) $\sqrt{2}(\cos\pi/4 + i\sin\pi/4)$ (ii) $2(\cos\pi/6 + i\sin\pi/6)$
   (iii) $\sqrt{2}(\cos\pi/4 - i\sin\pi/4)$ (iv) $(\cos\pi/2 - i\sin\pi/2)$
   (v) $\frac{1}{\sqrt{2}}(\cos\pi/4 - i\sin\pi/4)$ (vi) $\frac{1}{\sqrt{2}}(\cos 3\pi/4 + i\sin 3\pi/4)$
   (vii) $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ (viii) $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
2. $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$
3. (i) $1 + i\tan\alpha = \begin{cases} \sec\alpha(\cos\alpha + i\sin\alpha), & 0 \le \alpha \frac{\pi}{2} \\ -\sec\alpha\{\cos(\alpha-\pi) + i\sin(\alpha-\pi)\}, & \frac{\pi}{2} < \alpha \le \pi \end{cases}$

(ii) $\tan\alpha - i = \begin{cases} \sec\alpha\left\{\cos\left(\alpha-\frac{\pi}{2}\right)+i\sin\left(\alpha-\frac{\pi}{2}\right)\right\}, & 0\le\alpha<\frac{\pi}{2} \\ -\sec\alpha\left\{\cos\left(\frac{\pi}{2}+\alpha\right)+i\sin\left(\frac{\pi}{2}+\alpha\right)\right\}, & \frac{\pi}{2}<\alpha\le\pi \end{cases}$

(iii) $(1-\sin\alpha)+i\cos\alpha = \begin{cases} \sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right\}, & \text{if } 0\le\alpha<\frac{\pi}{2} \\ -\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)+i\sin\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)\right\}, & \text{if } \frac{\pi}{2}<\alpha<\frac{3\pi}{2} \\ -\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)+i\sin\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)\right\}, & \text{if } \frac{3\pi}{2}<\alpha<2\pi \end{cases}$

(iv) $\dfrac{1-i}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}} = \sqrt{2}\left(\cos\frac{7\pi}{12}-i\sin\frac{7\pi}{12}\right)$

6. $2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)$

HINTS TO SELECTED PROBLEMS

1. (ii) Let $z=\sqrt{3}+i$. Then, $|z| = \sqrt{(\sqrt{3})^2+(1)^2}=2$. Let θ be the argument of z and α be the acute angle given by $\tan\alpha = \dfrac{|\text{Im}(z)|}{|\text{Re}(z)|}$. Then, $\tan\alpha = \dfrac{1}{\sqrt{3}} \Rightarrow \alpha=\dfrac{\pi}{6}$.

Clearly, z lies in the first quadrant. So, $\arg(z)=\alpha=\dfrac{\pi}{6}$.

(iii) Let $z=1-i$. Then, $|z|=\sqrt{1^2+(-1)^2}=\sqrt{2}$. Let α be the acute angle given by $\tan\alpha = \dfrac{|\text{Im}(z)|}{|\text{Re}(z)|}$. Then, $\tan\alpha = \dfrac{|-1|}{|1|}=1 \Rightarrow \alpha=\dfrac{\pi}{4}$.

Clearly, z lies in the fourth quadrant. Therefore, $\arg(z)=-\alpha=-\dfrac{\pi}{4}$.

(viii) Let $z=\dfrac{-16}{1+i\sqrt{3}}=\dfrac{-16(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})}=\dfrac{-16(1-i\sqrt{3})}{1+3}=-4+4i\sqrt{3}$. Then,

$$|z| = \sqrt{(-4)^2+(4\sqrt{3})^2} = \sqrt{16+48} = 8$$

Let α the acute angle given by $\tan\alpha = \dfrac{|\text{Im}(z)|}{|\text{Re}(z)|}$. Then, $\tan\alpha = \dfrac{|4\sqrt{3}|}{|-4|} = \sqrt{3} \Rightarrow \alpha=\dfrac{\pi}{3}$

Clearly, z lies in the second quadrant. Therefore, $\arg(z)=\pi-\alpha=\pi-\dfrac{\pi}{3}=\dfrac{2\pi}{3}$.

3. (i) Let $z=1+i\tan\alpha$. Clearly, z is meaningful for $\alpha\ne(2n-1)\dfrac{\pi}{2}$, $n\in Z$. Also, $\tan\alpha$ is a periodic function with period π. So, let us take α lying in the interval $[0,\pi/2)\cup(\pi/2,\pi]$.
Following cases arise:

Case I When $\alpha\in[0,\pi/2)$: We have, $z=1+i\tan\alpha$

$\therefore \quad |z|=\sqrt{1+\tan^2\alpha}=\sqrt{\sec^2\alpha}=|\sec\alpha|=\sec\alpha \qquad \left[\because \frac{\pi}{2}<\alpha<\pi \;\therefore\; \sec\alpha<0\right]$

Let β be an acute angle given by $\tan\beta=\dfrac{|\text{Im}(z)|}{|\text{Re}(z)|}$. Then, $\tan\beta=|\tan\alpha|=\tan\alpha \Rightarrow \beta=\alpha$.

As z is represented by a point lying in first quadrant. Therefore, $\arg(z) = \beta = \alpha$.

So, the polar form of z is $\sec\alpha\,(\cos\alpha + i\sin\alpha)$

Case II When $\alpha \in (\pi/2, \pi]$: We have, $z = 1 + i\tan\alpha$

$$\therefore \quad |z| = \sqrt{1+\tan^2\alpha} = |\sec\alpha| = -\sec\alpha \qquad \left[\because \frac{\pi}{2} < \alpha < \pi \ \therefore \sec\alpha < 0\right]$$

Let β be an acute angle given by $\tan\beta = \dfrac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan\beta = |\tan\alpha| = -\tan\alpha \qquad [\because \alpha \in (\pi/2, \pi)]$$

$$\Rightarrow \quad \tan\beta = \tan(\pi - \alpha) \Rightarrow \beta = \pi - \alpha$$

We observe that z is represented by a point in fourth quadrant.

$$\therefore \quad \arg(z) = -\beta = \alpha - \pi.$$

Thus, z in polar form is $-\sec\alpha\,\{\cos(\alpha - \pi) + i\sin(\alpha - \pi)\}$.

(ii) Let $z = \tan\alpha - i$. Since $\tan\alpha$ is periodic with period π. So, let us take $\alpha \in [0, \pi/2) \cup (\pi/2, \pi]$.

Case I When $\alpha \in [0, \pi/2)$: We have, $z = \tan\alpha - i$

$$\therefore \quad |z| = \sqrt{\tan^2\alpha + 1} = |\sec\alpha| = \sec\alpha$$

Let β be the acute angle given by $\tan\beta = \dfrac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan\beta = \frac{1}{|\tan\alpha|} = |\cot\alpha| = \cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right) \Rightarrow \beta = \frac{\pi}{2} - \alpha$$

Clearly, $\operatorname{Re}(z) > 0$ and $\operatorname{Im}(z) < 0$. So, z lies in the fourth quadrant. Therefore, $\arg(z) = -\beta = \alpha - \dfrac{\pi}{2}$. Thus, z in polar form is given by $z = \sec\alpha\left\{\cos\left(\alpha - \dfrac{\pi}{2}\right) + i\sin\left(\alpha - \dfrac{\pi}{2}\right)\right\}$

Case II When $\alpha \in (\pi/2, \pi]$: we have, $z = \tan\alpha - i \Rightarrow |z| = \sqrt{\tan^2\alpha + 1} = |\sec\alpha| = -\sec\alpha$

Let β be the acute angle given by $\tan\beta = \dfrac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan\beta = \frac{1}{|\tan\alpha|} = |\cot\alpha| = -\cot\alpha = \tan\left(\alpha - \frac{\pi}{2}\right) \Rightarrow \beta = \alpha - \frac{\pi}{2}$$

Clearly, $\operatorname{Re}(z) < 0$ and $\operatorname{Im}(z) < 0$. So, z lies in third quadrant. Therefore, $\arg(z) = \pi + \beta = \dfrac{\pi}{2} + \alpha$

Thus, the polar form of z is $-\sec\alpha\left\{\cos\left(\dfrac{\pi}{2} + \alpha\right) + i\sin\left(\dfrac{\pi}{2} + \alpha\right)\right\}$.

(iii) Let $z = (1 - \sin\alpha) + i\cos\alpha$. Since sine and cosine functions are periodic functions with period 2π. So, let us take α lying in the interval $[0, 2\pi]$.

Now, $z = 1 - \sin\alpha + i\cos\alpha$

$$\Rightarrow \quad |z| = \sqrt{(1-\sin\alpha)^2 + \cos^2\alpha} = \sqrt{2 - 2\sin\alpha} = \sqrt{2}\,\sqrt{1-\sin\alpha} = \sqrt{2}\sqrt{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2}$$

$$\Rightarrow \quad |z| = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right|$$

Let β be the acute angle given by $\tan\beta = \dfrac{|\text{Im}(z)|}{|\text{Re}(z)|}$. Then,

$$\tan\beta = \frac{|\cos\alpha|}{|1-\sin\alpha|} = \left|\frac{\cos\alpha}{1-\sin\alpha}\right| = \left|\frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2}\right| = \left|\frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}\right|$$

$$\Rightarrow \quad \tan\beta = \left|\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right| = \left|\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right|$$

Following cases arise:

Case I When $0 \le \alpha < \frac{\pi}{2}$: In this case, we have $\cos\frac{\alpha}{2} > \sin\frac{\alpha}{2}$ and $\frac{\pi}{4}+\frac{\alpha}{2} \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$\therefore \quad |z| = \sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right) \text{and, } \tan\beta = \left|\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right| = \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \Rightarrow \beta = \frac{\pi}{4}+\frac{\alpha}{2}$$

Clearly, z lies in the first quadrant. Therefore, $\arg(z) = \frac{\pi}{4}+\frac{\alpha}{2}$.

Hence, the polar form of z is $\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right\}$.

Case II When $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$: In this case, we have $\cos\frac{\alpha}{2} < \sin\frac{\alpha}{2}$ and $\frac{\pi}{4}+\frac{\alpha}{2} \in \left(\frac{\pi}{2}, \pi\right)$

$$\therefore \quad |z| = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right| = -\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$$

and, $\tan\beta = \left|\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right| = -\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) = \tan\left\{\pi - \left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right\} = \tan\left(\frac{3\pi}{4}-\frac{\alpha}{2}\right) \Rightarrow \beta = \frac{3\pi}{4}-\frac{\alpha}{2}$

Since $1-\sin\alpha > 0$ and $\cos\alpha < 0$. So, z lies in fourth quadrant. Therefore, $\arg(z) = -\beta = \frac{\alpha}{2}-\frac{3\pi}{4}$.

Hence, the polar form of z is $-\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right) + i\sin\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)\right\}$.

Case III When $\frac{3\pi}{2} < \alpha < 2\pi$ In this case, we have

$$\cos\frac{\alpha}{2} < \sin\frac{\alpha}{2} \text{ and } \left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \in \left(\pi, \frac{5\pi}{4}\right)$$

$$\therefore \quad |z| = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right| = -\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$$

and, $\tan\beta = \left|\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right| = \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) = -\tan\left\{\pi - \left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right\} = \tan\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right) \Rightarrow \beta = \frac{\alpha}{2}-\frac{3\pi}{4}$

Clearly, $\text{Re}(z) < 0$ and $\text{Im}(z) > 0$. So, z lies in the first quadrant. Therefore, $\arg(z) = \beta = \frac{\alpha}{2}-\frac{3\pi}{4}$

Hence, the polar form of z is $-\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right) + i\sin\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)\right\}$.

(iv) Let $z = \dfrac{1-i}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}} = \dfrac{\sqrt{2}\,(\cos \pi/4 - i \sin \pi/4)}{(\cos \pi/3 + i \sin \pi/3)}$

$$\Rightarrow \quad z = \sqrt{2}\left\{\frac{\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}\right\} = \sqrt{2}\left\{\cos\left(-\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{4} - \frac{\pi}{3}\right)\right\}$$

$$\Rightarrow \quad z = \sqrt{2}\left\{\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right\}$$

ALITER We have, $z = \dfrac{z_1}{z_2}$, where $z_1 = 1 - i$ and $z_2 = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$.

Now, $z_1 = 1 - i = \sqrt{2}\left\{\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right\}$ and $z_2 = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

$$\Rightarrow \quad |z_1| = \sqrt{2} \text{ and } \theta_1 = \arg(z_1) = -\frac{\pi}{4}, |z_2| = 1, \theta_2 = \arg(z_2) = \frac{\pi}{3}$$

$$\therefore \quad \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\} = \sqrt{2}\left\{\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right\}$$

4. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$|z_1| = r_1$, $\arg(z_1) = \theta_1$, $|z_2| = \theta_2$ and $\arg(z_2) = \theta_2$

It is given that

$|z_2| = |z_1|$ and $\arg(z_1) + \arg(z_2) = \pi \Rightarrow r_1 = r_2$ and $\theta_1 + \theta_2 = \pi \Rightarrow r_1 = r_2$ and $\theta_1 = \pi - \theta_2$

$\therefore \quad z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\Rightarrow \quad z_1 = r_2\{\cos(\pi - \theta_2) + i\sin(\pi - \theta_2)\} = r_2(-\cos\theta_2 + i\sin\theta_2) = -r_2(\cos\theta_2 - i\sin\theta_2) = -\overline{z}_2$

5. Let $\arg(z_1) = \theta_1$ and $\arg(z_3) = \theta_2$

It is given that $z_2 = \overline{z}_1$ and $z_4 = \overline{z}_3$. Therefore, $\arg(z_2) = -\theta_1$ and $\arg(z_4) = -\theta_2$

Hence, $\arg\left(\dfrac{z_1}{z_4}\right) + \arg\left(\dfrac{z_2}{z_3}\right) = \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3) = \theta_1 + \theta_2 - \theta_1 - \theta_2 = 0.$

6. Let $z = \sin\dfrac{\pi}{5} + i\left(1 - \cos\dfrac{\pi}{5}\right) = 2\sin\dfrac{\pi}{10}\cos\dfrac{\pi}{10} + 2i\sin^2\dfrac{\pi}{10} = 2\sin\dfrac{\pi}{10}\left(\cos\dfrac{\pi}{10} + i\sin\dfrac{\pi}{10}\right)$

Clearly, z is the polar form with $|z| = 2\sin\dfrac{\pi}{10}$ and $\arg(z) = \dfrac{\pi}{10}$.

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The principal value of the argument of the complex number $1 - i$ is ....
2. The polar form of $(i^{25})^3$ is ....
3. The value of $\sqrt{-25} \times \sqrt{-9}$ is ....
4. The complex number $\dfrac{(1-i)^3}{1-i^3}$ in polar form is ....
5. The sum of the series $i + i^2 + i^3 + \ldots\ldots$ upto 1000 terms is ....
6. The multiplicative inverse of $(1 + i)$ is ....
7. If $|z| = 4$ and $\arg(z) = \dfrac{5\pi}{6}$, then $z =$ ....

8. If z_1 and z_2 are two complex numbers such that $z_1 + z_2$ is a real number, then z_2 =................ .
9. For any non-zero complex number z, arg (z) + arg $(\bar{z})$ = ...
10. If $|z+4| \le 3$, then the greatest and least values of $|z+1|$ are ....and ....
11. The modulus and argument of $\sin\frac{\pi}{5} + i\left(1-\cos\frac{\pi}{5}\right)$ are .... and .... respectively.
12. If $\left|\frac{z-2}{z+2}\right| = \frac{\pi}{6}$, then the locus of z is ....
13. If $|z+2i| = |z-2i|$, then the locus of z is ....
14. If $|z+2| = |z-2|$, then the locus of z is................................ .
15. If $z = -1 + \sqrt{-3}$, then arg (z) = ...
16. If $x < 0$ is a real number, then arg (x) = ...
17. The real value of 'a' for which $3i^3 - 2ai^2 + (1-a)\,i + 5$ is real is ....
18. If $|z| = 2$ and arg $(z) = \frac{\pi}{4}$, then z = ...
19. The value of $(-\sqrt{-1})^{4n-3}$, where $n \in N$, is ....
20. The locus of z satisfying arg $(z) = \frac{\pi}{3}$ is ....
21. The conjugate of the complex number $\frac{1-i}{1+i}$ is ....
22. If $(2+i)\,(2+2i)\,(2+3i)\ldots\ldots(2+ni) = x+iy$, then $5.8.13\ldots(4+n^2)$ = ...
23. If the point representing a complex number lies in the third quadrant, then the point representing its conjugate lies in the ....
24. The multiplication of a non-zero complex number by i rotates it through .... in the anti-clockwise direction.
25. The complex number $\cos\theta + i\sin\theta$ .... be zero for any θ.
26. The argument of the complex number $(-1+i\sqrt{3})\,(1+i)\,(\cos\theta + i\sin\theta)$ is ....
27. If a complex number coincides with its conjugate, then it lies on ....
28. The points representing the complex number z for which $|z+1| < |z-1|$ lie on the left side of ....
29. If three complex numbers z_1, z_2 and z_3 are in A.P., then points representing them lie on ....
30. The principal argument of i^{-1097} is ....
31. The value of $\frac{i^{4n+1} - i^{4n-1}}{2}$ is ....
32. If $z_1 = \sqrt{3} + i\sqrt{3}$ and $z_2 = \sqrt{3} + i$, then the point representing $\frac{z_1}{z_2}$ lies in ....
33. If $0 < \arg(z) < \pi$, then arg (z) − arg $(-z)$ = ...
34. For any two complex numbers z_1, z_2 and any real numbers a, b, $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$ = ...
35. Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then arg (z_1) − arg (z_2) = ...

36. Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$, then $\arg(z_1) - \arg(z_2) =$ ....

37. If $|z_1| = |z_2|$ and $\arg\left(\frac{z_1}{z_2}\right) = \pi$, then $z_1 + z_2 =$ ....

ANSWERS

1. $\frac{-\pi}{4}$
2. $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$
3. -15
4. $2(\cos\pi + i\sin\pi)$
5. 0
6. $\frac{1}{2} - \frac{i}{2}$
7. $-2\sqrt{3} + 2i$
8. $\bar{z}_1$
9. $2n\pi, n \in Z$
10. 6 and 0
11. $2\sin\frac{\pi}{10}, \frac{\pi}{10}$
12. circle
13. perpendicular bisector of the segment joining $(0, -2)$ and $(0, 2)$
14. perpendicular bisector of the segment joining $(-2, 0)$ and $(2, 0)$
15. $\frac{2\pi}{3}$
16. π
17. -2
18. $\sqrt{2}(1+i)$
19. $-i$
20. $y = \sqrt{3}x$ in the first quadrant except the origin
21. i
22. $x^2 + y^2$
23. Second quadrant
24. a right angle
25. cannot be
26. $\frac{11\pi}{12} + \theta$
27. x-axis
28. y-axis
29. a straight line
30. $-\frac{\pi}{2}$
31. i
32. first quadrant
33. π
34. $(a^2 + b^2)(|z_1|^2 + |z_2|^2)$
35. 0
36. $\frac{\pi}{2}$
37. 0

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the values of the square root of i.
2. Write the values of the square root of $-i$.
3. If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, then write the value of $(x^2 + y^2)^2$.
4. If $\pi < \theta < 2\pi$ and $z = 1 + \cos\theta + i\sin\theta$, then write the value of $|z|$.
5. If n is any positive integer, write the value of $\frac{i^{4n+1} - i^{4n-1}}{2}$.
6. Write the value of $\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$.
7. Write $1 - i$ in polar form.
8. Write $-1 + i\sqrt{3}$ in polar form.
9. Write the argument of $-i$. [NCERT]
10. Write the least positive integral value of n for which $\left(\frac{1+i}{1-i}\right)^n$ is real.

11. Find the principal argument of $(1+i\sqrt{3})^2$.
12. Find z, if $|z|=4$ and $\arg(z)=\frac{5\pi}{6}$.
13. If $|z-5i|=|z+5i|$, then find the locus of z.
14. If $\frac{(a^2+1)^2}{2a-i}=x+iy$, find the value of x^2+y^2.
15. Write the value of $\sqrt{-25}\times\sqrt{-9}$.
16. Write the sum of the series $i+i^2+i^3+\ldots$ upto 1000 terms.
17. Write the value of $\arg(z)+\arg(\bar{z})$.
18. If $|z+4|\le 3$, then find the greatest and least values of $|z+1|$.
19. For any two complex numbers z_1 and z_2 and any two real numbers a, b, find the value of $|az_1-bz_2|^2+|az_2+bz_1|^2$.
20. Write the conjugate of $\frac{2-i}{(1-2i)^2}$.
21. If $n\in N$, then find the value of $i^n+i^{n+1}+i^{n+2}+i^{n+3}$.
22. Find the real value of a for which $3i^3-2ai^2+(1-a)i+5$ is real.
23. If $|z|=2$ and $\arg(z)=\frac{\pi}{4}$, find z.
24. Write the argument of $(1+\sqrt{3})(1+i)(\cos\theta+i\sin\theta)$.

ANSWERS

1. $\pm\frac{1}{\sqrt{2}}(1+i)$
2. $\pm\frac{1}{\sqrt{2}}(1-i)$
3. $\frac{a^2+b^2}{c^2+d^2}$
4. $-2\cos\frac{\theta}{2}$
5. i
6. -2
7. $\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$
8. $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$
9. $\frac{3\pi}{2}$ or $-\frac{\pi}{2}$
10. 2
11. $\frac{2\pi}{3}$
12. $-2\sqrt{3}+2i$
13. Real axis
14. $\frac{(a^2+1)^4}{4a^2+1}$
15. -15
16. 0
17. 0
18. 6 and 0
19. $(a^2+b^2)(|z_1|^2+|z_2|^2)$
20. $-\frac{2}{25}-\frac{11}{25}i$
21. 0
22. $a=2$
23. $\sqrt{2}(1+i)$
24. $\frac{7\pi}{12}+\theta$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The value of $(1+i)(1+i^2)(1+i^3)(1+i^4)$ is

   (a) 2 (b) 0 (c) 1 (d) i

2. If $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is a real number and $0<\theta<2\pi$, then $\theta=$

   (a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

3. If $(1+i)(1+2i)(1+3i)\ldots(1+ni) = a+ib$, then $2\times5\times10\times\ldots\times(1+n^2)$ is equal to
(a) $\sqrt{a^2+b^2}$ (b) $\sqrt{a^2-b^2}$ (c) a^2+b^2 (d) a^2-b^2 (e) $a+b$

4. If $\sqrt{a+ib} = x+iy$, then possible value of $\sqrt{a-ib}$ is
(a) x^2+y^2 (b) $\sqrt{x^2+y^2}$ (c) $x+iy$ (d) $x-iy$ (e) $\sqrt{x^2-y^2}$

5. If $z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{6}$, then
(a) $|z| = 1$, $\arg(z) = \frac{\pi}{4}$ (b) $|z| = 1$, $\arg(z) = \frac{\pi}{6}$
(c) $|z| = \frac{\sqrt{3}}{2}$, $\arg(z) = \frac{5\pi}{24}$ (d) $|z| = \frac{\sqrt{3}}{2}$, $\arg(z) = \tan^{-1}\frac{1}{\sqrt{2}}$

6. The polar form of $(i^{25})^3$ is
(a) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ (b) $\cos\pi + i\sin\pi$ (c) $\cos\pi - i\sin\pi$ (d) $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

7. If $i^2 = -1$, then the sum $i + i^2 + i^3 + \ldots$ upto 1000 terms is equal to
(a) 1 (b) -1 (c) i (d) 0

8. If $z = \frac{-2}{1+i\sqrt{3}}$, then the value of arg(z) is
(a) π (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$

9. If $a = \cos\theta + i\sin\theta$, then $\frac{1+a}{1-a} =$
(a) $\cot\frac{\theta}{2}$ (b) $\cot\theta$ (c) $i\cot\frac{\theta}{2}$ (d) $i\tan\frac{\theta}{2}$

10. If $(1+i)(1+2i)(1+3i)\ldots\ldots(1+ni) = a+ib$, then $2.5.10.17\ldots\ldots(1+n^2) =$
(a) $a-ib$ (b) a^2-b^2 (c) a^2+b^2 (d) none of these

11. If $\frac{(a^2+1)^2}{2a-i} = x+iy$, then x^2+y^2 is equal to
(a) $\frac{(a^2+1)^4}{4a^2+1}$ (b) $\frac{(a+1)^2}{4a^2+1}$ (c) $\frac{(a^2-1)^2}{(4a^2-1)^2}$ (d) none of these

12. The principal value of the amplitude of $(1+i)$ is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{12}$ (c) $\frac{3\pi}{4}$ (d) π

13. The least positive integer n such that $\left(\frac{2i}{1+i}\right)^n$ is a positive integer, is
(a) 16 (b) 8 (c) 4 (d) 2

14. If z is a non-zero complex number, then $\left|\frac{|\bar{z}|^2}{z\bar{z}}\right|$ is equal to
(a) $\left|\frac{\bar{z}}{z}\right|$ (b) $|z|$ (c) $|\bar{z}|$ (d) none of these

15. If $a = 1+i$, then a^2 equals
(a) $1-i$ (b) $2i$ (c) $(1+i)(1-i)$ (d) $i-1$.

16. If $(x+iy)^{1/3} = a+ib$, then $\frac{x}{a}+\frac{y}{b}=$

(a) 0 (b) 1 (c) -1 (d) none of these

17. $(\sqrt{-2})(\sqrt{-3})$ is equal to

(a) $\sqrt{6}$ (b) $-\sqrt{6}$ (c) $i\sqrt{6}$ (d) none of these

18. The argument of $\frac{1-i\sqrt{3}}{1+i\sqrt{3}}$ is

(a) 60° (b) 120° (c) 210° (d) 240°.

19. If $z=\left(\frac{1+i}{1-i}\right)$, then z^4 equals

(a) 1 (b) -1 (c) 0 (d) none of these

20. If $z=\frac{1+2i}{1-(1-i)^2}$, then arg (z) equals

(a) 0 (b) $\frac{\pi}{2}$ (c) π (d) none of these

21. If $z=\frac{1}{(2+3i)^2}$, then $|z|=$

(a) $\frac{1}{13}$ (b) $\frac{1}{5}$ (c) $\frac{1}{12}$ (d) none of these

22. If $z=\frac{1}{(1-i)(2+3i)}$, then $|z|=$

(a) 1 (b) $1/\sqrt{26}$ (c) $5/\sqrt{26}$ (d) none of these

23. If $z=1-\cos\theta+i\sin\theta$, then $|z|=$

(a) $2\sin\frac{\theta}{2}$ (b) $2\cos\frac{\theta}{2}$ (c) $2\left|\sin\frac{\theta}{2}\right|$ (d) $2\left|\cos\frac{\theta}{2}\right|$

24. If $x+iy=(1+i)(1+2i)(1+3i)$, then $x^2+y^2=$

(a) 0 (b) 1 (c) 100 (d) none of these

25. If $z=\frac{1}{1-\cos\theta-i\sin\theta}$, then Re (z) =

(a) 0 (b) $\frac{1}{2}$ (c) $\cot\frac{\theta}{2}$ (d) $\frac{1}{2}\cot\frac{\theta}{2}$

26. If $x+iy=\frac{3+5i}{7-6i}$, then $y=$

(a) 9/85 (b) $-9/85$ (c) 53/85 (d) none of these

27. If $\frac{1-ix}{1+ix}=a+ib$, then $a^2+b^2=$

(a) 1 (b) -1 (c) 0 (d) none of these

28. If θ is the amplitude of $\frac{a+ib}{a-ib}$, then $\tan\theta=$

(a) $\frac{2a}{a^2+b^2}$ (b) $\frac{2ab}{a^2-b^2}$ (c) $\frac{a^2-b^2}{a^2+b^2}$ (d) none of these

29. If $z=\frac{1+7i}{(2-i)^2}$, then

(a) $|z|=2$ (b) $|z|=\frac{1}{2}$ (c) amp $(z)=\frac{\pi}{4}$ (d) amp $(z)=\frac{3\pi}{4}$

30. The amplitude of $\frac{1}{i}$ is equal to

(a) 0 (b) $\frac{\pi}{2}$ (c) $-\frac{\pi}{2}$ (d) π.

31. The argument of $\frac{1-i}{1+i}$ is

(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) $\frac{5\pi}{2}$

32. The amplitude of $\frac{1+i\sqrt{3}}{\sqrt{3}+i}$ is

(a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $-\frac{\pi}{6}$

33. The value of $(i^5+i^6+i^7+i^8+i^9)/(1+i)$ is

(a) $\frac{1}{2}(1+i)$ (b) $\frac{1}{2}(1-i)$ (c) 1 (d) $\frac{1}{2}$

34. $\frac{1+2i+3i^2}{1-2i+3i^2}$ equals

(a) i (b) -1 (c) $-i$ (d) 4

35. The value of $\frac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1$ is

(a) -1 (b) -2 (c) -3 (d) -4

36. The value of $(1+i)^4+(1-i)^4$ is

(a) 8 (b) 4 (c) -8 (d) -4

37. If $z=a+ib$ lies in third quadrant, then $\frac{\bar{z}}{z}$ also lies in the third quadrant if

(a) $a>b>0$ (b) $a<b<0$ (c) $b<a<0$ (d) $b>a>0$

[NCERT EXEMPLAR]

38. If $f(z)=\frac{7-z}{1-z^2}$, where $z=1+2i$, then $|f(z)|$ is

(a) $\frac{|z|}{2}$ (b) $|z|$ (c) $2|z|$ (d) none of these

[NCERT EXEMPLAR]

39. A real value of x satisfies the equation $\frac{3-4ix}{3+4ix}=a-ib\ (a,b\in R)$, if $a^2+b^2=$

(a) 1 (b) -1 (c) 2 (d) -2

[NCERT EXEMPLAR]

40. The complex number z which satisfies the condition $\left|\frac{i+z}{i-z}\right|=1$ lies on

(a) circle $x^2+y^2=1$ (b) the x-axis (c) the y-axis (d) the line $x+y=1$

[NCERT EXEMPLAR]

41. If z is a complex number, then

(a) $|z|^2 > |z|^2$ (b) $|z|^2 = |z|^2$ (c) $|z|^2 < |z|^2$ (d) $|z|^2 \geq |z|^2$

[NCERT EXEMPLAR]

42. Which of the following is correct for any two complex numbers z_1 and z_2?

(a) $|z_1 z_2| = |z_1||z_2|$ (b) $\arg(z_1 z_2) = \arg(z_1) \arg(z_2)$

(c) $|z_1 + z_2| = |z_1| + |z_2|$ (d) $|z_1 + z_2| \geq |z_1| + |z_2|$

[NCERT EXEMPLAR]

43. If the complex number $z = x + iy$ satisfies the condition $|z+1| = 1$, then z lies on

(a) x-axis (b) circle with centre $(-1, 0)$ and radius 1

(c) y-axis (d) none of these [NCERT EXEMPLAR]

44. $\sin x + i\cos 2x$ and $\cos x - i \sin 2x$ are conjugate to each other for

(a) $x = n\pi$ (b) $x = \left(n + \frac{1}{2}\right)\frac{\pi}{2}$ (c) $x = 0$ (d) No value of x

[NCERT EXEMPLAR]

45. The real value of α for which the expression $\frac{1 - i\sin\alpha}{1 + 2i\sin\alpha}$ is purely real, is

(a) $(n+1)\frac{\pi}{2}$ (b) $(2n+1)\frac{\pi}{2}$ (c) $n\pi$ (d) none of these

where $n \in N$. [NCERT EXEMPLAR]

46. The value of $(z + 3)(\bar{z} + 3)$ is equivalent to

(a) $|z+3|^2$ (b) $|z-3|$ (c) $z^2 + 3$ (d) none of these

[NCERT EXEMPLAR]

47. If $\left(\frac{1+i}{1-i}\right)^n = 1$, then $n =$

(a) $2m+1$ (b) $4m$ (c) $2m$ (d) $4m+1$

where $m \in N$

[NCERT EXEMPLAR]

48. The vector represented by the complex number $2 - i$ is rotated about the origin through an angle $\frac{\pi}{2}$ in the clockwise direction, the new position of point is

(a) $1 + 2i$ (b) $-1 - 2i$ (c) $2 + i$ (d) $-1 + 2i$

[NCERT EXEMPLAR]

49. The real value of θ for which the expression $\frac{1 + i\cos\theta}{1 - 2i\cos\theta}$ is a real number, is

(a) $n\pi + \frac{\pi}{4}$ (b) $n\pi + (-1)^n \frac{\pi}{4}$ (c) $2n\pi \pm \frac{\pi}{2}$ (d) none of theses

[NCERT EXEMPLAR]

50. $|z_1 + z_2| = |z_1| + |z_2|$ is possible if

(a) $z_2 = \bar{z}_1$ (b) $z_2 = \frac{1}{z_1}$

(c) $\arg(z_1) = \arg(z_2)$ (d) $|z_1| = |z_2|$ [NCERT EXEMPLAR]

51. The equation $|z + 1 - i| = |z - 1 + i|$ represents a

(a) straight line (b) circle (c) parabola (d) hyperbola

[NCERT EXEMPLAR]

52. The area of the triangle on the complex plane formed by the complex numbers z, $-iz$ and $z+iz$ is

(a) $|z|^2$ (b) $|\bar{z}|^2$ (c) $\frac{1}{2}|z|^2$ (d) none of these

[NCERT EXEMPLAR]

ANSWERS

1. (b)	2. (a)	3. (c)	4. (d)	5. (d)	6. (d)	7. (d)	8. (c)
9. (c)	10. (c)	11. (a)	12. (a)	13. (b)	14. (a)	15. (b)	16. (d)
17. (b)	18. (d)	19. (a)	20. (a)	21. (a)	22. (b)	23. (c)	24. (c)
25. (b)	26. (c)	27. (a)	28. (b)	29. (d)	30. (c)	31. (a)	32. (c)
33. (a)	34. (c)	35. (b)	36. (c)	37. (c)	38. (a)	39. (a)	40. (b)
41. (b)	42. (a)	43. (b)	44. (d)	45. (c)	46. (a)	47. (b)	48. (b)
49. (c)	50. (c)	51. (a)	52. (c)				

SUMMARY

1. $\sqrt{-1}$ is an imaginary quantity and is denoted by i which has the following properties: $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and, $i^{\pm n} = i^{\pm k}$, $n \in N$

   where k is the remainder when n is denoted by 4.
2. For any positive real number a, $\sqrt{-a} = i\sqrt{a}$.
3. For any two real numbers a and b, we have

$$\sqrt{a}\sqrt{b} = \begin{cases} \sqrt{ab}, & \text{if at least one of a and b is positive} \\ -\sqrt{ab}, & \text{if } a<0, b<0. \end{cases}$$

4. If a, b are real numbers, then a number $z = a+ib$ is called a complex number, real number a is known as the real part of z and b is known as its imaginary part. We write $a = \text{Re}(z)$, $b = \text{Im}(z)$.

   A complex number z is purely real iff $\text{Im}(z) = 0$ and z is purely imaginary iff $\text{Re}(z) = 0$
5. For any two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, we define

   Addition: $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

   Subtraction: $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$

   Multiplication: $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

   Reciprocal: $\dfrac{1}{z_1} = \dfrac{a_1}{a_1^2 + b_1^2} - i\dfrac{b_1}{a_1^2 + b_1^2}$

   Division: $\dfrac{z_1}{z_2} = z_1\left(\dfrac{1}{z_2}\right) = (a_1 + ib_1)\left(\dfrac{a_2}{a_2^2 + b_2^2} - i\dfrac{b_2}{a_2^2 + b_2^2}\right) = \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\dfrac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$

   Addition is commutative and associative. Complex number $0 = 0 + i0$ is the identity element for addition and every complex number $z = a + ib$ has its additive inverse $-z = -a - ib$.

   Multiplication is also commutative and associative. Complex number $1 = 1 + 0i$ is the identity element for multiplication. Every non-zero complex number $z = a + ib$ has its multiplicative inverse $1/z$ (also known as reciprocal of z) such that $\dfrac{1}{z} = \dfrac{a - ib}{a^2 + b^2} = \dfrac{\bar{z}}{|z|^2}$.

6. The conjugate of a complex number $z = a + ib$ is denoted by $\bar{z}$ and is equal to $a - ib$. For any three complex numbers z, z_1, z_2, we have

(i) $\overline{(\bar{z})} = z$
(ii) $z + \bar{z} = 2\,Re\,(z)$
(iii) $z - \bar{z} = 2i\,Im\,(z)$
(iv) $z = \bar{z} \Leftrightarrow z$ is purely real
(v) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary
(vi) $z\bar{z} = \{Re\,(z)\}^2 + \{Im\,(z)\}^2 = |z|^2$
(vii) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
(viii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
(ix) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

7. The modulus of a complex number $z = a + ib$ is denoted by $|z|$ and is defined as

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\{\text{Re}\,(z)\}^2 + \{\text{Im}\,(z)\}^2}$$

If z, z_1, z_2 are three complex numbers, then

(i) $|z| = 0 \Leftrightarrow z = 0$ i.e. $\text{Re}\,(z) = \text{Im}\,(z) = 0$
(ii) $|z| = |\bar{z}| = |-z|$
(iii) $-|z| \leq \text{Re}\,(z) \leq |z|\,;\; -|z| \leq \text{Im}\,(z) \leq |z|$
(iv) $z\bar{z} = |z|^2$
(v) $|\text{Im}\,(z^n)| \leq n\,|\text{Im}\,(z)|\,|z|^{n-1}, n \in N$
(vi) $|\text{Re}\,(z)| + |\text{Im}\,(z)| \leq \sqrt{2}\,|z|$

8. A complex number $z = x + iy$ can be represented by a point $P(x, y)$ (see Fig. 13.8) on the plane which is known as the Argand or Gaussian or Complex plane. The length of the line segment OP is called the modulus of z and is denoted by $|z|$.

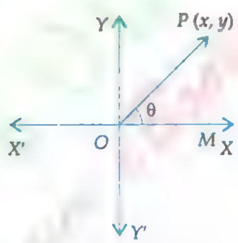


Fig.12.8

Clearly, $|z| = \sqrt{x^2 + y^2} = \sqrt{\{\text{Re}\,(z)\}^2 + \{\text{Im}\,(z)\}^2}$

The angle θ which OP makes with the positive direction of x-axis in anti-clockwise sense is called the argument or amplitude of z and is denoted by $arg\,(z)$ or amp (z).

Clearly, $\tan\theta = \dfrac{\text{Im}\,(z)}{\text{Re}\,(z)}$.

Let $OP = r$ and $\angle XOP = \theta$. Then, $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore \quad z = x + iy = r(\cos\theta + i\sin\theta)$

This is known as the polar form of complex number z. The Euler's notations are

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$\therefore \quad z = r(\cos\theta + i\sin\theta)$

or, $\quad z = re^{i\theta}$, which is known as the Eulerian form of z.

CHAPTER 13

QUADRATIC EQUATIONS

13.1 INTRODUCTION

In earlier classes, we have studied about quadratic equations with real coefficients and real roots only. In this chapter, we shall study about quadratic equations with real coefficients and complex roots. We shall also discuss quadratic equations with complex coefficients and their solutions in the complex number system. But, let us first recall some definitions and results.

13.2 SOME USEFUL DEFINITIONS AND RESULTS

REAL POLYNOMIAL *Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x is a real variable. Then,*

$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ *is called a real polynomial of real variable x with real coefficients.*

For example, $x^2 - 4x + 3$, $2x^3 - 6x^2 + 11x - 5$ etc. are real polynomials.

COMPLEX POLYNOMIAL *If $a_0, a_1, a_2, \ldots, a_n$ are complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is called a complex polynomial or a polynomial of complex variable with complex coefficients.*

For example, $2x^2 - (3+7i)x + (9i-3)$, $x^3 - 5i x^2 + (1-2i)x + (3+4i)$ etc are complex polynomials.

DEGREE OF A POLYNOMIAL *A polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, real or complex, is a polynomial of degree n, if $a_n \neq 0$.*

The polynomials $2x^3 - 7x^2 + x + 5$, $(3-2i)x^2 - ix + 5$ are polynomials of degree 3 and 2 respectively.

A polynomial of second degree is generally called a quadratic polynomial and polynomials of degree 3 and 4 are known as *cubic* and *biquadratic* polynomials.

POLYNOMIAL EQUATION *If $f(x)$ is a polynomial, then $f(x) = 0$ is called a polynomial equation.*

If $f(x)$ is a quadratic polynomial, then $f(x) = 0$ is called a quadratic equation. The general form of a *quadratic equation* is $ax^2 + bx + c = 0,\ a \neq 0$.

Here, x is the variable and a, b, c are called coefficients real or imaginary.

ROOTS OF AN EQUATION *The values of the variable satisfying a given equation are called its roots.*

Thus, $x = \alpha$, is a root of the equation $f(x) = 0$, if $f(\alpha) = 0$.

For example, $x = 1$ is a root of the equation $x^3 - 6x^2 + 11x - 6 = 0$, because

$$1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 1 - 6 + 11 - 6 = 0$$

Similarly, $x = \omega$ and $x = \omega^2$ are roots of the equation $x^2 + x + 1 = 0$ as they satisfy it.

SOLUTION SET *The set of all roots of an equation, in a given domain, is called the solution set of the equation.*

For example, the set $\{1, 2, 3\}$ is the solution set of the equation $x^3 - 6x^2 + 11x - 6 = 0$.

Solving an equation means finding its solution set. In other words, solving an equation is the process of obtaining its all roots.

IDENTITY *An expression involving equality and a variable is called an identity, if it is satisfied by every value of the variable.*

For example, $x^2 - 9 = (x-3)(x+3)$ is an identity as it is satisfied by every value of x.

and, $\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} = 1$ is also an identity as it holds good for all values of x.

FUNDAMENTAL THEOREM OF ALGEBRA *Every polynomial equation $f(x) = 0$ has at least one root, real or imaginary (complex).*

Thus, $x^7 - 3x^5 + 2x^2 = x + 2 = 0$ has at least one root. But, $f(x) = \sqrt{x} + 3 = 0$ has no root as this equation is not a polynomial equation. Fundamental theorem does not apply on this equation.

The fundamental theorem guarantees for one root of a polynomial equation. The following theorem states about the exact number of roots of a polynomial equation.

THEOREM *Every polynomial equation $f(x) = 0$ of degree n has exactly n roots real or imaginary.*

13.3 QUADRATIC EQUATION

The general form of a quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$ where a, b, c are numbers (real or complex) and x is a variable.

The following theorem suggests about the number of roots of a quadratic equation.

THEOREM *A quadratic equation cannot have more than two roots.*

PROOF If possible, let α, β, γ be three distinct roots of the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$. Then, each one of α, β, γ will satisfy this equation.

$\therefore \quad a\alpha^2 + b\alpha + c = 0$ ...(i)

$a\beta^2 + b\beta + c = 0$ ...(ii)

and, $a\gamma^2 + b\gamma + c = 0$ ...(iii)

Subtracting (ii) from (i), we obtain

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$$

$\Rightarrow \quad (\alpha - \beta)\{a(\alpha + \beta) + b\} = 0 \Rightarrow a(\alpha + \beta) + b = 0 \quad [\because \alpha \text{ and } \beta \text{ are distinct} \therefore \alpha - \beta \neq 0]$ ...(iv)

Subtracting (iii) from (ii), we obtain

$$a(\beta^2 - \gamma^2) + b(\beta - \gamma) = 0$$

$\Rightarrow \quad (\beta - \gamma)\{a(\beta + \gamma) + b\} = 0 \Rightarrow a(\beta + \gamma) + b = 0 \quad [\because \beta \text{ and } \gamma \text{ are distinct} \therefore \beta - \gamma \neq 0]$ ...(v)

Subtracting (v) from (iv), we get : $a(\alpha - \gamma) = 0$. But, this is not possible, because α and γ are distinct and $a \neq 0$. So, their product cannot be zero. Thus, the assumption that a quadratic equation has three distinct real roots is wrong. Hence, a quadratic equation cannot have more than 2 roots.

Q.E.D.

REMARK *It follows from the above theorem that if a quadratic equation is satisfied by more than two values of x, then it is satisfied by every value of x and so it is an identity.*

13.4 QUADRATIC EQUATIONS WITH REAL COEFFICIENTS

In earlier classes, we have solved quadratic equations with real coefficients and real roots either by factorization or by using Sridharacharya's formula. In this section, we shall mainly concentrate on quadratic equations with real coefficients and complex roots.

Consider the quadratic equation $ax^2 + bx + c = 0$ ...(i)

where $a, b, c \in R$ and $a \neq 0$.

Multiplying both sides of (i) by a, we obtain

$$a^2 x^2 + abx + ac = 0 \Rightarrow a^2 x^2 + abx + \frac{b^2}{4} = \frac{b^2}{4} - ac$$

$$\Rightarrow \left(ax + \frac{b}{2}\right)^2 = \frac{b^2 - 4ac}{4} \Rightarrow ax + \frac{b}{2} = \pm \frac{\sqrt{b^2 - 4ac}}{2}$$

$$\Rightarrow ax = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4ac}}{2} \Rightarrow ax = \frac{-b \pm \sqrt{b^2 - 4ac}}{2} \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$ has two roots, say α and β, given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Now, if we look at the expressions for these roots, we observe that the nature of these roots depend upon the value of the expressions $b^2 - 4ac$. This expression is generally denoted by D and is known as the discriminant of the quadratic equation (i). We also observe the following results:

RESULT I *If $b^2 - 4ac = 0$ i.e. $D = 0$, then $\alpha = \beta = -\frac{b}{2a}$.*

Thus, if $b^2 - 4ac = 0$, then the quadratic equation has real and equal roots each equal to $-b/2a$.

RESULT II *If a, b, c are rational numbers and $b^2 - 4ac$ is positive and a perfect square, then $\sqrt{b^2 - 4ac}$ is a rational number and hence α and β are rational and unequal.*

Thus, if $a, b, c \in Q$ and $b^2 - 4ac$ is positive and a perfect square, then roots are rational and unequal. If $a, b, c \in R$ and $b^2 - 4ac$ is positive and a perfect square, then roots are real and distinct.

RESULT III *If $b^2 - 4ac > 0$ i.e. $D > 0$ but it is not a perfect square, then roots are irrational and unequal.*

REMARK *If $a, b, c \in Q$ and $b^2 - 4ac$ is positive but not a perfect square, then roots are irrational and they always occur in conjugate pair like $2 + \sqrt{3}$ and $2 - \sqrt{3}$. However, if a, b, c are irrational numbers and $b^2 - 4ac$ is positive but not a perfect square, then the roots may not occur in conjugate pairs. For example, the roots of the equation $x^2 - (5 + \sqrt{2})\, x + 5\sqrt{2} = 0$ are 5 and $\sqrt{2}$ which do not form a conjugate pair.*

RESULT IV *If $b^2 - 4ac < 0$ i.e. $D < 0$, then $4ac - b^2 > 0$ and so the roots are imaginary and are given by*

$$\alpha = \frac{-b + i\sqrt{4ac - b^2}}{2a} \text{ and } \beta = \frac{-b - i\sqrt{4ac - b^2}}{2a}$$

Clearly, α and β are complex conjugate of each other i.e. $\alpha = \bar{\beta}$ and $\bar{\alpha} = \beta$.

REMARK *If $b^2 - 4ac < 0$, then the roots are complex conjugate of each other. In fact, complex roots of an equation with real coefficients always occur in conjugate pairs like $2 + 3i$ and $2 - 3i$. However, this may not be true in case of equations with complex coefficients. For example, $x^2 - 2ix - 1 = 0$ has both roots equal to i.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve the equation $4x^2 + 9 = 0$ by factorization method.*

SOLUTION We have,

$$4x^2 + 9 = 0$$

$$\Rightarrow \quad 4x^2 - 9i^2 = 0$$

$$\Rightarrow \quad (2x)^2 - (3i)^2 = 0$$

$$\Rightarrow \quad (2x + 3i)(2x - 3i) = 0 \Rightarrow 2x + 3i = 0 \text{ or, } 2x - 3i = 0 \Rightarrow x = -\frac{3}{2}i, \text{ or, } x = \frac{3}{2}i$$

Hence, the roots of the given equation are $\frac{3}{2}i$ and $-\frac{3}{2}i$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 2 *Solve the equation $x^2 - 4x + 13 = 0$ by factorization method.*

SOLUTION We have,

$$x^2 - 4x + 13 = 0$$

$$\Rightarrow \quad x^2 - 4x + 4 + 9 = 0$$

$$\Rightarrow \quad (x-2)^2 + 9 = 0$$

$$\Rightarrow \quad (x-2)^2 - 9i^2 = 0$$

$$\Rightarrow \quad (x-2)^2 - (3i)^2 = 0 \Rightarrow \{(x-2) - 3i\}\{(x-2) + 3i\} = 0 \Rightarrow (x - 2 - 3i)(x - 2 + 3i) = 0$$

$$\Rightarrow \quad x - 2 - 3i = 0, \text{ or } x - 2 + 3i = 0 \Rightarrow x = 2 + 3i, \text{ or } x = 2 - 3i$$

Hence, the roots of the given equation are $2 + 3i$ and $2 - 3i$.

EXAMPLE 3 *Solve the equation $9x^2 - 12x + 20 = 0$ by factorization method only.*

SOLUTION We have,

$$9x^2 - 12x + 20 = 0$$

$$\Rightarrow \quad 9x^2 - 12x + 4 + 16 = 0$$

$$\Rightarrow \quad (3x-2)^2 + 16 = 0$$

$$\Rightarrow \quad (3x-2)^2 - 16i^2 = 0$$

$$\Rightarrow \quad \{(3x-2) + 4i\}\{(3x-2) - 4i\} = 0 \Rightarrow (3x - 2 + 4i)(3x - 2 - 4i) = 0$$

$$\Rightarrow \quad 3x - 2 + 4i = 0, \text{ or } 3x - 2 - 4i = 0 \Rightarrow 3x = 2 - 4i, \text{ or } 3x = 2 + 4i \Rightarrow x = \frac{2}{3} - \frac{4}{3}i \text{ or } x = \frac{2}{3} + \frac{4}{3}i$$

Hence, the roots of the given equation are $\frac{2}{3} - \frac{4}{3}i$ and $\frac{2}{3} + \frac{4}{3}i$.

EXAMPLE 4 *Solve the quadratic equation $2x^2 - 4x + 3 = 0$ by using the general expressions for the roots of a quadratic equation.*

SOLUTION Comparing the given equation with the general form $ax^2 + bx + c = 0$, we get

$$a = 2, b = -4 \text{ and } c = 3$$

Substituting the values of a, b, c in $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$, we get

$$\alpha = \frac{4+\sqrt{16-24}}{4} \text{ and, } \beta = \frac{4-\sqrt{16-24}}{4}$$

$$\Rightarrow \quad \alpha = \frac{4+\sqrt{-8}}{4} \text{ and, } \beta = \frac{4-\sqrt{-8}}{4}$$

$$\Rightarrow \quad \alpha = \frac{4+2\sqrt{2}i}{4} \text{ and, } \beta = \frac{4-2\sqrt{2}i}{4} \Rightarrow \alpha = 1+\frac{1}{\sqrt{2}}i \text{ and, } \beta = 1-\frac{1}{\sqrt{2}}i$$

Hence, the roots of the given equation are $1+\frac{1}{\sqrt{2}}i$ and $1-\frac{1}{\sqrt{2}}i$.

EXAMPLE 5 *Solve the equation $25x^2 - 30x + 11 = 0$ by using the general expression for the roots of a quadratic equation.*

SOLUTION Comparing the given equation with the general form of the quadratic equation $ax^2 + bx + c = 0$, we get: $a = 25, b = -30$ and $c = 11$.

Substituting these values in $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$, we get

$$\alpha = \frac{30+\sqrt{900-1100}}{50} \text{ and, } \beta = \frac{30-\sqrt{900-1100}}{50}$$

$$\Rightarrow \quad \alpha = \frac{30+\sqrt{-200}}{50} \text{ and, } \beta = \frac{30-\sqrt{-200}}{50}$$

$$\Rightarrow \quad \alpha = \frac{30+10i\sqrt{2}}{50} \text{ and, } \beta = \frac{30-10i\sqrt{2}}{50} \Rightarrow \alpha = \frac{3}{5}+\frac{\sqrt{2}}{5}i \text{ and, } \beta = \frac{3}{5}-\frac{\sqrt{2}}{5}i$$

Hence, the roots of the given equation are $\frac{3}{5} \pm \frac{\sqrt{2}}{5}i$.

EXERCISE 13.1

BASIC

Solve the following quadratic equations by factorization method only (1-5):

1. $x^2 + 1 = 0$
2. $9x^2 + 4 = 0$
3. $x^2 + 2x + 5 = 0$
4. $4x^2 - 12x + 25 = 0$
5. $x^2 + x + 1 = 0$ [NCERT]

Solve the following quadratics (6-18):

6. $4x^2 + 1 = 0$
7. $x^2 - 4x + 7 = 0$
8. $x^2 + 2x + 2 = 0$
9. $5x^2 - 6x + 2 = 0$
10. $21x^2 + 9x + 1 = 0$
11. $x^2 - x + 1 = 0$
12. $x^2 + x + 1 = 0$ [NCERT]
13. $17x^2 - 8x + 1 = 0$
14. $27x^2 - 10x + 1 = 0$ [NCERT]
15. $17x^2 + 28x + 12 = 0$
16. $21x^2 - 28x + 10 = 0$ [NCERT]
17. $8x^2 - 9x + 3 = 0$
18. $13x^2 + 7x + 1 = 0$
19. $2x^2 + x + 1 = 0$
20. $\sqrt{3}x^2 - \sqrt{2}\,x + 3\sqrt{3} = 0$ [NCERT]
21. $\sqrt{2}\,x^2 + x + \sqrt{2} = 0$ [NCERT]
22. $x^2 + x + \frac{1}{\sqrt{2}} = 0$ [NCERT]
23. $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$ [NCERT]
24. $\sqrt{5}x^2 + x + \sqrt{5} = 0$ [NCERT]
25. $-x^2 + x - 2 = 0$ [NCERT]
26. $x^2 - 2x + \frac{3}{2} = 0$ [NCERT]
27. $3x^2 - 4x + \frac{20}{3} = 0$ [NCERT]

ANSWERS

1. $i, -i$
2. $\frac{2}{3}i, \frac{-2}{3}i$
3. $-1+2i, -1-2i$
4. $\frac{3}{2}+2i, \frac{3}{2}-2i$
5. $-\frac{1}{2}+i\frac{\sqrt{3}}{2}, -\frac{1}{2}-i\frac{\sqrt{3}}{2}$
6. $\frac{1}{2}i, \frac{-1}{2}i$
7. $2 \pm \sqrt{3}i$
8. $-1 \pm i$
9. $\frac{3}{5} \pm \frac{1}{5}i$
10. $\frac{-3}{14} \pm \frac{i\sqrt{3}}{42}$
11. $\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$
12. $-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$
13. $\frac{4}{17} \pm \frac{1}{17}i$
14. $\frac{5}{27} \pm \frac{\sqrt{2}}{27}i$
15. $\frac{-14}{17} \pm \frac{2\sqrt{2}}{17}i$
16. $\frac{2}{3} \pm \frac{\sqrt{14}}{21}i$
17. $\frac{9}{16} \pm \frac{\sqrt{15}}{16}i$
18. $\frac{-7}{26} \pm \frac{\sqrt{3}}{26}i$
19. $\frac{-1 \pm \sqrt{7}i}{4}$
20. $\frac{\sqrt{2} \pm \sqrt{34}i}{2\sqrt{3}}$
21. $\frac{-1 \pm \sqrt{7}i}{2\sqrt{2}}$
22. $\frac{-1 \pm \sqrt{2\sqrt{2}-1}\,i}{2}$
23. $\frac{-1 \pm \sqrt{7}i}{2\sqrt{2}}$
24. $\frac{-1 \pm \sqrt{19}i}{2\sqrt{5}}$
25. $\frac{-1 \pm \sqrt{7}i}{-2}$
26. $1 \pm \frac{1}{\sqrt{2}}i$
27. $\frac{2}{3} \pm \frac{4}{3}i$

HINTS TO SELECTED PROBLEMS

5. We have, $x^2+x+1 = 0$

$$\Rightarrow x^2+2(x)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1 = 0 \Rightarrow \left(x+\frac{1}{2}\right)^2+\frac{3}{4} = 0$$

$$\Rightarrow \left(x+\frac{1}{2}\right)^2-\frac{3}{4}i^2 = 0 \Rightarrow \left(x+\frac{1}{2}\right)^2-\left(\frac{\sqrt{3}}{2}i\right)^2 = 0$$

$$\Rightarrow \left(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(x+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = 0 \Rightarrow \left(x+\frac{1+\sqrt{3}i}{2}\right)\left(x+\frac{1-\sqrt{3}\,i}{2}\right) = 0$$

$$\Rightarrow x+\frac{1+\sqrt{3}i}{2} = 0 \text{ or, } x+\frac{1-\sqrt{3}i}{2} = 0 \Rightarrow x = -\frac{1+i\sqrt{3}}{2} \text{ or, } x = \frac{-1+i\sqrt{3}}{2}$$

12. We have, $x^2+x+1 = 0$ Comparing the given equation with the general form $ax^2+bx+c=0$, we get: $a = 1, b = 1$ and $c = 1$.

Substituting the values of a, b, c in $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$, we get

$$\alpha = \frac{-1+\sqrt{1-4}}{2} \text{ and } \beta = \frac{-1-\sqrt{1-4}}{2} \Rightarrow \alpha = \frac{-1+i\sqrt{3}}{2} \text{ and } \beta = \frac{-1-i\sqrt{3}}{2}$$

14. We have, $27x^2-10x+1 = 0$ Comparing this equation with $ax^2+bx+c=0$, we get: $a = 27, b = -10, c = 1$. Substituting the values of a, b, c in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{10+\sqrt{100-108}}{54} \text{ and } \beta = \frac{10-\sqrt{100-108}}{54}$$

$$\Rightarrow \quad \alpha = \frac{10+\sqrt{-8}}{54} \text{ and } \beta = \frac{10-\sqrt{-8}}{54} \Rightarrow \alpha = \frac{5+i\sqrt{2}}{27} \text{ and } \beta = \frac{5-i\sqrt{2}}{27}$$

16. The given equation is $21x^2 - 28x + 10 = 0$. Comparing this equation with $ax^2 + bx + c = 0$, we get: $a = 21, b = -28, c = 10$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we obtain}$$

$$\alpha = \frac{28+\sqrt{784-840}}{42} \text{ and } \beta = \frac{28-\sqrt{784-840}}{42}$$

$$\Rightarrow \quad \alpha = \frac{28+\sqrt{-56}}{42} \text{ and } \beta = \frac{28-\sqrt{-56}}{42} \Rightarrow \alpha = \frac{2}{3} + \frac{i\sqrt{14}}{21} \text{ and } \beta = \frac{2}{3} - \frac{i\sqrt{14}}{21}$$

20. The given equation is $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$. Comparing this equation with $ax^2 + bx + c = 0$, we get: $a = \sqrt{3}, b = -\sqrt{2}, c = 3\sqrt{3}$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we obtain}$$

$$\alpha = \frac{\sqrt{2}+\sqrt{2-36}}{2\sqrt{3}} \text{ and } \beta = \frac{\sqrt{2}-\sqrt{2-36}}{2\sqrt{3}} \Rightarrow \alpha = \frac{\sqrt{2}+i\sqrt{34}}{2\sqrt{3}} \text{ and } \beta = \frac{\sqrt{2}-i\sqrt{34}}{2\sqrt{3}}$$

21. The given equation is $\sqrt{2}x^2 + x + \sqrt{2} = 0$. Comparing this equation with $ax^2 + bx + c = 0$, we get: $a = \sqrt{2}, b = 1, c = \sqrt{2}$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we obtain}$$

$$\alpha = \frac{-1+\sqrt{1-8}}{2\sqrt{2}} \text{ and } \beta = \frac{-1-\sqrt{1-8}}{2\sqrt{2}} \Rightarrow \alpha = \frac{-1+i\sqrt{7}}{2\sqrt{2}} \text{ and } \beta = \frac{-1-i\sqrt{7}}{2\sqrt{2}}$$

22. The given equation is $x^2 + x + \frac{1}{\sqrt{2}} = 0$. Comparing this equation with $ax^2 + bx + c = 0$, we get: $a = 1, b = 1, c = \frac{1}{\sqrt{2}}$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we obtain}$$

$$\alpha = \frac{-1+\sqrt{1-2\sqrt{2}}}{2} \text{ and } \beta = \frac{-1-\sqrt{1-2\sqrt{2}}}{2} \Rightarrow \alpha = \frac{-1+i\sqrt{2\sqrt{2}-1}}{2} \text{ and } \beta = \frac{-1-i\sqrt{2\sqrt{2}-1}}{2}$$

23. The given equation is $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$. Comparing this equation with $ax^2 + bx + c = 0$, we get: $a = 1, b = \frac{1}{\sqrt{2}}$ and $c = 1$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we obtain}$$

$$\alpha = \frac{-\frac{1}{\sqrt{2}}+\sqrt{\frac{1}{2}-4}}{2} \text{ and } \beta = \frac{-\frac{1}{\sqrt{2}}-\sqrt{\frac{1}{2}-4}}{2} \Rightarrow \alpha = \frac{-1+i\sqrt{7}}{2\sqrt{2}} \text{ and } \beta = \frac{-1-i\sqrt{7}}{2\sqrt{2}}$$

24. The given equation is $\sqrt{5}x^2 + x + \sqrt{5} = 0$. Comparing this equation with $ax^2 + bx + c$, we get $a = \sqrt{5}, b = 1$ and $c = \sqrt{5}$. Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we obtain}$$

$$\alpha = \frac{-1 + \sqrt{1 - 20}}{2\sqrt{5}} \text{ and } \beta = \frac{-1 - \sqrt{1 - 20}}{2\sqrt{5}} \Rightarrow \alpha = \frac{-1 + i\sqrt{19}}{2\sqrt{5}} \text{ and } \beta = \frac{-1 - i\sqrt{19}}{2\sqrt{5}}$$

25. The given equation is $-x^2 + x - 2 = 0$. Comparing this equation with $ax^2 + bx + c$, we get: $a = -1, b = 1$ and $c = -2$. Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we obtain}$$

$$\alpha = \frac{-1 + \sqrt{1 - 8}}{-2} \text{ and } \beta = \frac{-1 - \sqrt{1 - 8}}{-2}$$

$$\Rightarrow \alpha = \frac{-1 + i\sqrt{7}}{-2} \text{ and } \beta = \frac{-1 - i\sqrt{7}}{-2} \Rightarrow \alpha = \frac{1}{2} - i\frac{\sqrt{7}}{2} \text{ and } \beta = \frac{1}{2} + i\frac{\sqrt{7}}{2}$$

26. The given equation is $x^2 - 2x + \frac{3}{2} = 0$. Comparing this equation with $ax^2 + bx + c = 0$, we get: $a = 1, b = -2$ and $c = 3/2$. Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we obtain}$$

$$\alpha = \frac{2 + \sqrt{4 - 6}}{2} \text{ and } \beta = \frac{2 - \sqrt{4 - 6}}{2} \Rightarrow \alpha = 1 + \frac{i}{\sqrt{2}} \text{ and } \beta = 1 - \frac{i}{\sqrt{2}}$$

27. The given equation is $3x^2 - 4x + \frac{20}{3} = 0$. Com,paring this equation with $ax^2 + bx + c = 0$, we get: $a = 3, b = -4$ and $c = \frac{20}{3}$. Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we obtain}$$

$$\alpha = \frac{4 + \sqrt{16 - 80}}{6} \text{ and } \beta = \frac{4 - \sqrt{16 - 80}}{6} \Rightarrow \alpha = \frac{4 + 8i}{6} = \frac{2}{3} + \frac{4}{3}i \text{ and } \beta = \frac{4 - 8i}{6} = \frac{2}{3} - \frac{4}{3}i$$

13.5 QUADRATIC EQUATIONS WITH COMPLEX COEFFICIENTS

Consider the quadratic equation $ax^2 + bx + c = 0$ ...(i)

where a, b, c are complex numbers and $a \neq 0$.

Proceeding as in section 13.4, we obtain that the roots of equation (i) are given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and, } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

These roots are complex as a, b, c are complex numbers.

Since the order relation is not defined in case of complex numbers. Therefore, we cannot assign positive or negative sign to the discriminant $D = b^2 - 4ac$. However, equation (i) has complex roots which are equal, if $D = b^2 - 4ac = 0$ and unequal roots if $D = b^2 - 4ac \neq 0$.

REMARK *In case of quadratic equations with real coefficients imaginary (complex) roots always occur in conjugate pairs. However, it is not true for quadratic equations with complex coefficients. For example, the equation* $4x^2 - 4ix - 1 = 0$ *has both roots equal to* $\frac{1}{2}i$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve the following quadratic equations by factorization method:*

(i) $x^2 - 5ix - 6 = 0$ (ii) $x^2 + 4ix - 4 = 0$

SOLUTION (i) The given equation is

$$x^2 - 5ix - 6 = 0$$

$$\Rightarrow \quad x^2 - 5ix + 6i^2 = 0$$

$$\Rightarrow \quad x^2 - 3ix - 2ix + 6i^2 = 0$$

$$\Rightarrow \quad x(x - 3i) - 2i(x - 3i) = 0$$

$$\Rightarrow \quad (x - 3i)(x - 2i) = 0 \Rightarrow x - 3i = 0, x - 2i = 0 \Rightarrow x = 3i, x = 2i$$

Hence, the roots of the given equation are $3i$ and $2i$.

(ii) Given equation is

$$x^2 + 4ix - 4 = 0 \Rightarrow x^2 + 4ix + 4i^2 = 0 \Rightarrow (x + 2i)^2 = 0$$

$$\Rightarrow \quad x + 2i = 0 \text{ (twice)} \Rightarrow x = -2i, -2i$$

Hence, both the roots of the equation are equal to $-2i$.

EXAMPLE 2 *Solve the following equations by factorization method*

(i) $x^2 - \sqrt{2}i\,x + 12 = 0$ (ii) $3x^2 + 7ix + 6 = 0$ (iii) $x^2 - (3\sqrt{2} + 2i)x + 6\sqrt{2}i = 0$

SOLUTION (i) We have,

$$x^2 - \sqrt{2}\,ix + 12 = 0$$

$$\Rightarrow \quad x^2 - 3\sqrt{2}i\,x + 2\sqrt{2}\,i\,x - 12i^2 = 0$$

$$\Rightarrow \quad x(x - 3\sqrt{2}i) + 2\sqrt{2}i(x - 3\sqrt{2}i) = 0$$

$$\Rightarrow \quad (x - 3\sqrt{2}\,i)(x + 2\sqrt{2}\,i) = 0$$

$$\Rightarrow \quad x - 3\sqrt{2}i = 0 \text{ or, } x + 2\sqrt{2}\,i = 0 \Rightarrow x = 3\sqrt{2}\,i \text{ or, } x = -2\sqrt{2}\,i$$

Hence, the roots of the given equation are $-2\sqrt{2}\,i$ and $3\sqrt{2}\,i$.

(ii) $$3x^2 + 7ix + 6 = 0$$

$$\Rightarrow \quad 3x^2 + 9ix - 2ix - 6i^2 = 0$$

$$\Rightarrow \quad 3x(x + 3i) - 2i(x + 3i) = 0$$

$$\Rightarrow \quad (x + 3i)(3x - 2i) = 0 \Rightarrow x + 3i = 0 \text{ or, } 3x - 2i = 0 \Rightarrow x = -3i \text{ or, } x = \frac{2}{3}i$$

Hence, the roots of the given equation are $-3i$ and $\frac{2}{3}i$.

(iii) $$x^2 - (3\sqrt{2} + 2i)x + 6\sqrt{2}\,i = 0$$

$$\Rightarrow \quad (x^2 - 3\sqrt{2}x) - (2ix - 6\sqrt{2}i) = 0$$

$$\Rightarrow \quad x(x - 3\sqrt{2}) - 2i(x - 3\sqrt{2}) = 0$$

$$\Rightarrow \quad (x-2i)(x-3\sqrt{2}) = 0 \Rightarrow x-2i = 0, x-3\sqrt{2} = 0 \Rightarrow x = 2i \text{ or, } 3\sqrt{2}$$

Hence, the roots of the given equation are $2i$ and $3\sqrt{2}$.

EXAMPLE 3 *Solve the following quadratic equations by using the general expressions for the roots of a quadratic equation:*

(i) $x^2 - (3\sqrt{2}-2i)\,x - 6\sqrt{2}\,i = 0$ (ii) $2x^2 + 3i\,x + 2 = 0$

SOLUTION (i) On comparing the given equation with the general equation $ax^2+bx+c=0$, we get: $a=1, b=-(3\sqrt{2}-2i)$ and $c=-6\sqrt{2}i$. Substituting the values of a, b, c in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we obtain}$$

$$\alpha = \frac{(3\sqrt{2}-2i)+\sqrt{(3\sqrt{2}-2i)^2+24\sqrt{2}i}}{2}, \text{ and } \beta = \frac{(3\sqrt{2}-2i)-\sqrt{(3\sqrt{2}-2i)^2+24\sqrt{2}i}}{2}$$

$$\Rightarrow \quad \alpha = \frac{(3\sqrt{2}-2i)+\sqrt{(3\sqrt{2}+2i)^2}}{2}, \quad \text{and } \beta = \frac{(3\sqrt{2}-2i)-\sqrt{13\sqrt{2}+2i)^2}}{2}$$

$$\Rightarrow \quad \alpha = \frac{3\sqrt{2}-2i+3\sqrt{2}+2i}{2}, \quad \text{and } \beta = \frac{(3\sqrt{2}-2i)-(3\sqrt{2}+2i)}{2}$$

$$\Rightarrow \quad \alpha = 3\sqrt{2}, \beta = -2i$$

Hence, the roots of the given equation are $3\sqrt{2}$ and $-2i$.

(ii) On comparing the given equation with the general equation $ax^2+bx+c=0$, we get $a=2, b=3i$ and $c=2$. Substituting these values of a, b, c in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{-3i+\sqrt{9i^2-16}}{4} \quad \text{and} \quad \beta = \frac{-3i-\sqrt{9i^2-16}}{4}$$

$$\Rightarrow \quad \alpha = \frac{-3i+\sqrt{-25}}{4} \quad \text{and} \quad \beta = \frac{-3i-\sqrt{-25}}{4}$$

$$\Rightarrow \quad \alpha = \frac{-3i+5i}{4} \quad \text{and} \quad \beta = \frac{-3i-5i}{4} \Rightarrow \alpha = \frac{i}{2} \text{ and } \beta = -2i$$

Hence, the roots of the given equation are $\frac{i}{2}$ and $-2i$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Solve:* $2x^2-(3+7i)\,x-(3-9i) = 0.$

SOLUTION On comparing the given equation with the general form $ax^2+bx+c=0$, we obtain $a=2, b=-(3+7i), c=-(3-9i)$. Substituting the values of a, b, c in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{(3+7i)+\sqrt{(3+7i)^2+8(3-9i)}}{4} \quad \text{and, } \beta = \frac{(3+7i)-\sqrt{(3+7i)^2+8(3-9i)}}{4}$$

$$\Rightarrow \quad \alpha = \frac{(3+7i)+\sqrt{9-49+42i+24-72i}}{4} \text{ and, } \beta = \frac{(3+7i)-\sqrt{9-49+42i+24-72i}}{4}$$

$$\Rightarrow \quad \alpha = \frac{3+7i+\sqrt{-16-30i}}{4} \quad \text{and,} \quad \beta = \frac{3+7i-\sqrt{-16-30i}}{4} \quad \text{...(i)}$$

Let us now find $\sqrt{-16-30\,i}$. Let $a+ib = \sqrt{-16-30i}$. Then,

$$a+ib = \sqrt{-16-30i}$$

$$\Rightarrow \quad a^2-b^2+2i\,ab = -16-30i$$

$$\Rightarrow \quad a^2-b^2 = -16 \quad \text{...(ii)} \quad \text{and,} \quad 2\,ab = -30 \quad \text{...(iii)}$$

$$\therefore \quad (a^2+b^2)^2 = (a^2-b^2)^2+4\,a^2\,b^2$$

$$\Rightarrow \quad (a^2+b^2)^2 = 256+900 = 1156 \Rightarrow a^2+b^2 = 34$$

Now, $a^2-b^2 = -16$ and $a^2+b^2 = 34 \Rightarrow a^2 = 9$ and $b^2 = 25 \Rightarrow a = \pm 3$ and $b = \pm 5$

From (iii), we find that a and b are of opposite signs.

$$\therefore \quad a = 3 \text{ and } b = -5 \text{ or, } a = -3 \text{ and } b = 5.$$

Hence, $\sqrt{-16-30i} = a+ib = 3-5i$ or, $-3+5i$. Substituting either of these values in (i), we get

$$\alpha = \frac{(3+7i)+(3-5i)}{4} \text{ and, } \beta = \frac{(3+7i)-(3-5i)}{4} \Rightarrow \alpha = \frac{3}{2}+\frac{1}{2}i \text{ and, } \beta = 3i$$

Hence, the roots of the given equation are $\frac{3}{2}+\frac{1}{2}i$ and $3i$

EXAMPLE 5 *Solve:* $x^2-(7-i)\,x+(18-i)=0$ *over C.*

SOLUTION Comparing the given equation with the general form $ax^2+bx+c=0$, we get $a=1, b=-(7-i)$ and $c=18-i$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and, } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{(7-i)+\sqrt{(7-i)^2-4\,(18-i)}}{2}, \quad \beta = \frac{(7-i)-\sqrt{(7-i)^2-4\,(18-i)}}{2}$$

$$\Rightarrow \quad \alpha = \frac{(7-i)+\sqrt{-24-10i}}{2}, \quad \beta = \frac{(7-i)-\sqrt{-24-10i}}{2} \quad \text{...(i)}$$

Let us now find $\sqrt{-24-10i}$. Let, $a+ib=\sqrt{-24-10i}$. Then,

$$(a+ib)^2 = -24-10i$$

$$\Rightarrow \quad (a^2-b^2)+2i\,ab = -24-10i$$

$$\Rightarrow \quad a^2-b^2 = -24 \quad \text{...(ii)} \quad \text{and,} \quad 2ab = -10 \quad \text{...(iii)}$$

$$\therefore \quad (a^2+b^2)^2 = (a^2-b^2)^2+4\,a^2\,b^2$$

$$\Rightarrow \quad (a^2+b^2)^2 = 576+100 = 676 \Rightarrow a^2+b^2 = 26 \quad \text{...(iv)}$$

Solving (ii) and (iv), we get $a=\pm 1$ and $b=\pm 5$. From (iii), we find that ab is negative.

$$\therefore \quad a = 1, b = -5 \text{ or, } a = -1, b = 5 \Rightarrow a+ib = 1-5i \text{ or, } -1+5i$$

Hence, $\sqrt{-24-10i} = a+ib = \pm(1-5i)$

Substituting either of these values in (i), we get

$$\alpha = \frac{7-i+1-5i}{2} = 4-3i \text{ and } \beta = \frac{(7-i)-(1-5i)}{2} = 3+2i$$

Hence, the roots of the given equation are $4-3i$ and $3+2i$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 6 *Find the value of P such that the difference of the roots of the equation* $x^2-Px+8=0$ *is* 2. [NCERT EXEMPLAR]

SOLUTION Let α, β be the roots of the equation $x^2 - Px + 8 = 0$. Then, $\alpha + \beta = P$ and $\alpha\beta = 8$.

It is given that

$$\alpha - \beta = 2 \Rightarrow (\alpha - \beta)^2 = 2^2 \Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 4 \Rightarrow P^2 - 4 \times 8 = 4 \Rightarrow P^2 = 36 \Rightarrow P = \pm 6.$$

EXAMPLE 7 *Find the real value of a such that the sum of the squares of the roots of the equation $x^2 - (a-2)x - (a+1) = 0$ is least.*

SOLUTION Let α, β be the roots of the equation $x^2 - (a-2)x - (a+1) = 0$. Then,

$$\alpha + \beta = a - 2 \text{ and } \alpha\beta = -(a+1)$$

$$\therefore \quad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (a-2)^2 + 2(a+1) = a^2 - 2a + 6 = (a-1)^2 + 5$$

Now, $(a-1)^2 \geq 0$ for all real values of $a \Rightarrow (a-1)^2 + 5 \geq 5$ for all $a \in R$. $\Rightarrow \alpha^2 + \beta^2 \geq 5$ for all $a \in R$.

Thus, the least value of $\alpha^2 + \beta^2$ is 5.

Now, $\alpha^2 + \beta^2 = 5 \Rightarrow (a-1)^2 + 5 = 5 \Rightarrow (a-1)^2 = 0 \Rightarrow a = 1$.

Hence, the sum of the squares of the roots is least when $a = 1$.

EXERCISE 13.2

1. Solving the following quadratic equations by factorization method:
   (i) $x^2 + 10ix - 21 = 0$ (ii) $x^2 + (1 - 2i)x - 2i = 0$
   (iii) $x^2 - (2\sqrt{3} + 3i)x + 6\sqrt{3}i = 0$ (iv) $6x^2 - 17ix - 12 = 0$
2. Solve the following quadratic equations:
   (i) $x^2 - (3\sqrt{2} + 2i)x + 6\sqrt{2}i = 0$ (ii) $x^2 - (5 - i)x + (18 + i) = 0$
   (iii) $(2 + i)x^2 - (5 - i)x + 2(1 - i) = 0$ (iv) $x^2 - (2 + i)x - (1 - 7i) = 0$
   (v) $ix^2 - 4x - 4i = 0$ (vi) $x^2 + 4ix - 4 = 0$
   (vii) $2x^2 + \sqrt{15}\,ix - i = 0$ [NCERT] (viii) $x^2 - x + (1 + i) = 0$
   (ix) $ix^2 - x + 12i = 0$ [NCERT] (x) $x^2 - (3\sqrt{2} - 2i)x - \sqrt{2}\,i = 0$
   (xi) $x^2 - (\sqrt{2} + i)x + \sqrt{2}i = 0$ [NCERT] (xii) $2x^2 - (3 + 7i)x + (9i - 3) = 0$

ANSWERS

1. (i) $-3i, -7i$ (ii) $-1, 2i$ (iii) $2\sqrt{3}, 3i$ (iv) $\frac{3}{2}i, \frac{4}{3}i$
2. (i) $3\sqrt{2}, 2i$ (ii) $3 - 4i, 2 + 3i$ (iii) $1 - i, \frac{4}{5} - \frac{2}{5}i$ (iv) $3 - i, -1 + 2i$
   (v) $-2i, -2i$ (vi) $-2i, -2i$ (vii) $\frac{1 + (4 - \sqrt{15})i}{4}, \frac{-1 - (\sqrt{15} + 4)i}{4}$
   (viii) $1 - i, i$ (ix) $-4i, 3i$ (x) $\frac{3\sqrt{2} - 2i}{2} \pm \frac{4 - \sqrt{2}i}{2}$
   (xi) $\sqrt{2}, i$ (xii) $\frac{3 + i}{2}, 3i$

HINTS TO SELECTED PROBLEMS

2. (vii) The given equation is $2x^2 + \sqrt{15}\,i\,x - i = 0$. Comparing this equation with the standard equation $ax^2 + bx + c = 0$, we get: $a = 2, b = \sqrt{15}i$ and $c = -i$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and, } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we obtain}$$

$$\alpha = \frac{-\sqrt{15}\,i+\sqrt{-15+8i}}{4} \text{ and, } \beta = \frac{-\sqrt{15}\,i-\sqrt{-15+8\,i}}{4} \quad \ldots \text{(i)}$$

Let $\sqrt{-15+8i} = a+ib$. Then,

$$-15+8i = (a+ib)^2 \Rightarrow -15+8i = a^2-b^2+2i\,ab \Rightarrow a^2-b^2 = -15 \text{ and } 2ab = 8$$

$$\therefore \quad (a^2+b^2)^2 = (a^2-b^2)^2+4a^2b^2 \Rightarrow (a^2+b^2)^2 = (-15)^2+64 = 289 \Rightarrow a^2+b^2 = 17$$

Solving $a^2-b^2 = -15$ and $a^2+b^2 = 17$, we obtain

$$a^2 = 1 \text{ and } b^2 = 16 \Rightarrow a = \pm 1 \text{ and } b = \pm 4$$

$$\Rightarrow \quad a = 1, b = 4 \text{ or } a = -1, b = -4 \quad [\because\ ab = 4 > 0 \ \therefore\ a \text{ and } b \text{ are of the same sign}]$$

$$\therefore \quad \sqrt{-15+8i} = a+ib = 1+4i, -1-4i$$

Substituting either of these values in (i), we obtain

$$\alpha = -\frac{1}{4}-\frac{1}{4}(\sqrt{15}+4)\,i, \beta = \frac{1}{4}+\frac{1}{4}(-\sqrt{15}+4)\,i$$

(ix) The given equation is $i\,x^2-x+12i=0$. Comparing this equation with the standard equation $ax^2+bx+c = 0$, we get $a=i, b = -1$ and $c=12i$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{1+\sqrt{1+48}}{2i} \text{ and } \beta = \frac{1-\sqrt{1+48}}{2i}$$

$$\Rightarrow \quad \alpha = \frac{1+7}{2i} \text{ and } \beta = \frac{1-7}{2i} \Rightarrow \alpha = \frac{4}{i} \text{ and } \beta = -\frac{3}{i} \Rightarrow \alpha = 0-4i \text{ and } \beta = 3i$$

(xi) The given equation is $x^2-(\sqrt{2}+i)\,x+\sqrt{2}\,i=0$. Comparing this equation with the standard equation $ax^2+bx+c = 0$, we get $a=1, b = -(\sqrt{2}+i)$ and $c=\sqrt{2}\,i$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{(\sqrt{2}+i)+\sqrt{(\sqrt{2}+i)^2-4\sqrt{2}\,i}}{2} \text{ and } \beta = \frac{(\sqrt{2}+i)-\sqrt{(\sqrt{2}+i)^2-4\sqrt{2}\,i}}{2}$$

$$\Rightarrow \quad \alpha = \frac{\sqrt{2}+i+\sqrt{(\sqrt{2}-i)^2}}{2} \text{ and } \beta = \frac{(\sqrt{2}+i)-\sqrt{(\sqrt{2}-i)^2}}{2}$$

$$\Rightarrow \quad \alpha = \frac{\sqrt{2}+i+\sqrt{2}-i}{2} \text{ and } \beta = \frac{(\sqrt{2}+i)-(\sqrt{2}-i)}{2} \Rightarrow \alpha = \sqrt{2} \text{ and } \beta = i$$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If $1-i$ is a root of the equation $x^2+ax+b=0$, where $a, b \in R$, then the values of a and b are.......................... .
2. If the difference of the roots of the equation $x^2-Px+8=0$ is 2, then P =............................. .

3. If the equation $2x^2 - kx + x + 8 = 0$ has real and equal roots, then $k =$ ....
4. The number of real roots of the equation $x^2 + 5|x| + 4 = 0$ is ....
5. If one root of the equation $x^2 + px + 12 = 0$ is 4, then the sum of the roots is ....
6. If α, β are roots of the equation $x^2 + x + 1 = 0$, then the equation whose roots are α^{19} and β^7 is..................... .
7. The value of $\sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots \text{to } \infty}}}$ is ....
8. If the equations $px^2 + 2qx + r = 0$ and $qx^2 - 2\sqrt{pr}\,x + q = 0$ have real roots, then $q^2 =$ ....
9. If the roots of the equation $x^2 - 8x + a^2 - 6a = 0$ are real, then 'a' lies in the interval ....
10. If the equations $x^2 + x + a = 0$ and $x^2 + ax + 1 = 0$, $a \neq 1$, have a common root, then $a =$.......... .
11. If the quadratic equation $2x^2 - (a^3 + 8a - 1)\, x + a^2 - 4a = 0$ possesses roots of opposite signs, then a lies in the interval ....

ANSWERS

1. $a = -2, b = 2$ 2. ± 6 3. $9, -7$ 4. 0 5. 7 6. $x^2 + x + 1 = 0$
7. 3 8. pr 9. $[-2, 8]$ 10. -2 11. $(0, 4)$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the number of real roots of the equation $(x-1)^2 + (x-2)^2 + (x-3)^2 = 0$.
2. If a and b are roots of the equation $x^2 - px + q = 0$, then write the value of $\frac{1}{a} + \frac{1}{b}$.
3. If roots α, β of the equation $x^2 - px + 16 = 0$ satisfy the relation $\alpha^2 + \beta^2 = 32$, then write the value of p.
4. If $2 + \sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, then write the values of p and q.
5. If the difference between the roots of the equation $x^2 + ax + 8 = 0$ is 2, write the values of a.
6. Write the roots of the equation $(a-b)\, x^2 + (b-c)\, x + (c-a) = 0$.
7. If a and b are roots of the equation $x^2 - x + 1 = 0$, then write the value of $a^2 + b^2$.
8. Write the number of quadratic equations, with real roots, which do not change by squaring their roots.
9. If α, β are roots of the equation $x^2 + lx + m = 0$, write an equation whose roots are $-\frac{1}{\alpha}$ and $-\frac{1}{\beta}$.
10. If α, β are roots of the equation $x^2 - a(x+1) - c = 0$, then write the value of $(1+\alpha)(1+\beta)$.

ANSWERS

1. No real root 2. $\frac{p}{q}$ 3. ± 8 4. $p = -4, q = 1$ 5. ± 8 6. $1, \frac{c-a}{a-b}$
7. -1 8. 3 9. $mx^2 - lx + 1 = 0$ 10. $1 - c$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The complete set of values of k, for which the quadratic equation $x^2 - kx + k + 2 = 0$ has equal roots, consists of
(a) $2 + \sqrt{12}$ (b) $2 \pm \sqrt{12}$ (c) $2 - \sqrt{12}$ (d) $-2 - \sqrt{2}$
2. For the equation $|x|^2 + |x| - 6 = 0$, the sum of the real roots is
(a) 1 (b) 0 (c) 2 (d) none of these
3. If a, b are the roots of the equation $x^2 + x + 1 = 0$, then $a^2 + b^2 =$
(a) 1 (b) 2 (c) -1 (d) 3
4. If α, β are roots of the equation $4x^2 + 3x + 7 = 0$, then $1/\alpha + 1/\beta$ is equal to
(a) $7/3$ (b) $-7/3$ (c) $3/7$ (d) $-3/7$
5. The values of x satisfying $\log_3 (x^2 + 4x + 12) = 2$ are
(a) $2, -4$ (b) $1, -3$ (c) $-1, 3$ (d) $-1, -3$
6. The number of real roots of the equation $(x^2 + 2x)^2 - (x + 1)^2 - 55 = 0$ is
(a) 2 (b) 1 (c) 4 (d) none of these
7. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then $\dfrac{1}{a\alpha + b} + \dfrac{1}{a\beta + b} =$
(a) c/ab (b) a/bc (c) b/ac (d) none of these
8. If α, β are the roots of the equation $x^2 + px + 1 = 0$; γ, δ the roots of the equation $x^2 + qx + 1 = 0$, then $(\alpha - \gamma)(\alpha + \delta)(\beta - \gamma)(\beta + \delta) =$
(a) $q^2 - p^2$ (b) $p^2 - q^2$ (c) $p^2 + q^2$ (d) none of these
9. The number of real solutions of $|2x - x^2 - 3| = 1$ is
(a) 0 (b) 2 (c) 3 (d) 4
10. The number of solutions of $x^2 + |x - 1| = 1$ is
(a) 0 (b) 1 (c) 2 (d) 3
11. If x is real and $k = \dfrac{x^2 - x + 1}{x^2 + x + 1}$, then
(a) $k \in [1/3, 3]$ (b) $k \geq 3$ (c) $k \leq 1/3$ (d) none of these
12. If the roots of $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c$ is
(a) 0 (b) 1 (c) 2 (d) none of these
13. The value of a such that $x^2 - 11x + a = 0$ and $x^2 - 14x + 2a = 0$ may have a common root is
(a) 0 (b) 12 (c) 24 (d) 32.
14. The values of k for which the quadratic equation $kx^2 + 1 = kx + 3x - 11x^2$ has real and equal roots are
(a) $-11, -3$, (b) $5, 7$ (c) $5, -7$ (d) none of these
15. If the equations $x^2 + 2x + 3\lambda = 0$ and $2x^2 + 3x + 5\lambda = 0$ have a non-zero common roots, then $\lambda =$

(a) 1 (b) -1 (c) 3 (d) none of these

16. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, the value of q is
(a) 49/4 (b) 4/49 (c) 4 (d) none of these

17. The value of p and q $(p \neq 0, q \neq 0)$ for which p, q are the roots of the equation $x^2 + px + q\ ab = 0$ are
(a) $p = 1, q = -2$ (b) $p = -1, q = -2$
(c) $p = -1, q = 2$ (d) $p = 1, q = 2$

18. The set of all values of m for which both the roots of the equation $x^2 - (m+1)x + m + 4 = 0$ are real and negative, is
(a) $(-\infty, -3] \cup [5, \infty)$ (b) $[-3, 5]$
(c) $(-4, -3]$ (d) $(-3, -1]$

19. The number of roots of the equation $\frac{(x+2)(x-5)}{(x-3)(x+6)} = \frac{x-2}{x+4}$ is
(a) 0 (b) 1 (c) 2 (d) 3

20. If α and β are the roots of $4x^2 + 3x + 7 = 0$, then the value of $\frac{1}{\alpha} + \frac{1}{\beta}$ is
(a) $\frac{4}{7}$ (b) $-\frac{3}{7}$ (c) $\frac{3}{7}$ (d) $-\frac{3}{4}$

21. If, α, β are the roots of the equation $x^2 + px + q = 0$, then $-\frac{1}{\alpha}, -\frac{1}{\beta}$ are the roots of the equation
(a) $x^2 - px + q = 0$ (b) $x^2 + px + q = 0$
(c) $qx^2 + px + 1 = 0$ (d) $qx^2 - px + 1 = 0$

22. If the difference of the roots of $x^2 - px + q = 0$ is unity, then
(a) $p^2 + 4q = 1$ (b) $p^2 - 4q = 1$
(c) $p^2 + 4q^2 = (1 + 2q)^2$ (d) $4p^2 + q^2 = (1 + 2p)^2$

23. If α, β are the roots of the equation $x^2 - p(x+1) - c = 0$, then $(\alpha + 1)(\beta + 1) =$
(a) c (b) $c - 1$ (c) $1 - c$ (d) none of these

24. The least value of k which makes the roots of the equation $x^2 + 5x + k = 0$ imaginary is
(a) 4 (b) 5 (c) 6 (d) 7

25. The equation of the smallest degree with real coefficients having $1 + i$ as one of the roots is
(a) $x^2 + x + 1 = 0$ (b) $x^2 - 2x + 2 = 0$
(c) $x^2 + 2x + 2 = 0$ (d) $x^2 + 2x - 2 = 0$

ANSWERS

1. (b) 2. (b) 3. (c) 4. (d) 5. (d) 6. (a) 7. (c) 8. (a)
9. (a) 10. (a) 11. (a) 12. (b) 13. (c) 14. (c) 15. (b) 16. (a)
17. (c) 18. (a) 19. (b) 20. (b) 21. (d) 22. (b), (c) 23. (c)
24. (d) 25. (b)

ACTIVITY

OBJECTIVE Graphically to obtain a quadratic function with the help of two linear functions.

MATERIALS REQUIRED Cardboard, drawing sheet, wires, thumbpins, adhesive etc.

METHOD OF CONSTRUCTION

Step I Take a cardboard and paste a drawing sheet on it.

Step II Draw two mutually perpendicular lines on the drawing sheet as the coordinate axes as shown in Fig. 14.1.

Step III Take a wire and fix it on the drawing sheet in such a way that it cuts OX and OY' at a distance 'a' from the origin.

Step IV Take another wire and fix it on the drawing sheet by using thumbpins in such a way that it cuts OX and OY' at a distance 'b' from the origin.

Step V Take one more wire, bend it in parabolic shape and fix it in such a way that it passes through $A(a, 0)$ and $B(b, 0)$.

DEMONSTRATION

Step I The coordinates of the two points where first wire cuts OX and OY' are $A\,(a, 0)$ and $A'(0, -a)$ respectively and the equation of the line along the wire is $y = x - a$.

Step II The coordinates of the points where second wire cuts OX and $OY'$are $B\,(b, 0)$ and $B'\,(0, -b)$ respectively and the equation of the line along the wire is $y = x - b$.

Step III The product of linear functions $y = x - a$ and $y = x - b$ is $y = (x-a)\,(x-b)$, which represents a quadratic function. It cuts the x-axis at $A\,(a, 0)$ and $B\,(b, 0)$. Thus, the parabolic shaped wire represents the quadratic function $y = (x-a)\,(x-b)$.

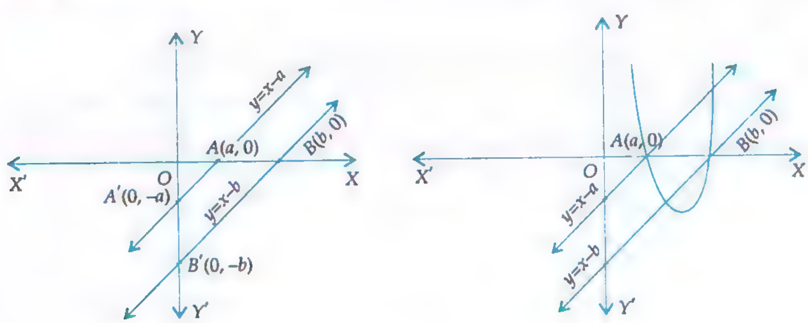


Fig.14.1

SUMMARY

1. Fundamental Theorem of Algebra: Every polynomial equation $f(x) = 0$ has at least one root, real or imaginary (complex).
2. Every polynomial equation $f(x) = 0$ of degree n has exactly n roots real or imaginary.
3. A quadratic equation cannot have more than two roots.
4. If $ax^2 + bx + c = 0, a \neq 0$ is a quadratic equation with real coefficients, then its roots α and β given by

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and, } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a} \text{ or, } \alpha = \frac{-b+\sqrt{D}}{2a} \text{ and, } \beta = \frac{-b-\sqrt{D}}{2a}$$

where $D = b^2 - 4ac$ is as the discriminant of the equation.

(i) If $D = 0$, then $\alpha = \beta = -\frac{b}{2a}$

So, the equation has real and equal roots each equal to $-\frac{b}{2a}$.

(ii) If $a, b, c \in Q$ and D is positive and a perfect square, then roots are rational and unequal.

(iii) If $a, b, c \in R$ and D is positive and a perfect square, then the roots are real and distinct.

(iv) If $D > 0$ but it is not a perfect square, then roots are irrational and unequal.

(v) If $D < 0$, then the roots are imaginary and are given by

$$\alpha = \frac{-b+i\sqrt{4ac-b^2}}{2a} \text{ and } \beta = \frac{-b-i\sqrt{4ac-b^2}}{2a}$$

(vi) If $a = 1, b, c \in I$ and the roots are rational numbers, then these roots must be integers.

(vii) If a quadratic equation in x has more than two roots, then it is an identity in x that is $a = b = c = 0$.

(viii) Complex roots of an equation with real coefficients always occur in pairs. However, this may not be true in case of equations with complex coefficients. For example, $x^2 - 2ix - 1 = 0$ has both roots equal to i.

(ix) Surd root of an equation with rational coefficients always occur in pairs like $2+\sqrt{3}$ and $2-\sqrt{3}$. However, Surd roots of an equation with irrational coefficients may not occur in pairs. For example, $x^2 - 2\sqrt{3}\,x + 3 = 0$ has both roots equal to $\sqrt{3}$.

CHAPTER 14

LINEAR INEQUATIONS

14.1 INTRODUCTION

In this chapter, we will study linear inequations in one and two variables. The knowledge of linear inequations is very helpful in solving problems in Science, Mathematics, Engineering, Linear Programming etc.

14.2 INEQUATIONS

In earlier classes, we have studied equations in one and two variables. An equation is defined as a statement involving variable (s) and the sign of equality (=). Similarly, we define the term inequation as follows:

INEQUATION *A statement involving variable (s) and the sign of inequality viz, >, <, ≥ or ≤ is called an inequation or an inequality.*

An inequation may contain one or more variables. Also, it may be linear or quadratic or cubic etc.

Following are some examples of inequations:

(i) $3x-2<0$ (ii) $2x+3\le 0$ (iii) $5x-3>0$

(iv) $4x+5\ge 0$ (v) $2x+3y<1$ (vi) $5x+4y\le 3$

(vii) $4x-6y>5$ (viii) $2x+5y\ge 4$ (ix) $2x^2+3x+4>0$

(x) $x^2-3x+2\ge 0$ (xi) $x^2+3x+2<0$ (xii) $x^2-5x+4\le 0$

(xiii) $x^3-6x^2+11x-6>0$ (xiv) $x^3+6x^2+11x+6\le 0$

LINEAR INEQUATION IN ONE VARIABLE *Let a be a non-zero real number and x be a variable. Then inequations of the form $ax+b<0$, $ax+b\le 0$, $ax+b>0$ and $ax+b\ge 0$ are known as linear inequations in one variable x.*

For example, $9x-15>0$, $5x-4\ge 0$, $3x+2<0$ and $2x-3\le 0$ are linear inequations in one variable.

LINEAR INEQUATIONS IN TWO VARIABLES *Let a, b be non-zero real numbers and x, y be variables. Then inequations of the form $ax+by<c$, $ax+by\le c$, $ax+by>c$ and $ax+by\ge c$ are known as linear inequations in two variables x* and *y*.

For example, $2x+3y\le 6$, $3x-2y\ge 12$, $x+y<4$, $2x+y\ge 6$ are linear inequations in two variables x and y.

QUADRATIC INEQUATION *Let a be a non-zero real number. Then an inequation of the form $ax^2+bx+c<0$, or $ax^2+bx+c\le 0$, or $ax^2+bx+c>0$, or $ax^2+bx+c\ge 0$ is known as a quadratic inequation.*

For example, $x^2+x-6<0$, $x^2-3x+2\ge 0$, $2x^2+3x+1>0$ and $x^2-5x+4\le 0$ are quadratic inequations.

In this chapter, we shall study linear inequations in one and two variables only.

14.3 SOLUTIONS OF AN INEQUATION

DEFINITION *A solution of an inequation is the value (s) of the variable (s) that makes it a true statement.*

Consider the inequation $\frac{3-2x}{5} < \frac{x}{3} - 4$.

Left hand side (LHS) of this inequation is $\frac{3-2x}{5}$ and right hand side (RHS) is $\frac{x}{3} - 4$.

We observe that:

For $x = 9$, we have

$$\text{LHS} = \frac{3-2\times 9}{5} = -3 \text{ and, RHS} = \frac{9}{3} - 4 = -1$$

Clearly, $-3 < -1$

$\Rightarrow$ LHS < RHS, which is true.

So, $x = 9$ is a solution of the given inequation.

For $x = 6$, we have

$$\text{LHS} = \frac{3-2\times 6}{5} = -\frac{9}{5} \text{ and RHS} = \frac{6}{3} - 4 = -2$$

Because, $-\frac{9}{5} < -2$ is not true. So, $x = 6$ is not a solution of the given inequation.

We can verify that any real number greater than 7 is a solution of the given inequation.

Let us now consider the inequation $x^2 + 1 < 0$.

We know that

$$x^2 \geq 0 \text{ for all } x \in R$$

$$\therefore \quad x^2 + 1 \geq 1 \text{ for all } x \in R$$

$$\Rightarrow \quad x^2 + 1 \nless 0 \text{ for any } x \in R.$$

So, there is no real value of x which makes the given inequation a true statement. Hence, it has no solution.

It follows from the above discussion that an inequation may or may not have a solution. However, if an inequation has a solution it may have infinitely many solutions.

SOLVING AN INEQUATION *It is the process of obtaining all possible solutions of an inequation.*

SOLUTION SET *The set of all posible solutions of an inequation is known as its solution set.*

For example, the solution set of the inequation $x^2 + 1 \geq 0$ is the set R of all real numbers whereas the solution set of the inequation $x^2 + 1 < 0$ is the null set ϕ.

14.4 SOLVING LINEAR INEQUATIONS IN ONE VARIABLE

As mentioned in the previous section that solving an inequation is the process of obtaining its all possible solutions. In the process of solving an inequation, we use mathematical simplifications which are governed by the following rules:

Rule 1 *Same number may be added to (or subtracted from) both sides of an inequation without changing the sign of inequality.*

Rule 2 *Both sides of an inequation can be multiplied (or divided) by the same positive real number without changing the sign of inequality. However, the sign of inequality is reversed when both sides of an inequation are multiplied or divided by a negative number.*

Rule 3 *Any term of an inequation may be taken to the other side with its sign changed without affecting the sign of inequality.*

A linear inequation in one variable is of the form

$$ax + b < 0 \text{ or, } ax + b \leq 0 \text{ or, } ax + b > 0 \text{ or, } ax + b \geq 0.$$

We follow the following algorithm to solve a linear inequation in one variable.

ALGORITHM

Step I *Obtain the linear inequation.*

Step II *Collect all terms involving the variable on one side of the inequation and the constant terms on the other side.*

Step III *Simplify both sides of inequality in their simplest forms to reduce the inequation in the form* $ax < b$, or $ax \leq b$, or $ax > b$, or $ax \geq b$

Step IV *Solve the inequation obtained in step III by dividing both sides of the inequation by the coefficient of the variable.*

Step V *Write the solution set obtained in step IV in the form of an interval on the real line.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I SOLVING EQUATIONS OF THE FORM: $ax + b > cx + d$, or, $ax + b \geq cx + d$, or, $ax + b < cx + d$ or, $ax + b \leq cx + d$

EXAMPLE 1 *Solve the following linear inequations:*

(i) $2x - 4 \leq 0$ (ii) $-3x + 12 < 0$ (iii) $4x - 12 \geq 0$ (iv) $7x + 9 > 30$

SOLUTION (i) We have,

$$2x - 4 \leq 0$$

$\Rightarrow$ $(2x - 4) + 4 \leq 0 + 4$ [Adding 4 on both sides]

$\Rightarrow$ $2x \leq 4 \Rightarrow \frac{2x}{2} \leq \frac{4}{2} \Rightarrow x \leq 2$

Hence, any real number less than or equal to 2 is a solution of the given inequation.
These solutions can be graphed on real line as shown in Fig. 14.1

Fig. 14.1

The solution set of the given inequation is $(-\infty, 2]$

(ii) We have,

$$-3x + 12 < 0$$

$\Rightarrow$ $-3x < -12$ [Transposing 12 on right side]

$\Rightarrow$ $\frac{-3x}{-3} > \frac{-12}{-3} \Rightarrow x > 4$ [Dividing both sides by -3]

Thus, any real number greater than 4 is a solution of the given inequation.
Hence, the solution set of the given inequation is $(4, \infty)$. This solution set can be graphed on real line as shown in Fig. 14.2

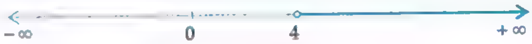


Fig. 14.2

(iii) We have,

$$4x - 12 \geq 0$$

$$\Rightarrow \quad 4x \geq 12 \qquad \text{[Transposing 12 on RHS]}$$

$$\Rightarrow \quad \frac{4x}{4} \geq \frac{12}{4} \qquad \text{[Dividing both sides by 4]}$$

$$\Rightarrow \quad x \geq 3 \Rightarrow x \in [3, \infty)$$

Hence, the solution set of the given inequation is $[3, \infty)$. This solution set can be graphed on real line as shown in Fig. 14.3

Fig. 14.3

(iv) We have,

$$7x + 9 > 30$$

$$\Rightarrow \quad 7x > 30 - 9$$

$$\Rightarrow \quad 7x > 21 \qquad \text{[Transposing 9 on RHS]}$$

$$\Rightarrow \quad \frac{7x}{7} > \frac{21}{7} \Rightarrow x > 3 \Rightarrow x \in (3, \infty)$$

Hence, $(3, \infty)$ is the solution set of the given inequation. This can be graphed on real line as shown in Fig. 14.4.

Fig.14.4

EXAMPLE 2 *Solve:* $5x - 3 < 3x + 1$ *when (i) x is a real number (ii) x is integer number (iii) x is a natural number.*

SOLUTION We have,

$$5x - 3 < 3x + 1$$

$$\Rightarrow \quad 5x - 3x < 3 + 1 \qquad \text{[Transposing } 3x \text{ on LHS and } -3 \text{ on RHS]}$$

$$\Rightarrow \quad 2x < 4$$

$$\Rightarrow \quad \frac{2x}{2} < \frac{4}{2} \Rightarrow x < 2 \qquad \left[\text{Multiplying both sides by } \frac{1}{2}\right]$$

(i) If $x \in R$, then $x < 2 \Rightarrow x \in (-\infty, 2)$.

Hence, the solution set is $(-\infty, 2)$ as shown in Fig. 14.5.

Fig.14.5

(ii) If $x \in Z$, then $x < 2 \Rightarrow x = 1, 0, -1, -2, -3, -4, \ldots$

So, the solution set is $\{\ldots, -4, -3, -2, -1, 0, 1\}$

(iii) If $x \in N$, then $x < 2 \Rightarrow x = 1$. So, the solution set is $\{1\}$.

EXAMPLE 3 *Solve the following equations:*

(i) $3x + 17 \leq 2(1 - x)$ (ii) $2(2x + 3) - 10 \leq 6(x - 2)$

SOLUTION (i) We have,

$$3x + 17 \leq 2(1 - x)$$

$$\Rightarrow \quad 3x + 17 \leq 2 - 2x$$

$$\Rightarrow \quad 3x + 2x \leq 2 - 17 \qquad \text{[Transposing } -2x \text{ to LHS and 17 to RHS]}$$

$\Rightarrow \quad 5x \le -15$

$\Rightarrow \quad \dfrac{5x}{5} \le \dfrac{-15}{5} \Rightarrow x \le -3 \Rightarrow x \in (-\infty, -3]$

Hence, the solution set of the given inequation is $(-\infty, -3]$, which can be graphed on real line as shown in Fig. 14.6.

Fig.14.6

(ii)We have,

$$2(2x+3) - 10 \le 6(x-2)$$

$\Rightarrow \quad 4x + 6 - 10 \le 6x - 12$

$\Rightarrow \quad 4x - 4 \le 6x - 12$

$\Rightarrow \quad 4x - 6x \le -12 + 4$ [Transposing -4 to RHS and $6x$ to LHS]

$\Rightarrow \quad -2x \le -8 \Rightarrow \dfrac{-2x}{-2} \ge \dfrac{-8}{-2} \Rightarrow x \ge 4 \Rightarrow x \in [4, \infty)$

Hence, the solution set of the given inequation is $[4, \infty)$ which can be graphed on real line as shown in Fig. 14.7.

Fig.14.7

EXAMPLE 4 *Solve the following inequations:*

(i) $\dfrac{2x-3}{4} + 9 \ge 3 + \dfrac{4x}{3}$ (ii) $\dfrac{5x-2}{3} - \dfrac{7x-3}{5} > \dfrac{x}{4}$

(iii) $\dfrac{1}{2}\left(\dfrac{3}{5}x + 4\right) \ge \dfrac{1}{3}(x-6)$ (iv) $\dfrac{3(x-2)}{5} \ge \dfrac{5(2-x)}{3}$

SOLUTION (i) We have,

$$\frac{2x-3}{4} + 9 \ge 3 + \frac{4x}{3}$$

$\Rightarrow \quad \dfrac{2x-3}{4} - \dfrac{4x}{3} \ge 3 - 9$ [Transposing $\dfrac{4x}{3}$ to LHS and 9 to RHS]

$\Rightarrow \quad \dfrac{3(2x-3) - 16x}{12} \ge -6$

$\Rightarrow \quad \dfrac{6x - 9 - 16x}{12} \ge -6$

$\Rightarrow \quad \dfrac{-9 - 10x}{12} \ge -6$

$\Rightarrow \quad -9 - 10x \ge -72$ [Multiplying both sides by 12]

$\Rightarrow \quad -10x \ge -72 + 9$

$\Rightarrow \quad -10x \ge -63 \Rightarrow \dfrac{-10x}{-10} \le \dfrac{-63}{-10} \Rightarrow x \le \dfrac{63}{10} \Rightarrow x \in \left(-\infty, \dfrac{63}{10}\right]$

Hence, the solution set of the given inequation is $\left(-\infty, \dfrac{63}{10}\right]$. This can be graphed on real line as shown in Fig. 14.8.

Fig.14.8

(ii) We have,

$$\frac{5x-2}{3}-\frac{7x-3}{5}>\frac{x}{4}$$

$$\Rightarrow \quad \frac{5(5x-2)-3(7x-3)}{15}>\frac{x}{4}$$

$$\Rightarrow \quad \frac{25x-10-21x+9}{15}>\frac{x}{4}$$

$$\Rightarrow \quad \frac{4x-1}{15}>\frac{x}{4}$$

$\Rightarrow \quad 4(4x-1)>15x$ [Multiplying both sides by 60 i.e. lcm of 15 and 4]

$\Rightarrow \quad 16x-4>15x$

$\Rightarrow \quad 16x-15x>4$ [Transposing $15x$ to LHS and -4 to RHS]

$\Rightarrow \quad x>4 \Rightarrow x \in (4, \infty)$

Hence, the solution set of the given inequation is $(4, \infty)$. This can be graphed on the real line as shown in Fig. 14.9.

Fig.14.9

(iii) We have,

$$\frac{1}{2}\left(\frac{3}{5}x+4\right)\geq\frac{1}{3}(x-6)$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{3x+20}{5}\right)\geq\frac{1}{3}(x-6)$$

$$\Rightarrow \quad \frac{3x+20}{10}\geq\frac{x-6}{3}$$

$\Rightarrow \quad 3(3x+20)\geq 10(x-6)$ [Multiplying both sides by 30 i.e. the lcm of 3 and 10]

$\Rightarrow \quad 9x+60\geq 10x-60$

$\Rightarrow \quad 9x-10x\geq -60-60$ [Transposing $10x$ on LHS and 60 on RHS]

$\Rightarrow \quad -x\geq -120$

$\Rightarrow \quad x\leq 120$ [Multiplying both sides by -1]

$\Rightarrow \quad x\in(-\infty, 120]$

Hence, the solution set of the given in equation is $(-\infty, 120]$ which can be graphed on real line as shown in Fig. 14.10.

Fig.14.10

(iv) We have,

$$\frac{3(x-2)}{5}\geq\frac{5(2-x)}{3}$$

$$\Rightarrow \quad \frac{3x-6}{5}\geq\frac{10-5x}{3}$$

$\Rightarrow \quad 3(3x-6)\geq 5(10-5x)$ [Multiplying both sides by 15 i.e. the lcm of 5 and 3]

$\Rightarrow \quad 9x-18\geq 50-25x$

$\Rightarrow \quad 9x+25x\geq 50+18$ [Transposing $-25x$ to LHS and 18 to RHS]

$$\Rightarrow \quad 34x\geq 68 \Rightarrow \frac{34x}{34}\geq\frac{68}{34} \Rightarrow x\geq 2 \Rightarrow x\in[2, \infty)$$

Hence, $[2, \infty)$ is the solution set of the given inequation. This solution set can be graphed on real line as shown in Fig. 14.11.

Fig. 14.11

EXAMPLE 5 *Solve the following inequations:*

(i) $\dfrac{1}{x-2} < 0$ (ii) $\dfrac{x+1}{x+2} \geq 1$

SOLUTION (i) We have,

$$\frac{1}{x-2} < 0$$

$$\Rightarrow \quad x - 2 < 0 \qquad \left[\because \frac{a}{b} < 0 \text{ and } a > 0 \Rightarrow b < 0\right]$$

$$\Rightarrow \quad x < 2 \Rightarrow x \in (-\infty, 2)$$

Hence, the solution set of the given inequation is $(-\infty, 2)$.

(ii) We have,

$$\frac{x+1}{x+2} \geq 1 \Rightarrow \frac{x+1}{x+2} - 1 \geq 0 \Rightarrow \frac{x+1-x-2}{x+2} \geq 0 \Rightarrow \frac{-1}{x+2} \geq 0$$

$$\Rightarrow \quad x + 2 < 0 \qquad \left[\because \frac{a}{b} > 0 \text{ and } a < 0 \Rightarrow b < 0\right]$$

$$\Rightarrow \quad x < -2 \Rightarrow x \in (-\infty, -2)$$

Hence, the solution set of the given inequation is $(-\infty, -2)$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type II EQUATIONS OF THE FORM

$$\frac{ax+b}{cx+d} > k, \text{ or } \frac{ax+b}{cx+d} \geq k, \text{ or } \frac{ax+b}{cx+d} < k, \text{ or } \frac{ax+b}{cx+d} \leq k$$

In order to solve this type of inequation, we use the following algorithm.

ALGORITHM

Step I *Obtain the inequation.*

Step II *Transpose all terms on LHS.*

Step III *Simplify LHS of the inequation obtained in step II to obtain an inequation of the form*

$$\frac{px+q}{rx+s} > 0, \text{ or } \frac{px+q}{rx+s} \geq 0, \text{ or } \frac{px+q}{rx+s} < 0, \text{ or } \frac{px+q}{rx+s} \leq 0.$$

Step IV *Make coefficient x positive in numerator and denominator if they are not.*

Step V *Equate numerator and denominator separately to zero and obtain the values of x. These values of x are generally called critical points.*

Step VI *Plot the critical points obtained in step V on real line. These points will divide the real line in three regions.*

Step VII *In the right most region the expression on LHS of the inequation obtained in step IV will be positive and in other regions it will be alternatively negative and positive. So, mark positive sign in the right most region and then mark alternatively negative and positive signs in other regions.*

Step VIII *Select appropriate region on the basis of the sign of the inequation obtained in step IV. Write these regions in the form of intervals to obtain the desired solution sets of the given inequation.*

EXAMPLE 6 *Solve the following linear inequations:*

(i) $\dfrac{x-3}{x-5} > 0$ (ii) $\dfrac{x-2}{x+5} > 2$

SOLUTION (i) We have,

$$\frac{x-3}{x-5} > 0 \qquad \text{...(i)}$$

Equating $x-3$ and $x-5$ to zero, we obtain $x = 3, 5$ as critical points. Plot these points on real line as shown in Fig. 14.12. The real line is divided into three regions. In the right most region the expression on LHS of (i) is positive and in the remaining two regions it is alternatively negative and positive as shown in Fig. 14.12.

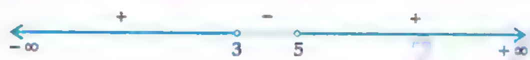


Fig. 14.12

Since the expression in (i) is positive, so the solution set of the given inequation is the union of regions containing positive signs. Hence, from Fig. 14.12

$$\frac{x-3}{x-5} > 0 \Rightarrow x \in (-\infty, 3) \cup (5, \infty)$$

Hence, the solution set of the given inequation is $(-\infty, 3) \cup (5, \infty)$ as shown in Fig. 14.12.

(ii) We have,

$$\frac{x-2}{x+5} > 2 \Rightarrow \frac{x-2}{x+5} - 2 > 0 \Rightarrow \frac{x-2-2(x+5)}{x+5} > 0 \Rightarrow \frac{x-2-2x-10}{x+5} > 0 \Rightarrow \frac{-x-12}{x+5} > 0$$

$$\Rightarrow \frac{x+12}{x+5} < 0 \qquad \left[\begin{array}{l}\text{Multiplying by } -1 \text{ to make coefficient of} \\ x \text{ positive in the expression in numerator}\end{array}\right] \quad \text{...(i)}$$

On equating $x+12$ and $x+5$ to zero, we obtain $x = -12, -5$ as critical points. These points are plotted on number line as shown in Fig. 14.12. The real line is divided into three regions and the signs of LHS of inequation (i) are marked. Since the inequation in (i) possesses less than sign which means that LHS of the inequation is negative. So, the solution set of the given inequation is the union of the regions ontaining negative sign in Fig. 14.13. Hence, the solution set of the given inequation is $(-12, -5)$.

Fig. 14.13

EXAMPLE 7 *Solve the following inequations:*

(i) $\dfrac{2x+4}{x-1} \geq 5$ (ii) $\dfrac{x+3}{x-2} \leq 2$ **[NCERT EXEMPLAR]**

SOLUTION (i) We have,

$$\frac{2x+4}{x-1} \geq 5$$

$$\Rightarrow \frac{2x+4}{x-1} - 5 \geq 0 \Rightarrow \frac{2x+4-5(x-1)}{x-1} \geq 0 \Rightarrow \frac{2x+4-5x+5}{x-1} \geq 0 \Rightarrow \frac{-3x+9}{x-1} \geq 0$$

$$\Rightarrow \frac{3x-9}{x-1} \leq 0 \qquad \text{[Multiplying both sides by } -1]$$

$$\Rightarrow \frac{3(x-3)}{(x-1)} \leq 0$$

$$\Rightarrow \frac{x-3}{x-1} \leq 0 \qquad \text{[Dividing both sides by 3]}$$

$$\Rightarrow 1 < x \leq 3 \qquad \text{[See Fig. 14.14]}$$

Fig. 14.14

$\Rightarrow \quad x \in (1, 3]$

Hence, the solution set of the given inequation is $(1, 3]$.

(ii) We have,

$$\frac{x+3}{x-2} \leq 2$$

$$\Rightarrow \quad \frac{x+3}{x-2} - 2 \leq 0 \Rightarrow \frac{x+3-2x+4}{x-2} \leq 0 \Rightarrow \frac{-x+7}{x-2} \leq 0$$

$$\Rightarrow \quad \frac{x-7}{x-2} \geq 0 \qquad \text{[Multiplying both sides by } -1]$$

$$\Rightarrow \quad x \in (-\infty, 2) \cup [7, \infty) \qquad \text{[See Fig. 14.15]}$$

Fig. 14.15

Hence, the solution set of the given inequation is $(-\infty, 2) \cup [7, \infty)$.

EXERCISE 14.1

BASIC

1. Solve: $12x < 50$, when
   (i) $x \in R$ (ii) $x \in Z$ (iii) $x \in N$
2. Solve: $-4x > 30$, when
   (i) $x \in R$ (ii) $x \in Z$ (iii) $x \in N$
3. Solve: $4x - 2 < 8$, when
   (i) $x \in R$ (ii) $x \in Z$ (iii) $x \in N$

Solve the following linear inequations in R. (4 – 28):

4. $3x - 7 > x + 1$
5. $x + 5 > 4x - 10$
6. $3x + 9 \geq -x + 19$
7. $2(3 - x) \geq \frac{x}{5} + 4$
8. $\frac{3x-2}{5} \leq \frac{4x-3}{2}$
9. $-(x-3) + 4 < 5 - 2x$
10. $\frac{x}{5} < \frac{3x-2}{4} - \frac{5x-3}{5}$
11. $\frac{2(x-1)}{5} \leq \frac{3(2+x)}{7}$
12. $\frac{5x}{2} + \frac{3x}{4} \geq \frac{39}{4}$
13. $\frac{x-1}{3} + 4 < \frac{x-5}{5} - 2$
14. $\frac{2x+3}{4} - 3 < \frac{x-4}{3} - 2$
15. $\frac{5-2x}{3} < \frac{x}{6} - 5$
16. $\frac{4+2x}{3} \geq \frac{x}{2} - 3$
17. $\frac{2x+3}{5} - 2 < \frac{3(x-2)}{5}$
18. $x - 2 \leq \frac{5x+8}{3}$

BASED ON LOTS

19. $\frac{6x-5}{4x+1} < 0$
20. $\frac{2x-3}{3x-7} > 0$
21. $\frac{3}{x-2} < 1$
22. $\frac{1}{x-1} \leq 2$
23. $\frac{4x+3}{2x-5} < 6$
24. $\frac{5x-6}{x+6} < 1$
25. $\frac{5x+8}{4-x} < 2$
26. $\frac{x-1}{x+3} > 2$
27. $\frac{7x-5}{8x+3} > 4$
28. $\frac{x}{x-5} > \frac{1}{2}$

ANSWERS

1. (i) $(-\infty, 25/6)$ (ii) $\{\ldots -3, -2, -1, 0, 1, 2, 3, 4\}$ (iii) $\{1, 2, 3, 4\}$
2. (ii) $(-\infty, -15/2)$ (ii) $\{\ldots\ldots\ldots\ldots, -9, -8\}$ (iii) ϕ
3. (i) $(-\infty, 5/2)$ (ii) $\{\ldots\ldots, -2, -1, 0, 1, 2\}$ (iii) $\{1, 2\}$
4. $(4, \infty)$
5. $(-\infty, 5)$
6. $[5/2, \infty)$
7. $(-\infty, 10/11]$
8. $[11/14, \infty)$
9. $(-\infty, -2)$
10. $(-\infty, 2/9)$
11. $[-44, \infty)$
12. $[3, \infty)$
13. $(-\infty, -50)$
14. $(-\infty, -13/2)$
15. $(8, \infty)$
16. $[-26, \infty)$
17. $(-1, \infty)$
18. $[-7, \infty)$
19. $(-1/4, 5/6)$
20. $(-\infty, 3/2) \cup (7/3, \infty)$
21. $(-\infty, 2) \cup (5, \infty)$
22. $(-\infty, 1) \cup [3/2, \infty)$
23. $(-\infty, 5/2) \cup (33/8, \infty)$
24. $(-6, 3)$
25. $(-\infty, 0) \cup (4, \infty)$
26. $(-7, -3)$
27. $(-17/25, -3/8)$
28. $(-\infty, -5) \cup (5, \infty)$

14.5 SOLUTION OF SYSTEM OF LINEAR INEQUATIONS IN ONE VARIABLE

In the previous section, we have learnt how to solve a linear inequation in one variable. In this section, we shall use it to solve a system of linear inequations in one variable. Recall that the solution set of a linear inequation is the set of all points on real line satisfying the given inequation. Therefore, the solution set of a system of linear inequations in one variable is the intersection of the solution sets of the linear inequations in the given system.

We use the following algorithm to solve a system of linear inequations in one variable.

ALGORITHM

Step I *Obtain the system of linear inequations.*

Step II *Solve each inequation and obtain their solution sets. Also, represent them on real time.*

Step III *Find the intersection of the solution sets obtained in step II by taking the help of the graphical representation of the solution sets in step II.*

Step IV *The set obtained in step III is the required solution set of the given system of inequations.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve the following system of linear inequations:*

$$3x - 6 \geq 0$$
$$4x - 10 \leq 6$$

SOLUTION The given system of inequations is

$$3x - 6 \geq 0 \quad \text{...(i)}$$
$$4x - 10 \leq 6 \quad \text{...(ii)}$$

Now, $3x - 6 \geq 0 \Rightarrow 3x \geq 6 \Rightarrow \frac{3x}{3} \geq \frac{6}{3} \Rightarrow x \geq 2$. So, the Solution set of inequation (i) is $[2, \infty)$

and, $4x - 10 \leq 6 \Rightarrow 4x \leq 16 \Rightarrow x \leq 4$. So, the solution set of inequation (ii) is $(-\infty, 4]$

Fig.14.16 (i)

Fig.14.16 (ii)

The solution sets of inequations (i) and (ii) are represented graphically on real line in Figs. 14.16 (i) and (ii) respectively. Clearly, the intersection of these solution sets is the set [2, 4]. Hence, the solution set of the given system of inequations is the interval [2, 4].

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 2 *Solve the following system of inequations:*

$$\frac{5x}{4} + \frac{3x}{8} > \frac{39}{8}$$

$$\frac{2x-1}{12} - \frac{x-1}{3} < \frac{3x+1}{4}$$

SOLUTION The given system of inequation is

$$\frac{5x}{4} + \frac{3x}{8} > \frac{39}{8} \qquad \text{...(i)}$$

$$\frac{2x-1}{12} - \frac{x-1}{3} < \frac{3x+1}{4} \qquad \text{...(ii)}$$

Now, $\frac{5x}{4} + \frac{3x}{8} > \frac{39}{8}$

$\Rightarrow$ $\frac{10x+3x}{8} > \frac{39}{8} \Rightarrow 13x > 39 \Rightarrow x > 3 \Rightarrow x \in (3, \infty)$

So, the solution set of inequation (i) is the interval $(3, \infty)$.

and, $\frac{2x-1}{12} - \frac{x-1}{3} < \frac{3x+1}{4}$

$\Rightarrow$ $\frac{(2x-1)-4(x-1)}{12} < \frac{3x+1}{4}$

$\Rightarrow$ $\frac{-2x+3}{12} < \frac{3x+1}{4}$

$\Rightarrow$ $-2x+3 < 3(3x+1)$ [Multiplying both sides by 12 i.e. the l.c.m. of 12 and 4]

$\Rightarrow$ $-2x+3 < 9x+3 \Rightarrow -2x-9x < 3-3 \Rightarrow -11x < 0 \Rightarrow x > 0 \Rightarrow x \in (0, \infty)$

So, the solution set of inequation (ii) is the interval $(0, \infty)$. Let us now represent the solution sets of inequations (i) and (ii) on real line. These solution sets are graphed on real line in Figs. 14.17 (i) and 14.17 (ii) respectively.

Fig. 14.17 (i)

Fig. 14.17 (ii)

From Figs. 14.17 (i) and (ii), we observe that the intersection of the solution sets of inequations (i) and (ii) is interval $(3, \infty)$ represented by common thick line.

Hence, the solution set of the given system of inequations is the interval $(3, \infty)$.

EXAMPLE 3 *Solve the following system of inequations:*

$$2(2x+3) - 10 < 6(x-2)$$

$$\frac{2x-3}{4} + 6 \geq 2 + \frac{4x}{3}$$

SOLUTION The given system of inequations is

$$2(2x+3)-10<6(x-2) \quad \text{...(i)}$$

$$\frac{2x-3}{4}+6 \geq 2+\frac{4x}{3} \quad \text{...(ii)}$$

Now, $2(2x+3)-10<6(x-2)$

$\Rightarrow \quad 4x+6-10<6x-12$

$\Rightarrow \quad 4x-4<6x-12 \Rightarrow 4x-6x<4-12 \Rightarrow -2x<-8 \Rightarrow x>4 \Rightarrow x \in (4, \infty)$

So, the solution set of the first inequation is the interval $(4, \infty)$.

and, $\quad \dfrac{2x-3}{4}+6 \geq 2+\dfrac{4x}{3}$

$\Rightarrow \quad \dfrac{2x-3+24}{4} \geq \dfrac{6+4x}{3}$

$\Rightarrow \quad \dfrac{2x+21}{4} \geq \dfrac{4x+6}{3}$

$\Rightarrow \quad 3(2x+21) \geq 4(4x+6)$

$\Rightarrow \quad 6x+63 \geq 16x+24$

$\Rightarrow \quad 6x-16x \geq 24-63 \Rightarrow -10x \geq -39 \Rightarrow x \leq \dfrac{39}{10} \Rightarrow x \leq 3.9 \Rightarrow x \in (-\infty, 3.9]$

So, the solution set of inequation (ii) is the interval $(-\infty, 3.9]$.

Fig. 14.18 (i)

Fig. 14.18 (ii)

The solution sets of inequations (i) and (ii) are graphed on real line in Figs. 14.18 (i) and (ii) respectively. We observe that there is no common solution of the two inequations. So, the given system of inequations has no solution.

EXAMPLE 4 *Solve:* $-11 \leq 4x-3 \leq 13$

SOLUTION We have,

$$-11 \geq 4x-3 \geq 13 \Leftrightarrow -11 \geq 4x-3 \text{ and } 4x-3 \geq 13$$

Thus, we have two inequations and we wish to solve them simultaneously. Instead of solving these inequations by using the method discussed in first three examples, let us solve them directly in a different way as given below.

We have,

$$-11 \leq 4x-3 \leq 13$$

$\Rightarrow \quad -11+3 \leq 4x-3+3 \leq 13+3$ [Adding 3 throughout]

$\Rightarrow \quad -8 \leq 4x \leq 16$

$\Rightarrow \quad \dfrac{-8}{4} \leq x \leq \dfrac{16}{4}$ [Dividing by 4 throughout]

$\Rightarrow \quad -2 \leq x \leq 4 \Rightarrow x \in [-2, 4]$

Hence, the interval $[-2, 4]$ is the solution set of the given system of inequations.

EXAMPLE 5 *Solve:* $-5 \leq \dfrac{2-3x}{4} \leq 9$ [NCERT EXEMPLAR]

SOLUTION We have,

$$-5 \le \frac{2-3x}{4} \le 9$$

$$\Rightarrow \quad -5 \times 4 \le \frac{2-3x}{4} \times 4 \le 9 \times 4 \qquad \text{[Multiplying throughout by 4]}$$

$$\Rightarrow \quad -20 \le 2 - 3x \le 36$$

$$\Rightarrow \quad -20 - 2 \le 2 - 3x - 2 \le 36 - 2 \qquad \text{[Subtracting 2 throughout]}$$

$$\Rightarrow \quad -22 \le -3x \le 34$$

$$\Rightarrow \quad \frac{-22}{-3} \ge \frac{-3x}{-3} \ge \frac{34}{-3} \qquad \text{[Dividing throughout by } -3\text{]}$$

$$\Rightarrow \quad \frac{22}{3} \ge x \ge \frac{-34}{3} \Rightarrow \frac{-34}{3} \le x \le \frac{22}{3} \Rightarrow x \in [-34/3, 22/3]$$

Hence, the interval $[-34/3, 22/3]$ is the solution set of the given system of inequations.

EXAMPLE 6 *Solve the system of inequations:* $\frac{x}{2x+1} \ge \frac{1}{4}$, $\frac{6x}{4x-1} < \frac{1}{2}$ **[NCERT EXEMPLAR]**

SOLUTION The given system of inequations is

$$\frac{x}{2x+1} \ge \frac{1}{4} \qquad \text{...(i)}$$

$$\frac{6x}{4x-1} < \frac{1}{2} \qquad \text{...(ii)}$$

Now, $\frac{x}{2x+1} \ge \frac{1}{4}$

$$\Rightarrow \quad \frac{x}{2x+1} - \frac{1}{4} \ge 0 \Rightarrow \frac{4x-(2x+1)}{4(2x+1)} \ge 0 \Rightarrow \frac{2x-1}{2x+1} \ge 0 \Rightarrow x \in (-\infty, -1/2) \cup [1/2, \infty)$$

[See Fig. 14.19 (i)]

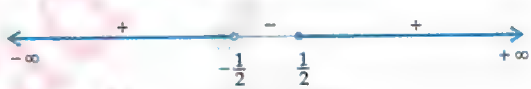


Fig. 14.19 (i)

Fig.14.19 (ii)

Thus, the solution set of inequation (i) is $(-\infty, -1/2) \cup [1/2, \infty)$ ... (iii)

And, $\frac{6x}{4x-1} < \frac{1}{2}$

$$\Rightarrow \quad \frac{6x}{4x-1} - \frac{1}{2} < 0$$

$$\Rightarrow \quad \frac{12x-(4x-1)}{2(4x-1)} < 0 \Rightarrow \frac{8x+1}{2(4x-1)} < 0 \Rightarrow \frac{8x+1}{4x-1} < 0 \Rightarrow x \in (-1/8, 1/4)$$

Thus, the solution set of inequation (ii) is $(-1/8, 1/4)$ ...(iv)

It is evident from Fig. 14.19 that the intersection of (iii) and (iv) is the null set.

Hence, the given system of equations has no solution.

EXERCISE 14.2

BASIC

Solve each of the following system of equations in R.

1. $x+3>0,\ 2x<14$
2. $2x-7>5-x,\ 11-5x\le 1$
3. $x-2>0,\ 3x<18$
4. $2x+6\ \ge 0,\ 4x-7<0$
5. $3x-6>0,\ 2x-5>\ 0$
6. $2x-3<7,\ 2x>-4$
7. $2x+5\le 0,\ x-3\le 0$
8. $5x-1<24,\ 5x+1>-24$
9. $3x-1\ge 5,\ x+2>-1$
10. $11-5x>-4,\ 4x+13\le -11$
11. $4x-1\le 0,\ 3-4x<0$
12. $x+5>2(x+1),\ 2-x<3(x+2)$
13. $2(x-6)<3x-7,\ 11-2x<6-x$
14. $5x-7<3(x+3),\ 1-\dfrac{3x}{2}\ge x-4$

BASED ON LOTS

14. $\dfrac{2x-3}{4}-2\ge\dfrac{4x}{3}-6,\ 2(2x+3)<6(x-2)+10$
16. $\dfrac{7x-1}{2}<-3,\ \dfrac{3x+8}{5}+11<0$
17. $\dfrac{2x+1}{7x-1}>5,\ \dfrac{x+7}{x-8}>2$ [NCERT EXEMPLAR]
18. $0<\dfrac{-x}{2}<3$
19. $10\le -5(x-2)<20$
20. $-5<2x-3<5$
21. $\dfrac{4}{x+1}\le 3\le\dfrac{6}{x+1},\ x>0$ [NCERT EXEMPLAR]

ANSWERS

1. $(-3,7)$ 2. $(4,\infty)$ 3. $(2,6)$ 4. $[-3,7/4)$
5. $(5/2,\ \infty)$ 6. $(-2\ 5)$ 7. $(-\infty,-5/2]$ 8. $(-5,5)$
9. $[2,\infty)$ 10. $(-\infty,\ -6]$ 11. No Solution 12. $(-1,\ 3)$
13. $(5,\ \infty)$ 14. $(-\infty,\ 2]$ 15. No Solution 16. $(-\infty,\ -21)$
17. No Solution 18. $(-6,\ 0)$ 19. $(-2,\ 0]$ 20. $(-1,\ 4)$ 21. $[1/3,\ 1]$

14.5.1 SOME IMPORTANT RESULTS

In this sub-section, let us discuss some results on inequations involving modulus of the variable. We state and prove these results as theorems.

THEOREM 1 *If a is a positive real number, then*

(i) $|x|<a \Leftrightarrow\ -a<x<a$ i.e. $x\in(-a,a)$ (ii) $|x|\le\ a\ \Leftrightarrow\ -a\le x\le a$ i.e. $x\in[-a,a]$

Fig.14.20 (i)

Fig.14.20 (ii)

PROOF (i) We know that: $|x|=\begin{cases} x, & \text{if } x\ge 0\\ -x, & \text{if } x<0\end{cases}$

So, we consider the following cases:

Case I When $x\ge 0$: In this case, $|x|\ =\ x$.

$\therefore$ $|x|<a\Rightarrow x<a$

Thus, in this case the solution set of the given inequation is given by

$x\ge 0$ and $x<a\ \Rightarrow\ 0\le x<a$ ...(i)

Case II When $x<0$: In this case, $|x|\ =-x$.

$\therefore \quad |x| < a \Rightarrow -x < a \Rightarrow x > -a$

Thus, in this case the solution set of the given inequation is given by

$x < 0$ and $x > -a \Rightarrow -a < x < 0$ ...(ii)

Combining (i) and (ii), we get

$|x| < a \Leftrightarrow -a < x < 0$ or, $0 \le x < a \Leftrightarrow -a < x < a.$

(ii) Proceeding exactly as in (i), we get

$|x| \le a \Rightarrow -a \le x \le a.$

THEOREM 2 *If a is a positive real number, then*

(i) $|x| > a \Leftrightarrow x < -a$ or $x > a$

Fig.14.21 (i)

(ii) $|x| \ge a \Leftrightarrow x \le -a$ or $x \ge a$

Fig.14.21 (ii)

PROOF Case I When $x > 0$: In this case, $|x| = x$

$\therefore \quad |x| > a \Rightarrow x > a$

Thus, in this case the solution set of the inequation $|x| > a$ is given by

$x > 0$ and $x > a \Rightarrow x > a$ $[\because a > 0]$ ...(i)

Case II *When $x < 0$:* In this case, $|x| = -x$

$\therefore \quad |x| > a \Rightarrow -x > a \Rightarrow x < -a$

Thus, in this case the solution set of the given inequation is given by

$x < 0$ and $x < -a \Rightarrow x < -a$ $[\because a > 0]$ ...(ii)

Combining (i) and (ii), we get: $|x| > a \Leftrightarrow x < -a$ or $x > a$

(ii) Proceeding as in (i), we get: $|x| \ge a \Leftrightarrow x \le -a$ or $x \ge a.$

THEOREM 3 *Let r be a positive real number and a be a fixed real number. Then,*

(i) $|x - a| < r \Leftrightarrow a - r < x < a + r$ i.e. $x \in (a - r, a + r)$

(ii) $|x - a| \le r \Leftrightarrow a - r \le x \le a + r$ i.e. $x \in [a - r, a + r]$

(iii) $|x - a| > r \Leftrightarrow x < a - r$, or $x > a + r$

(iv) $|x - a| \ge r \Leftrightarrow x \le a - r$, or $x \ge a + r$

PROOF (i) Using Theorem 1, we obtain

$|x - a| < r \Leftrightarrow -r < x - a < r \Leftrightarrow a - r < x - a + a < a + r \Leftrightarrow a - r < x < a + r$

(ii) Using Theorem 1 (ii), we obtain

$|x - a| \le r \Leftrightarrow -r \le x - a \le r \Leftrightarrow a - r \le x - a + a \le a + r \Leftrightarrow a - r \le x \le a + r$

(iii) Using Theorem 2(i), we obtain

$|x - a| > r \Leftrightarrow x - a < r$, or $x - a > r \Leftrightarrow x < a - r$, or $x > a + r$

(iv) Using Theorem 2 (ii), we obtain,

$|x - a| \ge r \Leftrightarrow x - a \le -r$, or $x - a \ge r \Leftrightarrow x \le a - r$, or $x \ge a + r$

NOTE *These results may be used directly for solving linear inequations involving absolute values.*

THEOREM 4 *Let a, b be positive real numbers. Then*

(i) $a < |x| < b \Leftrightarrow x \in (-b, -a) \cup (a, b)$

(ii) $a \le |x| \le b \Leftrightarrow x \in [-b, -a] \cup [a, b]$

(iii) $a \le |x - c| \le b \Leftrightarrow x \in [-b + c, -a + c] \cup [a + c, b + c]$

(iv) $a < |x - c| < b \Leftrightarrow x \in (-b + c, -a + c) \cup (a + c, b + c)$

PROOF (i) $a < |x| < b \Leftrightarrow |x| > a$ and $|x| < b \Leftrightarrow (x < -a$ or $x > a)$ and $(-b < x < b)$

$\Leftrightarrow x \in (-b, -a) \cup (a, b)$

Similarly, we can prove other results.

ILLUSTRATIVE EXAMPLES

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 1 $|3x-2| \le \frac{1}{2}$

SOLUTION We know that: $|x-a| \le r \Leftrightarrow a-r \le x \le a+r$

$\therefore \quad |3x-2| \le \frac{1}{2} \Leftrightarrow 2-\frac{1}{2} \le 3x \le 2+\frac{1}{2} \Leftrightarrow \frac{3}{2} \le 3x \le \frac{5}{2} \Leftrightarrow \frac{1}{2} \le x \le \frac{5}{6} \Leftrightarrow x \in [1/2,\ 5/6]$

Hence, the solution set of the given inequation is the interval $[1/2, 5/6]$.

EXAMPLE 2 *Solve:* $|x-2| \ge 5$

SOLUTION We know that: $|x-a| \ge r \Leftrightarrow x \le a-r$, or $x \ge a+r$

$\therefore \quad |x-2| \ge 5$

$\Leftrightarrow \quad x \le 2-5$, or $x \ge 2+5$

$\Leftrightarrow \quad x \le -3$ or $x \ge 7 \Leftrightarrow x \in (-\infty, -3]$ or $x \in [7, \infty) \Leftrightarrow x \in (-\infty, -3] \cup [7, \infty)$

Hence the solution set of the given inequation is $(-\infty, -3] \cup [7, \infty)$

EXAMPLE 3 *Solve* : $1 \le |x - 2| \le 3$. **[NCERT EXEMPLAR]**

SOLUTION We know that :

$a \le |x-c| \le b \Leftrightarrow x \in [-b+c,\ -a+c] \cup [a+c,\ b+c]$

$\therefore \quad 1 \le |x - 2| \le 3 \Leftrightarrow x \in [-3+2, -1+2] \cup [1+2,\ 3+2] \Leftrightarrow x \in [-1, 1] \cup [3, 5]$

Hence, the solution set of the given inequation is $[-1, 1] \cup [3, 5]$.

EXAMPLE 4 *Solve the following system of inequations:* $|x-1| \le 5,\quad |x| \ge 2$ **[NCERT EXEMPLAR]**

SOLUTION The given system of inequations is

$|x-1| \le 5$ ...(i)

$|x| \ge 2$ ...(ii)

Now, $|x-1| \le 5$

$\Rightarrow \quad 1-5 \le x \le 1+5$ $\quad [\because |x-a| \le r \Leftrightarrow a-r \le x \le a+r]$

$\Rightarrow \quad -4 \le x \le 6 \Rightarrow x \in [-4, 6]$

Thus, the solution set of (i) is the interval $x \in [-4, 6]$.

and, $|x| \ge 2 \Leftrightarrow x \le -2$, or $x \ge 2 \Leftrightarrow x \in (-\infty, -2] \cup [2, \infty)$ $\quad [\because |x| \ge a \Rightarrow x \le -a \text{ or } x \le a]$

Thus, the solution set of (ii) is $(-\infty, -2] \cup [2, \infty)$.

The solution sets of inequations (i) and (ii) are represented graphically in Figures 14.22 (i) and 14.22 (ii) respectively. The intersection of these two is $[-4, -2] \cup [2, 6]$

Fig.14.22 (i)

Fig.14.22 (ii)

Hence, the solution set of the given system of inequations is $[-4, -2] \cup [2, 6]$.

EXAMPLE 5 *Solve* : $\frac{|x|-1}{|x|-2} \ge 0,\ x \in R,\ x \ne \pm 2$.

SOLUTION We have,

$$\frac{|x|-1}{|x|-2} \ge 0$$

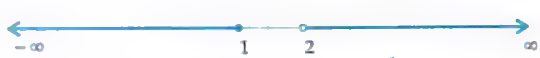


Fig.14.23 Solution set of $\frac{y-1}{y-2} \ge 0$

$\Rightarrow \quad \frac{y-1}{y-2} \geq 0$, where $y = |x|$

$\Rightarrow \quad y \leq 1$ or $y > 2$ [See Fig. 14.23]

$\Rightarrow \quad |x| \leq 1$ or $|x| > 2$

$\Rightarrow \quad (-1 \leq x \leq 1)$ or $(x < -2$ or $x > 2)$

$\Rightarrow \quad x \in [-1, 1]$ or $x \in (-\infty, -2) \cup (2, \infty)$

$\Rightarrow \quad x \in [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$

Hence, the solution set of the given inequation is $[-1, 1] \cup (-\infty, -2) \cup (2, \infty)$.

EXAMPLE 6 *Solve:* $\frac{-1}{|x|-2} \geq 1$, *where* $x \in R$, $x \neq \pm 2$

SOLUTION We have, $\frac{-1}{|x|-2} \geq 1$

$\Rightarrow \quad \frac{-1}{|x|-2} - 1 \geq 0$

$\Rightarrow \quad \frac{-1-(|x|-2)}{|x|-2} \geq 0$

$\Rightarrow \quad \frac{1-|x|}{|x|-2} \geq 0$

$\Rightarrow \quad \frac{|x|-1}{|x|-2} \leq 0$

$\Rightarrow \quad \frac{y-1}{y-2} \leq 0$, where $y = |x|$

$\Rightarrow \quad 1 \leq y < 2$ [See Fig. 14.24]

$\Rightarrow \quad 1 \leq |x| < 2$ $[\because \; y = |x|]$

$\Rightarrow \quad x \in (-2, -1] \cup [1, 2)$ $[\because \; a < |x| \leq b \Leftrightarrow x \in [-b, -a) \cup (a, b]]$

Fig.14.24 Solution set of $\frac{y-1}{y-2} \leq 0$

Hence, the solution set of the given inequation is $(-2, -1] \cup [1, 2)$

EXAMPLE 7 *Solve the inequation:* $\left|\frac{2}{x-4}\right| > 1$, $x \neq 4$.

SOLUTION We have,

$$\left|\frac{2}{x-4}\right| > 1 \quad x \neq 4$$

$\Rightarrow \quad \frac{2}{|x-4|} > 1$ $\left[\because \; \left|\frac{a}{b}\right| = \frac{|a|}{|b|}\right]$

$\Rightarrow \quad 2 > |x-4|$ $[\because \; |x-4| > 0$ for all $x \neq 4]$

$\Rightarrow \quad 4-2 < x < 4+2$ $[\because \; |x-a| < r \Leftrightarrow a-r < x < a+r]$

$\Rightarrow \quad 2 < x < 6 \Rightarrow x \in (2, 6)$

But, $x \neq 4$. Hence, the solution set of the given inequation is $(2, 4) \cup (4, 6)$.

EXAMPLE 8 *Solve:* $\frac{|x+3|+x}{x+2} > 1$ **[NCERT EXEMPLAR]**

SOLUTION We have, $\frac{|x+3|+x}{x+2} > 1$. Clearly, LHS of this inequation is meaningful for $x \neq -2$.

Now, $\dfrac{|x+3|+x}{x+2} > 1$

$\Rightarrow \quad \dfrac{|x+3|+x}{x+2} - 1 > 0 \Rightarrow \dfrac{|x+3|+x-x-2}{x+2} > 0 \Rightarrow \dfrac{|x+3|-2}{x+2} > 0.$

Now two cases arise:

Case I *When* $x+3 \geq 0$ *i.e.* $x \geq -3$: In this case, $|x+3| = x+3$.

$\therefore \quad \dfrac{|x+3|-2}{x+2} > 0$

Fig.14.25

$\Rightarrow \quad \dfrac{x+3-2}{x+2} > 0$

$\Rightarrow \quad \dfrac{x+1}{x+2} > 0 \Rightarrow x \in (-\infty, -2) \cup (-1, \infty)$ [See Fig. 14.25]

But, $x \geq -3$. Therefore, the solution set of the given inequation in this case is $[-3, -2) \cup (-1, \infty)$.

Case II *When* $x+3 < 0$ *i.e.* $x < -3$: In this case, $|x+3| = -(x+3)$.

$\therefore \quad \dfrac{|x+3|-2}{x+2} > 0$

Fig.14.26

$\Rightarrow \quad \dfrac{-(x+3)-2}{x+2} > 0$

$\Rightarrow \quad \dfrac{-(x+5)}{x+2} > 0 \Rightarrow \dfrac{x+5}{x+2} < 0 \Rightarrow x \in (-5-2)$ [See Fig. 14.26]

But, $x < -3$. Therefore, the solution set of the given inequation in this case is the interval $(-5, -3)$.

From Case I and Case II, we obtain that the solution set of the given inequation is

$$[-3, -2) \cup (-1, \infty) \cup (-5, -3) = (-5, -2) \cup (-1, \infty).$$

EXAMPLE 9 *Solve:* $|x-1| + |x-2| \geq 4$

SOLUTION On the LHS of the given inequation there are two terms both containing modulus. By equating the expressions within the modulus to zero, we get $x = 1, 2$ as critical points. These points divide real line in three parts viz. $(-\infty, 1]$, $[1, 2]$ and $[2, \infty)$. So, we consider the following three cases.

Fig.14.27

Case I *When* $-\infty < x < 1$: In this case, we have $|x-1| = -(x-1)$ and $|x-2| = -(x-2)$

$\therefore \quad |x-1| + |x-2| \geq 4$

$\Rightarrow \quad -(x-1) - (x-2) \geq 4 \Rightarrow -2x + 3 \geq 4 \Rightarrow -2x \geq 1 \Rightarrow x \leq -\dfrac{1}{2}$

But, $-\infty < x < 1$. Therefore, in this case the solution set of the given inequation is $(-\infty, -1/2]$

Case II *When* $1 \leq x < 2$: In this case, we have $|x-1| = (x-1)$ and $|x-2| = -(x-2)$

$\therefore \quad |x-1| + |x-2| \geq 4$

$\Rightarrow \quad x-1-(x-2) \geq 4 \Rightarrow 1 \geq 4$, which is an absurd result.

So, the given inequation has no solution for $x \in [1, 2)$.

Case III *When* $x \geq 2$: In this case, we have $|x-1| = x-1$ and $|x-2| = x-2$

$\therefore \quad |x-1| + |x-2| \geq 4$

$$\Rightarrow \quad x-1+x-2 \geq 4 \Rightarrow 2x-3 \geq 4 \Rightarrow 2x \geq 7 \Rightarrow x \geq \frac{7}{2}$$

But, $x > 2$. Therefore, in this case the solution set of the given inequation is $[7/2, \infty)$.

Combining Case I and Case II, we obtain that the solution set of the given inequation is

$$(-\infty, -1/2] \cup [7/2, \infty)$$

EXAMPLE 10 *Solve:* $\dfrac{|x-1|}{x+2} < 1.$

SOLUTION We have, $\dfrac{|x-1|}{x+2} < 1 \Rightarrow \dfrac{|x-1|}{x+2} - 1 < 0 \Rightarrow \dfrac{|x-1|-(x+2)}{x+2} < 0$

Now the following cases arise.

Case I *When* $x-1 \geq 0$ *i.e.* $x \geq 1$: In this case, we have $|x-1| = x-1$

$$\therefore \quad \frac{|x-1|-(x+2)}{x+2} < 0$$

$$\Rightarrow \quad \frac{(x-1)-(x+2)}{x+2} < 0 \Rightarrow \frac{-3}{x+2} < 0 \Rightarrow x+2 > 0 \Rightarrow x > -2$$

But, $x \geq 1$. Therefore, $x > -2$ and $x \geq 1$. implies that $x \geq 1$. Thus, in this case the solution set of the given inequation is $[1, \infty)$.

Case II *When* $x-1 < 0$ *i.e.* $x < 1$: In this case, we have $|x-1| = -(x-1)$.

$$\therefore \quad \frac{|x-1|-(x+2)}{x+2} < 0$$

$$\Rightarrow \quad \frac{-(x-1)-(x+2)}{x+2} < 0$$

$$\Rightarrow \quad -\frac{2x+1}{x+2} < 0$$

$$\Rightarrow \quad \frac{2x+1}{x+2} > 0 \Rightarrow x \in (-\infty \;\; -2) \cup (-1/2, \infty)$$ [See Fig. 15.28]

Fig.14.28

But, $x < 1$. Therefore, $x \in (-\infty, -2) \cup (-1/2, \infty)$ and $x < 1$ implies that $x \in (-\infty, -2) \cup \left(-\dfrac{1}{2}, 1\right)$.

Thus, in this case the solution set of the given inequation is $(-\infty, -2) \cup (-1/2, 1)$.

Combining Case I and Case II, we obtain that the solution set of the given inequation is

$$(-\infty, -2) \cup (-1/2, \infty)$$

EXAMPLE 11 *Solve the inequation:* $\dfrac{1}{|x|-3} \leq \dfrac{1}{2}.$ [NCERT EXEMPLAR]

SOLUTION We have, $\dfrac{1}{|x|-3} \leq \dfrac{1}{2}$

Fig.14.29 Signs of $\dfrac{|x|-5}{|x|-3}$

$$\Rightarrow \quad \frac{1}{|x|-3} - \frac{1}{2} \leq 0$$

$$\Rightarrow \quad \frac{2-|x|+3}{2(|x|-3)} \leq 0 \Rightarrow \frac{5-|x|}{|x|-3} \leq 0 \Rightarrow \frac{|x|-5}{|x|-3} \geq 0 \Rightarrow |x| < 3 \text{ or } |x| \geq 5$$

$\Rightarrow \quad x \in (-3, 3)$ or $x \in (-\infty, -5] \cup [5, \infty) \Rightarrow x \in (-\infty, -5] \cup (-3, 3] \cup [5, \infty)$

EXAMPLE 12 *Solve the inequation:* $\dfrac{|x-2|-1}{|x-2|-2} \le 0.$ **[NCERT EXEMPLAR]**

SOLUTION Let $y = |x-2|$, then

$$\frac{|x-2|-1}{|x-2|-2} \le 0 \Rightarrow \frac{y-1}{y-2} \le 0$$

Fig. 14.30 (i) Signs of $\dfrac{y-1}{y-2} \le 0$

Fig. 14.30 (ii) Solution set of $|x-2| \ge 1$

Fig. 14.30 (iii) Solution set of $|x-2| < 2$

$\Rightarrow \quad 1 \le y < 2$

$\Rightarrow \quad 1 \le |x-2| < 2$

$\Rightarrow \quad 1 \le |x-2|$ and $|x-2| < 2$

Now, $\quad 1 \le |x-2| \Rightarrow |x-2| \ge 1$

$\Rightarrow x \le 1$ or $x \ge 3 \Rightarrow x \in (-\infty, 1] \cup [3, \infty)$ ...(i)

and, $\quad |x-2| < 2 \Rightarrow 2-2 < x < 2+2 \Rightarrow 0 < x < 4$ ...(ii)

Hence, the solution set of the given inequation is

$$((-\infty, 1] \cup [3, \infty)) \cap (0, 4) = (0, 1] \cup [3, 4)$$

EXERCISE 14.3

BASED ON HOTS

Solve each of the following system of equations in R.

1. $\left|x + \frac{1}{3}\right| > \frac{8}{3}$ **2.** $|4 - x| + 1 < 3$ **3.** $\left|\frac{3x-4}{2}\right| \le \frac{5}{12}$ **4.** $\frac{|x-2|}{x-2} > 0$

5. $\frac{1}{|x|-3} < \frac{1}{2}$ **6.** $\frac{|x+2|-x}{x} < 2$ **7.** $\left|\frac{2x-1}{x-1}\right| > 2$

8. $|x-1| + |x-2| + |x-3| \ge 6$ **9.** $|x+1| + |x| > 3$ **[NCERT EXEMPLAR]**

10. $1 \le |x-2| \le 3$ **11.** $|3-4x| \ge 9$ **[NCERT EXEMPLAR]**

ANSWERS

1. $(-\infty, -3) \in (7/3, \infty)$ **2.** $(2, 6)$ **3.** $[19/18, 29/18]$

4. $(2, \infty)$ **5.** $(-\infty, -5) \cup (-3, 3) \cup (5, \infty)$ **6.** $(-\infty, 0) \cup (1, \infty)$

7. $(3/4, 1) \cup (1, \infty)$ **8.** $(-\infty, 0] \cup [4, \infty)$ **9.** $(-\infty, -2) \cup (1, \infty)$

10. $[-1, 1] \cup [3, 5]$ **11.** $(-\infty, -3/2] \cup [3, \infty)$

14.6 SOME APPLICATIONS OF LINEAR IN EQUATIONS IN ONE VARIABLE

In this section, we shall utilize the knowledge of solving linear in equations in one variable in solving different problems from various fields such as science, engineering, economics etc. Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find all pairs of consecutive odd positive integers, both of which are smaller than 18, such that their sum is more than 20.*

SOLUTION Let x be the smaller of the two consecutive odd positive integers. Then, the other odd integer is $x+2$.

It is given that both the integers are smaller than 18 and their sum is more than 20. Therefore,

$$x+2<18 \text{ and, } x+(x+2)>20$$

$$\Rightarrow \quad x<16 \text{ and } 2x+2>20$$

$$\Rightarrow \quad x<16 \text{ and } 2x>18$$

$$\Rightarrow \quad x<16 \text{ and } x>9 \Rightarrow 9<x<16 \Rightarrow x=11, 13, 15 \qquad [\because x \text{ is an odd integer}]$$

Hence, the required pairs of odd integers are (11, 13), (13, 15) and (15, 17).

EXAMPLE 2 *Find all pairs of consecutive even positive integers, both of which are larger than 8, such that their sum is less than 25.*

SOLUTION Let x be the smaller of the two consecutive even positive integers. Then, the other even integer is $x+2$.

It is given that both the integers are larger than 8 and their sum is less than 25. Therefore,

$$x>8 \text{ and } x+x+2<25$$

$$\Rightarrow \quad x>8 \text{ and } 2x+2<25$$

$$\Rightarrow \quad x>8 \text{ and } 2x<23$$

$$\Rightarrow \quad x>8 \text{ and } x<\frac{23}{2} \Rightarrow 8<x<\frac{23}{2} \Rightarrow x=10 \qquad [\because x \text{ is an even integer}]$$

Hence, the required pair of even integers is (10, 12).

EXAMPLE 3 *The cost and revenue functions of a product are given by $C(x)=2x+400$ and $R(x)=6x+20$ respectively, where x is the number of items produced by the manufacturer. How many items the manufacturer must sell to realize some profit?*

SOLUTION We know that: Profit = Revenue – Cost. Therefore, to earn some profit, we must have

Revenue > Cost

$$\Rightarrow \quad 6x+20>2x+400 \Rightarrow 6x-2x>400-20 \Rightarrow 4x>380 \Rightarrow x>\frac{380}{4}=95$$

Hence, the manufacturer must sell more than 95 items to realize some profit.

EXAMPLE 4 *IQ of a person is given by the formula: $IQ=\frac{MA}{CA}\times 100$, where MA is mental age and CA is chronological age. If $80 \le IQ \le 140$ for a group of 12 year children, find the range of their mental age.*

SOLUTION We have: $CA = 12$ years

$$\therefore \quad IQ=\frac{MA}{CA}\times 100 \Rightarrow IQ=\frac{MA}{12}\times 100=\frac{25}{3}MA$$

Now, $80 \le IQ \le 140$

$$\Rightarrow \quad 80 \le \frac{25}{3}MA \le 140 \Rightarrow 240 \le 25\,MA \le 420 \Rightarrow \frac{240}{25} \le MA \le \frac{420}{25} \Rightarrow 9.6 \le MA \le 16.8$$

EXAMPLE 5 *In the first four papers each of 100 marks, Rishi got 95, 72, 73, 83 marks. If he wants an average of greater than or equal to 75 marks and less than 80 marks, find the range of marks he should score in the fifth paper.*

SOLUTION Suppose scores x marks in the fifth paper. Then,

$$75 \leq \frac{95+72+73+83+x}{5} < 80$$

$$\Rightarrow \quad 75 \leq \frac{323+x}{5} < 80 \Rightarrow 375 < 323+x < 400 \Rightarrow 52 < x < 77$$

Hence, Rishi must score between 52 and 77 marks.

EXAMPLE 6 *A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18% ?*

SOLUTION Let x litres of 30% acid solution be added to 600 litres of 12% solution of acid. Then,

Total quantity of mixture $= (600 + x)$ litres

Total acid content in the $(600 + x)$ litres of mixture $= \frac{30x}{100} + \frac{12}{100} \times 600$

It is given that acid content in the resulting mixture must be more than 15% and less than 18%.

$$\therefore \quad 15\% \text{ of } (600+x) < \left(\frac{30x}{100} + \frac{12}{100} \times 600\right) < 18\% \text{ of } (600+x)$$

$$\Rightarrow \quad \frac{15}{100} \times (600+x) < \frac{30x}{100} + \frac{12}{100} \times 600 < \frac{18}{100} \times (600+x)$$

$$\Rightarrow \quad 15(600+x) < 30x + 12 \times 600 < 18(600+x) \qquad \text{[Multiplying through out by 100]}$$

$$\Rightarrow \quad 9000 + 15x < 30x + 7200 < 10800 + 18x$$

$$\Rightarrow \quad 9000 + 15x < 30x + 7200 \text{ and } 30x + 7200 < 10800 + 18x$$

$$\Rightarrow \quad 9000 - 7200 < 30x - 15x \text{ and } 30x - 18x < 10800 - 7200$$

$$\Rightarrow \quad 1800 < 15x \text{ and } 12x < 3600$$

$$\Rightarrow \quad 15x > 1800 \text{ and } 12x < 3600$$

$$\Rightarrow \quad x > 120 \text{ and } x < 300$$

$$\Rightarrow \quad 120 < x < 300$$

Hence, the number of litres of the 30% solution of acid must be more than 120 but less than 300.

EXAMPLE 7 *A man wants to cut three lengths from a single piece of board of length 91 cm. The second length is to be 3 cm longer than the shortest and third length is to be twice as long as the shortest. What are the possible lengths for the shortest board if third piece is to be at least 5 cm longer than the second?*

SOLUTION Let the length of the shortest piece be x cm. Then, the lengths of the second and third piece are $x + 3$ cm and $2x$ cm respectively. Then,

$$x + (x+3) + 2x \leq 91 \text{ and } 2x \geq (x+3) + 5$$

$$\Rightarrow \quad 4x + 3 \leq 91 \text{ and } 2x \geq x + 8$$

$$\Rightarrow \quad 4x \leq 88 \text{ and } x \geq 8 \Rightarrow x \leq 22 \text{ and } x \geq 8 \Rightarrow 8 \leq x \leq 22.$$

Hence, the shortest piece must be at least 8 cm long but not more than 22 cm long.

EXERCISE 14.4

BASIC

1. Find all pairs of consecutive odd positive integers, both of which are smaller than 10, such that their sum is more than 11.
2. Find all pairs of consecutive odd natural number, both of which are larger than 10, such that their sum is less than 40.
3. Find all pairs of consecutive even positive integers, both of which are larger than 5, such that their sum is less than 23.

4. The marks scored by Rohit in two tests were 65 and 70. Find the minimum marks he should score in the third test to have an average of at least 65 marks.
5. A solution is to be kept between 86° and 95°F. What is the range of temperature in degree Celsius, if the Celsius (C)/Fahrenheit (F) conversion formula is given by $F = \frac{9}{5}C + 32$.
6. A solution is to be kept between 30°C and 35°C. What is the range of temperature in degee Fahrenheit?
7. To receive grade 'A' in a cource, one must obtain an average of 90 marks or more in five papers each of 100 marks. If Shikha scored 87, 95, 92 and 94 marks in first four papers, find the minimum marks that she must score in the last paper to get grade 'A' in the course.
8. A company manufactures cassettes and its cost and revenue functions for a week are $C = 300 + \frac{3}{2}x$ and $R = 2x$ respectively, where x is the number of cassettes produced and sold in a week. How many cassettes must be sold for the company to realize a profit?
9. The longest side of a triangle is three times the shortest side and the third side is 2 cm shorter than the longest side if the perimeter of the triangles at least 61 cm, Find the minimum length of the shortest-side.
10. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content?
11. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be more than 4% but less than 6% boric acid. If there are 640 litres of the 8% solution, how many litres of 2% solution will have to be added?
12. The water acidity in a pool is considered normal when the average pH reading of three daily measurements is between 7.2 and 7.8. If the first two pH reading are 7.48 and 7.85, find the range of pH value for the third reading that will result in the acidity level being normal.
13. In drilling worlds's deepest hole it was found that the temperature T in degree celcius, x km below the earth's surface was given by $T = 30 + 25\,(x - 3)$, $3 \le x \le 15$. At what depth will the temperature be between 155°C and 205°C? [NCERT EXEMPLAR]

ANSWERS

1. (5,7), (7, 9)
2. (11, 13), (13, 15), (15, 17), (17, 19)
3. (6, 8), (8, 10), (10, 12)
4. 60
5. Between 30°C and 35°C
6. Between 86°F and 95°F
7. 82 marks
8. More than 600
9. 9 cm
10. More than 562.5 litres but less than 900 litres
11. More than 320 litres but less than 1280 litres
12. Between 6.27 and 8.07
13. Between 8 km and 10 km.

14.7 GRAPHICAL SOLUTION OF LINEAR INEQUATIONS IN TWO VARIABLES

If a, b, c are real numbers, then the equation $ax + by + c = 0$ is called a linear equation in two variables x and y whereas the inequalities $ax + by \le c$, $ax + by \ge c$, $ax + by < c$ and $ax + by > c$ are called linear inequations in two variables x and y.

We have studied in coordinate geometry that the graph of the equation $ax + by = c$ is a straight line which divides the xy-plane into two parts which are represented by $ax + by \le c$ and $ax + by \ge c$. These two parts are known as the closed half-spaces. The regions represented by $ax + by < c$ and $ax + by > c$ are known as the open half spaces. In set theoretical notations, the set $\{(x, y) : ax + by = c\}$ is the straight line, sets $\{(x, y) : ax + by \le c\}$ and $\{(x, y) : ax + by \ge c\}$ are closed half spaces and the sets $\{(x, y) : ax + by < c\}$ and $\{(x, y) : ax + by > c\}$ are open half-spaces.

These half spaces are also known as the solution sets of the corresponding inequations.

In order to find the solution set of a linear inequation in two variables, we follow the following algorithm.

ALGORITHM

Step I *Convert the given inequation, say $ax + by \le c$, into the equation $ax + by = c$ which represents a straight line in xy-plane.*

Step II *Put $y = 0$ in the equation obtained in step I to get the point where the line meets with x-axis. Similarly, put $x = 0$ to obtain a point where the line meets with y-axis.*

Step III *Join the points obtained in step II to obtain the graph of the line obtained from the given inequation. In case of a strict inequality i.e. $ax + by < c$ or $ax + by > c$, draw the dotted line, otherwise mark it thick line.*

Step IV *Choose a point, if possible (0, 0), not lying on this line : Substitute its coordinates in the inequation. If the inequation is satisfied, then shade the portion of the plane which contains the chosen point; otherwise shade the portion which does not contain the chosen point.*

Step V *The shaded region obtained in step IV represents the desired solution set.*

REMARK *In case of the inequalities $ax + by \le c$ and $ax + by > c$ points on the line are also a part of the shaded region while in case of inequalities $ax + by < c$ and $ax + by > c$ points on the line $ax + by = c$ are not in the shaded region.*

The following examples illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve the following inequations graphically:*

(i) $2x + 3y \le 6$ (ii) $2x - y \ge 1$ (iii) $x \ge 2$ (iv) $y \le -3$

SOLUTION (i) Converting the given inequation into equation, we obtain $2x + 3y = 6$.

Putting $y = 0$ and $x = 0$ respectively in this equation, we get $x = 3$ and $y = 2$. So, this line meets x-axis at $A(3, 0)$ and y-axis at $B(0, 2)$. We plot these points and join them by a thick line. This line divides the xy-plane in two parts. To determine the region represented by the given inequality consider the point $O(0, 0)$. Clearly, (0, 0) satisfies the inequality. So, the region containing the origin is represented by the given inequation as shown in Fig. 14.31. This region represents the solution set of the given inequations.

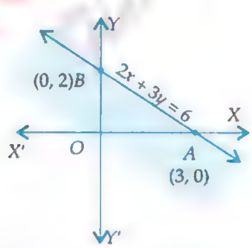


Fig 14.31

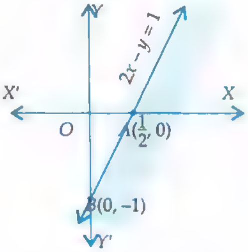


Fig. 14.32

(ii) Converting the given inequation into equation we obtain $2x - y = 1$. This line meets x and y-axes at $A(1/2, 0)$ and $B(0, -1)$ respectively. Joining these points by a thick line we obtain the line passing through A and B as shown in Fig. 14.32. This line divides the xy-plane into two regions viz. one lying above it and the other lying below it. Consider the point $O(0, 0)$. Clearly, (0, 0) does not satisfy the inequation $2x - y \ge 1$. So, the region not containing the origin is represented by the given inequation as shown in Fig. 14.32. Clearly it represents the solution set of the given inequation.

(iii) We have $x \geq 2$.Converting the inequation into equation, we obtain $x = 2$. Clearly, it is a line parallel to y-axis at a distance of 2 units from it. This line divides the xy-plane into two parts viz. one part on the LHS of $x = 2$ and the other on its RHS. We find that the point (0, 0) does not satisfy the inequation $x \geq 2$. So, the region represented by the given equation is the shaded region shown in Fig. 14.33. The shaded region is the required solution set of the given inequation.

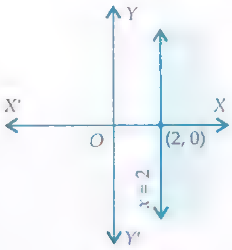


Fig. 14.33

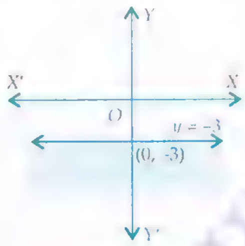


Fig. 14.34

(iv) We have $y \leq -3$. Converting the given inequation into equation we obtain $y = -3$. Clearly, it is a line parallel to x-axis at a distance of 3 units below it. The line $y = -3$ divides the xy-plane into two regions one below it and the other above it. Consider the point O (0, 0). We find that (0, 0) does not satisfy the inequation $y \leq -3$. So, the region represented by the given inequation is the region not containing the origin as show in Fig. 14.34. Clearly, it is the solution set of the given inequation.

EXAMPLE 2 *Solve the following inequations graphically:*

(i) $|x| \leq 3$ (ii) $|y - x| \leq 3$ (iii) $|x - y| \geq 1$

SOLUTION (i) Converting the given inequation into equation, we obtain $x = 3$. This equation represents a line parallel to y-axis at a distance of 3 units from it. The line given by $x = 3$ divides the xy-plane into two regions. Clearly, the point O (0, 0) satisfies $x \leq 3$. So, the graph of $x \leq 3$ is as shown in Fig. 14.35. The shaded region represents the solution set of this inequation.

(ii) We have, $|y - x| \leq 3$. This inequation is equivalent to

$$-3 \leq y - x \leq 3 \qquad [\because |x| \leq a \Leftrightarrow -a \leq x \leq a]$$

$$\Leftrightarrow \quad -3 \leq y - x \text{ and } y - x \leq 3$$

$$\Leftrightarrow \quad x - y - 3 \leq 0 \text{ and } x - y + 3 \geq 0$$

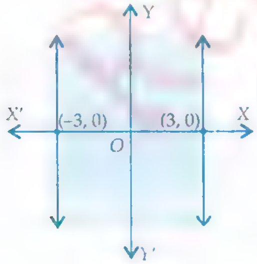


Fig. 14.35

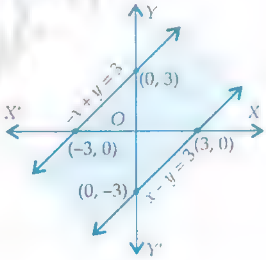


Fig. 14.36

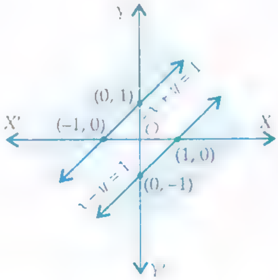


Fig. 14.37

The region represented by $|y - x| \leq 3$ is the region common to the regions represented by $x - y - 3 \leq 0$ and $x - y + 3 \geq 0$ as shown in Fig. 14.36. This shaded region represents the solution set of the given inequation.

(iii) We have,

$$|x - y| \geq 1 \Leftrightarrow x - y \geq 1 \text{ or } x - y \leq -1 \Leftrightarrow x - y - 1 > 0 \text{ or } x - y + 1 \leq 0$$

The required region is the union of regions represented by $x - y - 1 \geq 0$ and $x - y + 1 \leq 0$ as shown in Fig. 14.37. The shaded region represents the solution set of the given inequation.

EXERCISE 14.5

BASIC

Represent to solution set of each of the following inequations graphically in two dimensional plane:

1. $x + 2y - 4 \le 0$
2. $x + 2y \ge 6$
3. $x + 2 \ge 0$
4. $x - 2y < 0$
5. $-3x + 2y \le 6$
6. $x \le 8 - 4y$
7. $0 \le 2x - 5y + 10$
8. $3y > 6 - 2x$
9. $y > 2x - 8$
10. $3x - 2y \le x + y - 8$

14.8 SOLUTION OF SIMULTANEOUS LINEAR INEQUATIONS IN TWO VARIABLE

In this section, we will discuss the technique of finding the solution set of simultaneous linear inequations. Solving simultaneous linear inequations means finding the set of points (x, y) for which all the constraints are satisfied. Note that the solution set of simultaneous linear inequations may be an empty set or it may be the region bounded by the straight lines corresponding to linear inequations or it may be an unbounded region with straight line boundaries.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I ON FINDING THE SOLUTION SET REPRESENTED BY SIMULTANEOUS LINEAR INEQUATIONS

EXAMPLE 1 *Exhibit graphically the solution set of the linear inequations*

$$3x + 4y \le 12, \quad 4x + 3y \le 12, \quad x \ge 0, \quad y \ge 0$$

SOLUTION Converting the inequations into equations, the inequations reduce to

$$3x + 4y = 12, 4x + 3y = 12, x = 0 \text{ and } y = 0.$$

Region Represented by $3x + 4y \le 12$: The line $3x + 4y = 12$ meets the coordinate axes at A (4, 0) and B (0, 3). Draw a thick line joining A and B. We find that (0, 0) satisfies inequation $3x + 4y \le 12$. So, the portion containing the origin represents the solution set of the inequation $3x + 4y \le 12$.

Region Represented by $4x + 3y \le 12$: The line $4x + 3y = 12$ meets the x and y-axes at $A_1(3, 0)$ and $B_1(0, 4)$ respectively. Join these two points by a thick line. Clearly, the region containing the origin is represented by the inequation $4x + 3y \le 12$.

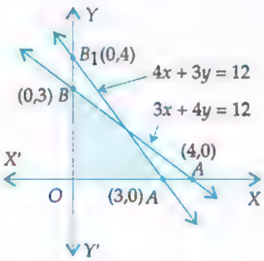


Fig. 14.38

Region Represented by $x \ge 0$ and $y \ge 0$: Clearly, $x \ge 0$ and $y \ge 0$ represent the first quadrant.

Hence, the shaded region given in Fig. 14.38 represents the solution set of the given linear inequations.

EXAMPLE 2 *Exhibit graphically the solution set of the linear inequations*

$$x + y \le 5, \; 4x + y \ge 4, \; x + 5y \ge 5, \; x \le 4, \; y \le 3$$

SOLUTION Converting the inequations into equations, we obtain

$$x + y = 5, \; 4x + y = 4, \; x + 5y = 5, \; x = 4, \; y = 3$$

Region Represented by $x + y \le 5$: The line $x + y = 5$ meets the coordinate axes at $A(5, 0)$ and $B(0, 5)$ respectively. Join these points by a thick line. Clearly, (0, 0) satisfies the inequality $x + y \le 5$. So, the portion containing the origin represents the solution set of the inequation $x + y \le 5$.

Region Represented by $4x + y \ge 4$: The line $4x + y = 4$ meets the coordinate axes at $A_1(1, 0)$ and B_1 (0, 4) respectively. Join these points by a thick line. Clearly, (0, 0) does not satisfy the

inequation $4x + y \geq 4$. So, the portion not containing the origin is represented by the inequation $4x + y \geq 4$.

Region Represented by $x + 5y \geq 5$: The line $x + 5y \geq 5$ meets the coordinate axes at A (5, 0) and B_2 (0, 1) respectively. Join these two points by a thick line. We find that (0, 0) does not satisfy the inequation $x + 5y \geq 5$. So, the portion not containing the origin is represented by the given inequation.

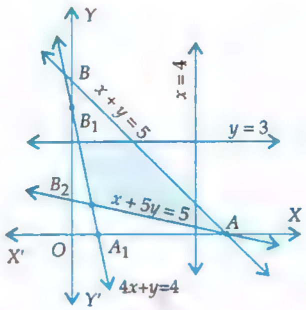


Fig. 14.39

Region Represented by $x \leq 4$: Clearly, $x = 4$ is a line parallel to y-axis at a distance of 4 units from the origin. Since (0, 0) satisfies the inequation $x \leq 4$. So, the portion lying on the left side of $x = 4$ is the region represented by $x \leq 4$.

Region Represented by $y \leq 3$: Clearly, $y = 3$ is a line parallel to x-axis at a distance 3 from it. Since (0, 0) satisfies $y \leq 3$. So, the portion containing the origin is represented by the given inequation.

The common region of the above five regions represents the solution set of the given linear constaints as shown in Fig. 14.39.

EXAMPLE 3 *Draw the diagram of the solution set of the linear inequations* $3x + 4y \geq 12$, $y \geq 1$, $x \geq 0$.

SOLUTION Converting the inequations into equations, we get $3x + 4y = 12$, $y = 1$, $x = 0$

Region Represented by $3x + 4y \geq 12$: The line $3x + 4y = 12$ meets the coordinate axes at A (4, 0) and B(0, 3) joining these points by a thick line we get the graph of $3x + 4y = 12$. Since (0, 0) does not satisfy the inequation $3x + 4y \geq 12$. So, the portion not containing the origin is represented by the inequation $3x + 4y \geq 12$.

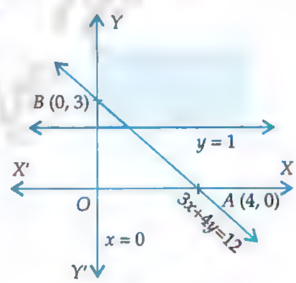


Fig. 14.40

Region Represented by $y \geq 1$: The line $y = 1$ is parallel to x-axis at a unit distance from it. Since (0, 0) does not satisfy the inequation $y \geq 1$. So, the region lying above the line $y = 1$ is represented by $y \geq 1$.

Region Represented by $x \geq 0$: Clearly, $x \geq 0$ represents the region lying on the right side of y-axis.

The solution set of the given linear constraints is the intersection of the above regions as shown in Fig. 14.40.

Type II ON FINDING THE LINEAR INEQUATIONS WHEN THEIR SOLUTION SET IS GIVEN

EXAMPLE 4 *Find the linear inequations for which the shaded area in Fig. 14.41 is the solution set.*

SOLUTION Consider the line $x + 2y = 8$. We observe that the shaded region and the origin are on the same side of the line $x + 2y = 8$ and (0, 0) satisfies the linear constraint $x + 2y \leq 8$. So, we must have one inequations as $x + 2y \leq 8$.

Now consider the line $2x + y = 2$. We find that the shaded region and the origin are on the opposite sides of the line $2x + y = 2$ and (0, 0) does not satisfy the inequation $2x + y \geq 2$. So, the second inequations is $2x + y \geq 2$.

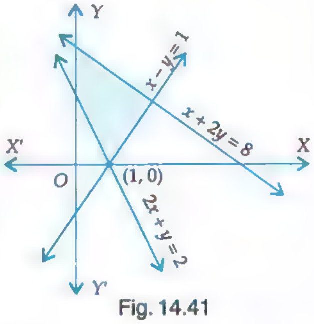


Fig. 14.41

Finally, consider the line $x - y = 1$. We observe that the shaded region and the origin are on the same side of the line $x - y = 1$. We observe that the shaded region and the origin are on the same side of the line $x - y = 1$ and (0, 0) satisfies $x - y \leq 1$. So, the third constraint is $x - y \leq 1$.

We also notice that the shaded region is above x-axis and is on the right side of y-axis. So, we must have $x \geq 0$ and $y \geq 0$.

Thus, the linear inequations corresponding to the given solution set are

$$x + 2y \le 8,\ \ 2x + y \ge 2,\ \ x - y \le 1,\ \ x \ge 0,\ \ y \ge 0$$

EXAMPLE 5 *Find the linear inequations for which the shaded region in Fig. 14.42 is the solution set.*

SOLUTION Consider the line $2x + 3y = 3$. We observe that the shaded region and the origin lie on the opposite side of this line and (0, 0) satisfies $2x + 3y \le 3$. Therefore, we must have $2x + 3y \ge 3$ as the linear inequations corresponding to the line $2x + 3y = 3$.

Consider the line $3x + 4y = 18$. Clearly, the shaded region and the origin lie on the same side of this line and (0, 0) satisfies the inequation $3x + 4y \le 18$. So, we must have $3x + 4y \le 18$ as the linear inequations corresponding to $3x + 4y = 18$.

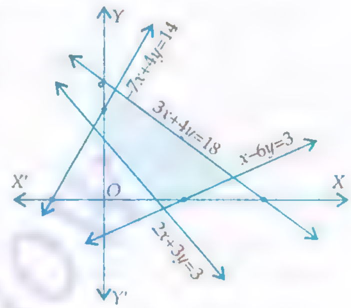


Fig. 14.42

Consider the line $x - 6y = 3$. It is evident from the figure that the origin and the shaded region lie on the same side of this line and (0, 0) satisfies $x - 6y \le 3$. So, $x - 6y \le 3$ is the corresponding inequations.

Consider the line $-7x + 4y = 14$. We find that the shaded region and the origin are on the same side of this line and (0, 0) satisfies the inequations $-7x + 4y \le 14$. So, the corresponding linear inequations is $-7x + 4y \le 14$.

Also, the shaded region is in first quadrant only. So, we must have $x \ge 0$ and $y \ge 0$.

Thus, the linear inequations comprising the given solution set are

$$2x + 3y \ge 3,\ \ 3x + 4y \le 18,\ \ -7x + 4y \le 14,\ \ x - 6y \le 3,\ \ x \ge 0,\ \ y \ge 0$$

EXERCISE 14.6

BASIC

1. Solve the following systems of linear inequations graphically:
   (i) $2x + 3y \le 6,\ 3x + 2y \le 6,\ x \ge 0,\ y \ge 0$ (ii) $2x + 3y \le 6,\ x + 4y \le 4,\ x \ge 0,\ y \ge 0$
   (iii) $x - y \le 1,\ x + 2y \le 8,\ 2x + y \ge 2,\ x \ge 0,\ y \ge 0$
   (iv) $x + y \ge 1,\ 7x + 9y \le 63,\ x \le 6,\ y \le 5,\ x \ge 0,\ y \ge 0$
   (v) $2x + 3y \le 35,\ y \ge 3,\ x \ge 2,\ x \ge 0,\ y \ge 0$
2. Show that the solution set of the following linear inequations is empty set :
   (i) $x - 2y \ge 0,\ 2x - y \le -2,\ x \ge 0,\ y \ge 0$ (ii) $x + 2y \le 3,\ 3x + 4y \ge 12,\ y \ge 1,\ x \ge 0,\ y \ge 0$
3. Find the linear inequations for which the shaded area in Fig. 14.43 is the solution set. Draw the diagram of the solution set of the linear inequations:

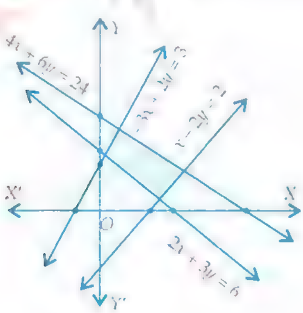


Fig. 14.43

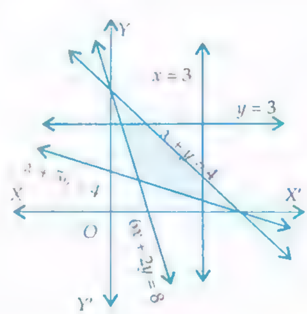


Fig. 14.44

4. Find the linear inequations for which the solution set is the shaded region given in Fig. 14.44.

5. Show that the solution set of the following linear in equations is an unbounded set:

$$x + y \geq 9,\ 3x + y \geq 12,\ x \geq 0,\ y \geq 0.$$

6. Solve the following systems of inequations graphically:

(i) $2x + y \geq 8,\ x + 2y \geq 8,\ x + y \leq 6$

(ii) $12x + 12y \leq 840,\ 3x + 6y \leq 300,\ 8x + 4y \leq 480\quad x \geq 0, y \geq 0$

(iii) $x + 2y \leq 40,\ 3x + y \geq 30,\ 4x + 3y \geq 60,\ x \geq 0,\ y \geq 0$

(iv) $5x + y \geq 10,\ 2x + 2y \geq 12,\ x + 4y \geq 12,\ x \geq 0,\ y \geq 0$

7. Show that the following system of linear equations has no solution:

$x + 2y \leq 3,\ 3x + 4y \geq 12,\ x \geq 0,\ y \geq 1.$ [NCERT EXEMPLAR]

8. Show that the solution set of the following system of linear inequalities is an unbounded region $2x + y \geq 8,\ x + 2y \geq 10,\ x \geq 0,\ y \geq 0.$ [NCERT EXEMPLAR]

9. Find the linear inequalities for which the shaded region in Fig. 14.45 is the solution set. [NCERT EXEMPLAR]

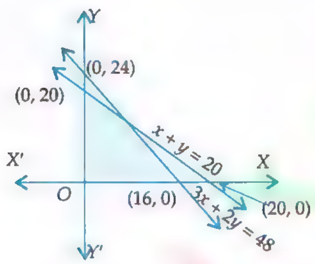


Fig. 14.45

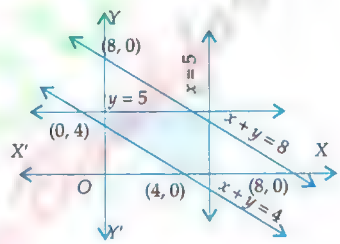


Fig. 14.46

10. Find the linear inequalities for which the shaded region in Fig. 14.46 is the solution set. [NCERT EXEMPLAR]

ANSWERS

3. $2x + 3y \geq 6,\ 4x + 6y \leq 24,\ -3x + 2y \leq 3,\ x - 2y \leq 2,\ x \geq 0,\ y \geq 0,$
4. $x + y \leq 4,\ y \leq 3,\ x \leq 3,\ x + 5y \geq 4,\ 6x + 2y \geq 8,\ x \geq 0,\ y \geq 0$
9. $x + y \leq 20,\ 3x + 2y \leq 48,\ x \geq 0,\ y \geq 0$
10. $x + y \leq 8,\ x + y \geq 4,\ x \leq 5,\ y \leq 5,\ x \geq 0,\ y \geq 0$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If $x \geq -3$, then $x + 5$............ 2.
2. If $-x \leq -4$, then $2x$.............. 8.
3. If $\frac{1}{x-2} < 0$, then x .... 2.
4. If $|x - 1| \leq 2$ then -1 .... $x < 3$.
5. If $|3x - 7| > 2$, then x .... $\frac{5}{3}$ or, x ....3.
6. If $-4x \geq 2$, then x............ -3.

7. If $-\frac{3x}{4} \leq -3$, then x...........4.

8. If $x > y$ and $z < 0$, then $-xz$ ....$-yz$.

9. The solution set of the inequation $|x+1| < 3$ is ....

10. The solution set of the inequation $|x+2| > 5$ is ....

11. If $\frac{x-3}{|x-3|} \geq 0$, then x belongs to the interval ....

12. The solution set of the inequation $\frac{|x|+1}{|x|-1} < 0$ is........................ .

ANSWERS

1. ≥ 2. ≥ 3. < 4. ≤, ≤ 5. <, > 6. ≤ 7. ≥ 8. >
9. $(-4, 2)$ 10. $(-\infty, -7) \cup (3, \infty)$ 11. $(3, \infty)$ 12. $(-1, 1)$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the solution set of the inequation $\frac{x^2}{x-2} > 0$.

2. Write the solution set of the inequation $x + \frac{1}{x} \geq 2$.

3. Write the set of values of x satisfying the inequation $(x^2 - 2x + 1)(x - 4) \geq 0$.

4. Write the solution set of the equation $|2 - x| = x - 2$.

5. Write the set of values of x satisfying $|x - 1| \leq 3$ and $|x - 1| \leq 1$.

6. Write the solution set of the inequation $\left|\frac{1}{x} - 2\right| < 4$.

7. Write the number of integral solutions of $\frac{x+2}{x^2+1} > \frac{1}{2}$.

8. Write the set of values of x satisfying the inequations $5x + 2 < 3x + 8$ and $\frac{x+2}{x-1} < 4$.

9. Write the solution set of $\left|x + \frac{1}{x}\right| > 2$.

10. Write the solution set of the inequation $|x - 1| \geq |x - 3|$.

ANSWERS

1. $[2, \infty)$ 2. $(0, \infty)$ 3. $(-\infty, 4)$ 4. $(2, \infty)$
5. $[2, 4]$ 6. $(-\infty, -1/2) \cup (1/6, \infty)$ 7. 3
8. $(-\infty, 1) \cup (2, 3)$ 9. $R - \{-1, 0, 1\}$ 10. $[2, \infty)$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $x < 7$, then
   (a) $-x < -7$ (b) $-x \leq -7$ (c) $-x > -7$ (d) $-x \geq -7$
2. If $-3x + 17 < -13$, then
   (a) $x \in (10, \infty)$ (b) $x \in [10, \infty)$ (c) $x \in (-\infty, 10]$ (d) $x \in [-10, 10)$
3. Given that x, y and b are real numbers and $x < y$, $b > 0$, then
   (a) $\frac{x}{b} < \frac{y}{b}$ (b) $\frac{x}{b} \leq \frac{y}{b}$ (c) $\frac{x}{b} > \frac{y}{b}$ (d) $\frac{x}{b} \geq \frac{y}{b}$
4. If x is a real number and $|x| < 5$, then
   (a) $x \geq 5$ (b) $-5 < x < 5$ (c) $x \leq -5$ (d) $-5 \leq x \leq 5$
5. If x and a are real numbers such that $a > 0$ and $|x| > a$, then
   (a) $x \in (-a, \infty)$ (b) $x \in [-\infty, a]$ (c) $x \in (-a, a)$ (d) $x \in (-\infty, -a) \cup (a, \infty)$
6. If $|x - 1| > 5$, then
   (a) $x \in (-4, 6)$ (b) $x \in [-4, 6]$
   (c) $x \in (-\infty, -4) \cup (6, \infty)$ (d) $x \in (-\infty, -4) \cup [6, \infty)$
7. If $|x + 2| \leq 9$, then
   (a) $x \in (-7, 11)$ (b) $x \in [-11, 7]$
   (c) $x \in (-\infty, -7) \cup (11, \infty)$ (d) $x \in (-\infty, -7) \cup [11, \infty)$
8. The inequality representing the following graph is
   (a) $|x| < 3$ (b) $|x| \leq 3$ (c) $|x| > 3$ (d) $|x| \geq 3$

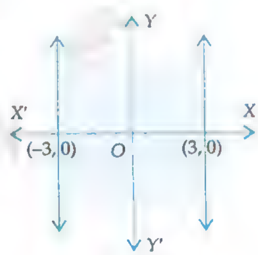


Fig. 14.47

9. The linear inequality representing the solution set given in Fig. 14.48 is
   (a) $|x| < 5$ (b) $|x| > 5$ (c) $|x| \geq 5$ (d) $|x| \leq 5$

Fig. 14.48

10. The solution set of the inequation $|x + 2| \leq 5$ is
    (a) $(-7, 5)$ (b) $[-7, 3]$ (c) $[-5, 5]$ (d) $(-7, 3)$
11. If $\frac{|x - 2|}{x - 2} \geq 0$, then
    (a) $x \in [2, \infty)$ (b) $x \in (2, \infty)$ (c) $x \in (-\infty, 2)$ (d) $x \in (-\infty, 2]$

12. If $|x+3| \geq 10$, then

(a) $x \in (-13, 7]$ (b) $x \in (-13, 7)$

(c) $x \in (-\infty, -13) \cup (7, \infty)$ (d) $x \in (-\infty, -13] \cup [7, \infty)$

13. Solution of a linear inequality in variable x is represented on the number line as shown in Fig. 14.49. The solution can also be described as

Fig. 14.49

(a) $x \in (-\infty, 5)$ (b) $x \in (-\infty, 5]$ (c) $x \in [5, \infty)$ (d) $x \in (5, \infty)$

14. The shaded part of the number line in Fig. 14.50 can also be represented as

Fig. 14.50

(a) $x \in \left(\frac{9}{2}, \infty\right)$ (b) $x \in \left[\frac{9}{2}, \infty\right)$ (c) $x \in \left[-\infty, \frac{9}{2}\right)$ (d) $x \in \left(-\infty, \frac{9}{2}\right]$

15. The shaded part of the number line in Fig. 14. 51 can also be described as

Fig. 14.51

(a) $(-\infty, 1) \cup (2, \infty)$ (b) $(-\infty, 1] \cup [2, \infty)$

(c) $(1, 2)$ (d) $[1, 2]$

ANSWERS

1. (c)	2. (a)	3. (a)	4. (b)	5. (d)	6. (c)	7. (b)	8. (b)
9. (c)	10. (b)	11. (b)	12. (d)	13. (d)	14. (b)	15. (a)	

CHAPTER 15

PERMUTATIONS

15.1 THE FACTORIAL

In this section, we shall introduce the term and notation of factorial which will be often used in this chapter and the next three chapters.

FACTORIAL *The continued product of first n natural numbers is called the "n factorial" and is denoted by $n!$ or $\lfloor n$.*

i.e $\quad n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n.$

Thus, $\quad 3! = 1 \times 2 \times 3 = 6; \quad 4! = 1 \times 2 \times 3 \times 4 = 24, \quad 5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$ etc.

Clearly, $n!$ is defined for positive integers only.

ZERO FACTORIAL *As we will require zero factorial in the later sections of this chapter and it does not make any sense to define it as the product of the integers from 1 to zero. So, we define $0! = 1$.*

NOTE *Factorials of proper fractions or negative integers are not defined. Factorial n is defined only for whole numbers.*

DEDUCTION We have,

$$n! = 1 \times 2 \times 3 \times 4 \ldots \times (n-1) \times n = [1 \times 2 \times 3 \times 4 \ldots \times (n-1)]\, n = [(n-1)!]\, n = n \times (n-1)!$$

Thus, $\quad n! = n \times (n-1)!$

Similarly,

$$n! = n(n-1)(n-2)! = n(n-1)(n-2)(n-3)! = n(n-1)(n-2)(n-3)(n-4)! \text{ and so on.}$$

For example, $8! = 8(7!)$, $\quad 5! = 5(4!)$ $\quad$ and $\quad 2! = 2(1!)$

Following examples will illustrate the use of this property of factorial n.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Compute:* (i) $\dfrac{20!}{18!}$ $\qquad$ (ii) $\dfrac{10!}{6!\,4!}$

SOLUTION (i) We have,

$$\frac{20!}{18!} = \frac{20\,(19!)}{18!} = \frac{20 \times 19 \times 18!}{18!} = 20 \times 19 = 380 \qquad [\because\ n! = n \times (n-1)!]$$

(ii) $$\frac{10!}{6!\,4!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{6! \times (4 \times 3 \times 2 \times 1)} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

EXAMPLE 2 *Convert the following products into factorials:*

(i) $6 \cdot 7 \cdot 8 \cdot 9 \cdot 10$ $\qquad$ (ii) $2 \cdot 4 \cdot 6 \cdot 8 \cdot 10$

SOLUTION (i) $$6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{10!}{5!}$$

(ii) $$2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 = (2 \times 1)(2 \times 2)(2 \times 3)(2 \times 4)(2 \times 5) = 2^5 \times (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5) = 2^5 \times 5!$$

EXAMPLE 3 *Find the LCM of 4!, 5! and 6!*

SOLUTION We have, $5! = 5 \times 4!$ and $6! = 6 \times 5 \times 4!$

$\therefore$ L.C.M. of $4!, 5!, 6! = $ L.C.M. $\{4!, 5 \times 4!, 6 \times 5 \times 4!\} = (4!) \times 5 \times 6 = 6! = 720$

EXAMPLE 4 *If* $\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$, *find x.*

SOLUTION We have,

$$\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$$

$$\Rightarrow \frac{1}{9!} + \frac{1}{10 \times 9!} = \frac{x}{11 \times 10 \times 9!}$$

$$\Rightarrow \frac{1}{9!}\left(1 + \frac{1}{10}\right) = \left(\frac{x}{11 \times 10}\right) \times \frac{1}{9!} \Rightarrow 1 + \frac{1}{10} = \frac{x}{11 \times 10} \Rightarrow \frac{11}{10} = \frac{x}{11 \times 10} \Rightarrow x = 11 \times 11 = 121.$$

ALITER We have,

$$\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$$

Multiplying both sides by the LCM of 9!, 10! and 11! i.e. by 11!, we obtain

$$\frac{11!}{9!} + \frac{11!}{10!} = \frac{x}{11!} \times 11! \Rightarrow \frac{11 \times 10 \times 9!}{9!} + \frac{11 \times 10!}{10!} = x \Rightarrow 11 \times 10 + 11 = x \Rightarrow x = 121.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Find n, if:*

(i) $(n+2)! = 2550 \times n!$ (ii) $(n+1)! = 12 \times (n-1)!$

SOLUTION (i) We have,

$$(n+2)! = 2550 \times n!$$

$$\Rightarrow (n+2)(n+1) \times n! = 2550 \times n!$$

$$\Rightarrow (n+2) \times (n+1) = 2550$$

$\Rightarrow (n+2)(n+1) = 51 \times 50$ [Expressing 2550 as the product of two consecutive natural numbers]

$\Rightarrow n+2 = 51$ or, $n+1 = 50 \Rightarrow n = 49$ [By comparing]

(ii) We have,

$$(n+1)! = 12 \times (n-1)!$$

$$\Rightarrow (n+1) \times n \times (n-1)! = 12 \times (n-1)!$$

$\Rightarrow n(n+1) = 12 \Rightarrow (n+1)n = 4 \times 3 \Rightarrow n = 3$ [By comparing]

EXAMPLE 6 *If* $\frac{n!}{2!(n-2)!}$ and $\frac{n!}{4!(n-4)!}$ *are in the ratio 2 : 1, find the value of n.*

SOLUTION We have,

$$\frac{n!}{2!(n-2)!} : \frac{n!}{4!(n-4)!} = 2:1 \Rightarrow \frac{n!}{2!(n-2)!} \times \frac{4!(n-4)!}{n!} = \frac{2}{1}$$

$$\Rightarrow \frac{4!(n-4)!}{2!(n-2) \times (n-3) \times (n-4)!} = \frac{2}{1} \Rightarrow \frac{4 \times 3 \times 2!}{2!(n-2)(n-3)} = \frac{2}{1}$$

$$\Rightarrow (n-2)(n-3) = 6 \Rightarrow (n-2)(n-3) = 3 \times 2 \Rightarrow n-2 = 3 \text{ and } n-3 = 2 \Rightarrow n = 5$$

EXAMPLE 7 *Prove that:* $\frac{(2n)!}{n!} = \left\{1 \cdot 3 \cdot 5 \ldots (2n-1)\right\} 2^n$.

SOLUTION We have,

$$\frac{(2n)!}{n!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \ldots (2n-2)(2n-1)(2n)}{n!}$$

$$= \frac{\{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-1)\} \cdot \{2 \cdot 4 \cdot 6 \cdot 8 \ldots (2n-2)(2n)\}}{n!}$$

$$= \frac{\{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-1)\}\, 2^n \{1 \cdot 2 \cdot 3 \cdot 4 \ldots (n-1)\, n\}}{n!}$$

$$= \frac{\{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-1)\} \cdot 2^n \cdot n!}{n!} = \{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-1)\}\, 2^n$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 8 *Prove that $(n!+1)$ is not divisible by any natural number between 2 and n.*

SOLUTION Let m be divisible by k and r be any natural number between 1 and k. If $m+r$ is divided by k, then we obtain r as the remainder.

We have, $n! = 1 \cdot 2 \cdot 3 \cdot 4 \ldots (n-1) \cdot n$.

Therefore, $n!$ is divisible by every natural number between 2 and n. So, $(n!+1)$, when divided by any natural number between 2 and n, leaves 1 as the remainder.

Hence, $(n!+1)$ is not divisible by any natural number between 2 and n.

EXAMPLE 9 *Prove the inequalities $(n!)^2 \le n^n (n!) < (2n)!$ for all positive integers n.*

SOLUTION Clearly, $(n!)^2 = (n!)(n!) = (1 \cdot 2 \cdot 3 \cdot 4 \ldots (n-1)\, n)(n!)$

We know that

$$\left.\begin{array}{r} 1 \le n \\ 2 \le n \\ 3 \le n \\ \vdots \;\vdots\; \vdots \\ (n-1) \le n \\ n \le n \end{array}\right\} \Rightarrow 1 \cdot 2 \cdot (n-1)\, n \le \underbrace{n \cdot n \cdot n \ldots n}_{n-\text{times}} \Rightarrow n! \le n^n \Rightarrow (n!)(n!) \le n^n (n!) \Rightarrow (n!)^2 \le n^n (n!) \quad \ldots\text{(i)}$$

Now, $(2n)! = 1 \cdot 2 \cdot 3 \ldots (n-1)\, n\, (n+1)(n+2) \ldots (2n-1)(2n) = n!\,(n+1)(n+2) \ldots (2n-1)(2n)$

Now,
$$\left.\begin{array}{r} n+1 > n \\ n+2 > n \\ n+3 > n \\ \vdots \;\vdots\; \vdots \\ n+(n-1) > n \\ n+n > n \end{array}\right\} \Rightarrow (n+1)(n+2)(n+3) \ldots (2n-1)(2n) > n^n$$

$$\Rightarrow \quad n!\,(n+1)(n+2) \ldots (2n-1)(2n) > n!\, n^n \Rightarrow (2n)! > n!\, n^n \Rightarrow n!\, n^n < (2n)! \quad \ldots\text{(ii)}$$

From (i) and (ii), we obtain: $(n!)^2 \le n^n (n!) < (2n)!$

EXAMPLE 10 *Prove that 33! is divisible by 2^{15}. What is the largest integer n such that 33! is divisible by 2^n?*

SOLUTION Let $E_2(n)$ denote the index of 2 in n. Then,

$$E_2(33!) = E_2(1.2.3.4.5.6.....32.33)$$

$$\Rightarrow \quad E_2(33!) = E_2(2.4.6.8.....30.32)$$

$$\Rightarrow \quad E_2(33!) = 16 + E_2(1.2.3......15.16)$$

$$\Rightarrow \quad E_2(33!) = 16 + E_2(2.4.6....14.16)$$

$\Rightarrow \quad E_2(33!) = 16 + 8 + E_2(1.2.3\ldots 8)$

$\Rightarrow \quad E_2(33!) = 16 + 8 + E_2(2.4.6.8)$

$\Rightarrow \quad E_2(33!) = 16 + 8 + 4 + E_2(1.2.3.4)$

$\Rightarrow \quad E_2(33!) = 16 + 8 + 4 + E_2(2.4) = 16 + 8 + 4 + 3 = 31.$

Thus, exponent of 2 in 33! is 31 i.e. $33! = 2^{31} \times$ an integer

This shows that 33! is divisible by 2^{15} and the largest integer n such that 33! is divisible by 2^n is 31.

EXERCISE 15.1

BASIC

1. Compute:
   (i) $\frac{30!}{28!}$ (ii) $\frac{11!-10!}{9!}$ (iii) L.C.M. (6!, 7!, 8!)
2. Prove that $\frac{1}{9!}+\frac{1}{10!}+\frac{1}{11!}=\frac{122}{11!}$
3. Find x in each of the following:
   (i) $\frac{1}{4!}+\frac{1}{5!}=\frac{x}{6!}$ (ii) $\frac{x}{10!}=\frac{1}{8!}+\frac{1}{9!}$ [NCERT] (iii) $\frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!}$ [NCERT]
4. Convert the following products into factorials:
   (i) $5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10$ (ii) $3\cdot 6\cdot 9\cdot 12\cdot 15\cdot 18$
   (iii) $(n+1)(n+2)(n+3)\ldots(2n)$ (iv) $1\cdot 3\cdot 5\cdot 7\cdot 9\ldots(2n-1)$
5. Which of the following are true:
   (i) $(2+3)! = 2! + 3!$ (ii) $(2\times 3)! = 2!\times 3!$
6. Prove that : $n!(n+2) = n! + (n+1)!$

BASED ON LOTS

7. If $(n+2)! = 60[(n-1)!]$, find n.
8. If $(n+1)! = 90[(n-1)!]$, find n.
9. If $(n+3)! = 56[(n+1)!]$, find n.
10. If $\frac{(2n)!}{3!(2n-3)!}$ and $\frac{n!}{2!(n-2)!}$ are in the ratio 44 : 3, find n.
11. Prove that:
    (i) $\frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-(r-1))$ (ii) $\frac{n!}{(n-r)!\,r!}+\frac{n!}{(n-r+1)!\,(r-1)!}=\frac{(n+1)!}{r!\,(n-r+1)!}$
12. Prove that: $\frac{(2n+1)!}{n!} = 2^n\left\{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)\right\}$

ANSWERS

1. (i) 870 (ii) 100 (iii) 8! 3. (i) 36 (ii) 100 (iii) 64 4. (i) $\frac{10!}{4!}$ (ii) $3^6(6!)$
(iii) $\frac{(2n)!}{n!}$ (iv) $\frac{(2n)!}{2^n\, n!}$ 5. (i) False (ii) False 7. 3 8. 9 9. 5 10. 6

HINTS TO SELECTED PROBLEMS

3. (i) We have,

$$\frac{1}{4!}+\frac{1}{5!}=\frac{x}{6!} \Rightarrow \frac{6!}{4!}+\frac{6!}{5!}=x \qquad \text{[Multiplying both sides by 6!]}$$

$$\Rightarrow \frac{6\times5\times4!}{4!}+\frac{6\times5!}{5!}=x \Rightarrow 6\times5+6=x \Rightarrow x=36$$

(ii) We have,

$$\frac{x}{10!}=\frac{1}{8!}+\frac{1}{9!} \Rightarrow x=\frac{10!}{8!}+\frac{10!}{9!} \qquad \text{[Multiplying both sides by 10!]}$$

$$\Rightarrow x=\frac{10\times9\times8!}{8!}+\frac{10\times9!}{9!} \Rightarrow x=10\times9+10=100$$

(iii) We have,

$$\frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!} \Rightarrow \frac{8!}{6!}+\frac{8!}{7!}=x \qquad \text{[Multiplying both sides by 8!]}$$

$$\Rightarrow \frac{8\times7\times6!}{6!}+\frac{8\times7!}{7!}=x \Rightarrow 8\times7+8=x \Rightarrow x=64$$

7. $(n+2)!=60(n-1)!$

$\Rightarrow (n+2)\,(n+1)\,(n)\,(n-1)!=60\times(n-1)!$

$\Rightarrow (n+2)\,(n+1)\,(n)=5\times4\times3$ [Expressing 60 as the product of three consecutive integers]

$\Rightarrow n=3$ [On comparing two sides]

8. $(n+1)!=90\,(n-1)!$

$\Rightarrow \quad n+1\,(n)\,(n-1)!=90\,(n-1)!$

$\Rightarrow \quad (n+1)n=10\times9$ [Writing 90 as the product of consecutive integers]

$\Rightarrow \quad n=9$

9. $(n+3)!=56\,(n+1)!$

$\Rightarrow \quad (n+3)\,(n+2)\,(n+1)!=56\,(n+1)!$

$\Rightarrow \quad (n+3)\,(n+2)=8\times7$ [Writing 56 as the product of consecutive integers]

$\Rightarrow \quad n+2=7 \Rightarrow n=5$

15.2 FUNDAMENTAL PRINCIPLES OF COUNTING

In this section, we shall discuss two fundamental principles viz. principle of addition and principle of multiplication. These two principles will enable us to understand permutations and combinations. In fact these two principles form the base of permutations and combinations.

FUNDAMENTAL PRINCIPLE OF MULTIPLICATION *If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, second job can be completed in n ways; then the two jobs in succession can be completed in $m \times n$ ways.*

EXPLANATION If the first job is performed in any one of the m ways, we can associate with this any one of the n ways of performing the second job: and thus there are n ways of performing the two jobs without considering more than one way of performing the first; and so corresponding to each of the m ways of performing the first job, we have n ways of performing the second job. Hence, the number of ways in which the two jobs can be performed is $m \times n$.

ILLUSTRATION 1 *In a class there are 10 boys and 8 girls. The teacher wants to select a boy and a girl to represent the class in a function. In how many ways can the teacher make this selection ?*

SOLUTION Here the teacher is to perform two jobs:

(i) selecting a boy among 10 boys, and (ii) selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore by the fundamental principle of multiplication, the required number of ways is $10 \times 8 = 80$.

REMARK *The above principle can be extended for any finite number of jobs as stated below:*
If there are n jobs $J_1, J_2, \ldots J_n$ such that job J_i can be performed independently in m_i ways; $i = 1, 2, ..., n$. Then the total number of ways in which all the jobs can be performed is $m_1 \times m_2 \times m_3 \times \ldots \times m_n$.

FUNDAMENTAL PRINCIPLE OF ADDITION *If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m + n)$ ways.*

ILLUSTRATION 2 *In a class there are 10 boys and 8 girls. The teacher wants to select either a boy or a girl to represent the class in a function. In how many ways the teacher can make this selection?*

SOLUTION Here the teacher is to perform either of the following two jobs :

(i) selecting a boy among 10 boys. or, (ii) selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore, by fundamental principle of addition either of the two jobs can be performed in $(10 + 8) = 18$ ways. Hence, the teacher can make the selection of either a boy or a girl in 18 ways.

DIFFERENCE BETWEEN THE TWO PRINCIPLES *As we have discussed in the principle of multiplication a job is divided or decomposed into a number of sub-jobs which are unconnected to each other and the job is said to be performed if each sub-job is performed. While in the principle of addition there are a number of independent jobs and we have to perform one of them. So, the total number of ways of completing any one of the sub-jobs is the sum of the number of ways of completing each sub-jobs.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *There are 3 candidates for a Classical, 5 for a Mathematical, and 4 for a Natural science scholarship.*

(i) In how many ways can these scholarships be awarded?

(ii) In how many ways one of these scholarships be awarded?

SOLUTION Clearly, Classical scholarship can be awarded to any one of the three candidates. So, there are 3 ways of awarding the Classical scholarship.

Similarly, Mathematical and Natural science scholarships can be awarded in 5 and 4 ways respectively. So, by Fundamental Principle of multiplication,

Number of ways of awarding three scholarships $= 3 \times 5 \times 4 = 60$

By Fundamental Principle of addition,

Number of way of awarding one of the three scholarships $= 3 + 5 + 4 = 12$

EXAMPLE 2 *A room has 6 doors. In how many ways can a man enter the room through one door and come out through a different door ?*

SOLUTION Clearly, a person can enter the room through any one of the six doors. So, there are six ways of entering into the room. After entering into the room, the man can come out through any one of the remaining five doors. So, he can come out through a different door in 5 ways.

Hence, the number of ways in which a man can enter a room through one door and come out through a different door $= 6 \times 5 = 30$.

EXAMPLE 3 *The flag of a newly formed forum is in the form*  *of three blocks, each to be coloured differently. If there are six different colours on the whole to choose from, how many such designs are possible ?*

SOLUTION Since there are six colours to choose from, therefore, first block can be coloured in 6 ways. Now, the second block can be coloured by any one of the remaining colours in five ways. So, there are five ways to colour the second block.

After colouring first two blocks only four colours are left. The third block can now be coloured by any one of the remaining four colours. So, there are four ways to colour the third block.

Hence, by the fundamental principle of multiplication, the number of flag-designs is $6 \times 5 \times 4 = 120$.

EXAMPLE 4 *Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, when* [NCERT]
(i) *the repetition of the letters is not allowed.* (ii) *the repetition of the letters is allowed.*

SOLUTION (i) The total number of words is same as the number of ways of filling in 4 vacant places by the 4 letters. The first place can be filled in 4 different ways by any one of the 4 letters R, O, S, E. Since the repetition of letters is not allowed. Therefore, the second place can be filled in by any one of the remaining 3 letters in 3 different ways, following which the third place can be filled in by the remaining 2 letters in 2 different ways; following which the fourth place can be filled in by the remaining one letter in one way. Thus, by the fundamental principle of counting the required number of ways is $4 \times 3 \times 2 \times 1 = 24$.

Hence, required number of words = 24.

(ii) If the repetition of the letters is allowed, then each of the 4 vacant places can be filled in succession in 4 different ways.

Hence, required number of words = $4 \times 4 \times 4 \times 4 = 256$.

EXAMPLE 5 *Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?* [NCERT]

SOLUTION The total number of signals is equal to the number of ways of filling in 2 vacant places in succession by four flags of different colours. The upper cacant place can be filled in 4 different ways by any one of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by any one of the remaining the different flags.

Hence, by the fundamental principle of multiplication, the required number of signals is $4 \times 3 = 12$.

EXAMPLE 6 *Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.* [NCERT]

SOLUTION Since a signal may consist of either 2 flags, 3 flags, 4 flags or 5 flags. Therefore,

Total number of signals = Number of 2 flags signals

+ Nnumber of 3 flags signals

+ Number of 4 flags signals

+ Number of 5 flags signals

$= 5 \times 4 + 5 \times 4 \times 3 + 5 \times 4 \times 3 \times 2 + 5 \times 4 \times 3 \times 2 \times 1$

$= 20 + 60 + 120 + 120 = 320$

EXAMPLE 7 *In a monthly test, the teacher decides that there will be three questions, one from each of Exercises 7, 8 and 9 of the text book. If there are 12 questions in Exercise 7, 18 in Exercise 8 and 9 in Exercise 9, in how many ways can three questions be selected ?*

SOLUTION There are 12 questions in exercise 7. So, one question from exercise 7 can be selected in 12 ways. Exercise 8 contains 18 questions. So, second question can be selected in 18 ways. There are 9 questions in exercise 9. So, third question can be selected in 9 ways. Hence, three questions can be selected in $12 \times 18 \times 9 = 1944$ ways.

EXAMPLE 8 *How many words (with or without meaning) of three distinct letters of the English alphabets are there ?*

SOLUTION Here we have to fill up three places by distinct letters of the English alphabets. Since there are 26 letters of the English alphabet, the first place can be filled by any of these letters. So, there are 26 ways of filling up the first place. Now, the second place can be filled up by any of the remaining 25 letters. So, there are 25 ways of filling up the second place. After filling up the first two places only 24 letters are left to fill up the third place. So, the third place can be filled in 24 ways.

Hence, the required number of words $= 26 \times 25 \times 24 = 15600$

EXAMPLE 9 *There are 6 multiple choice questions in an examination. How many sequence of answers are possible, if the first three questions have 4 choices each and the next three have 5 each ?*

SOLUTION Here we have to perform 6 jobs of answering 6 multiple choice questions. Each one of the first three questions can be answered in 4 ways and each one of the next three can be answered in 5 different ways.

So, the total number of different sequences $= 4 \times 4 \times 4 \times 5 \times 5 \times 5 = 8000$

EXAMPLE 10 *Find the total number of ways of answering 5 objective type questions, each question having 4 choices.*

SOLUTION Since each question can be answered in 4 ways. So, the total number of ways of answering 5 questions is $4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

EXAMPLE 11 *How many three-digit numbers can be formed without using the digits 0, 2, 3, 4, 5 and 6 ?*

SOLUTION We have to determine the total number of three digit numbers formed by using the digits 1, 7, 8, 9. Clearly, the repetition of digits is allowed.

A three digit number has three places viz. units's, ten's and hundred's. Unit's place can be filled by any of the digits 1, 7, 8, 9. So, unit's place can be filled in 4 ways. Similarly, each one of the ten's and hundred's place can be filled in 4 ways.

$\therefore$ Total number of required numbers $= 4 \times 4 \times 4 = 64$.

EXAMPLE 12 *How many numbers are there between 100 and 1000 in which all the digits are distinct ?*

SOLUTION A number between 100 and 1000 has three digits. So, we have to form all possible 3-digit numbers with distinct digits. We cannot have 0 at the hundred's place. So, the hundred's place can be filled with any of the 9 digits 1, 2, 3, ..., 9. So, there are 9 ways of filling the hundred's place.

Now, 9 digits are left including 0. So, ten's place can be filled with any of the remaining 9 digits in 9 ways. Now, the unit's place can be filled with in any of the remaining 8 digits. So, there are 8 ways of filling the unit's place.

Hence, the total number of required numbers $= 9 \times 9 \times 8 = 648$.

EXAMPLE 13 *How many numbers are there between 100 and 1000 such that every digit is either 2 or 9 ?*

SOLUTION Every number between 100 and 1000 consists of three digits. So, we have to determine the total number of three digit numbers such that every digit is either 2 or 9.

Clearly, each one of the unit's, ten's and hundred's place can be filled in 2 ways.

So, the total number of required numbers $= 2 \times 2 \times 2 = 8$.

EXAMPLE 14 *How many numbers are there between 100 and 1000 such that 7 is in the unit's place.*

SOLUTION Every number between 100 and 1000 is a three digit number. So, we have to form 3-digit numbers with 7 at the unit's place by using the digits 0, 1, 2, ..., 9. Clearly, repetition of digits is allowed. The hundred's place can be filled with any of the digits from 1 to 9 (zero cannot be there at hundred's place). So, hundred's place can be filled in 9 ways. Now, the ten's place can be filled with any of the digits from 0 to 9. So, ten's place can be filled in 10 ways. Since all the numbers have digit 7 at the unit's place, so, unit's place can be filled in only one way. Hence, by the fundamental principle of counting the total number of numbers between 100 and 1000 having 7 at the unit's place $= 9 \times 10 \times 1 = 90$.

EXAMPLE 15 *A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them, if he has three servants to carry the cards ?*

SOLUTION Since a card can be sent by any one of the three servants, so the number of ways of sending the invitation card to the first friend = 3. Similarly, invitation cards can be sent to each of the six friends in 3 ways.

So, the required number of ways $= 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 = 729$.

EXAMPLE 16 *How many three-digit numbers more than 600 can be formed by using the digits 2, 3, 4, 6, 7.*

SOLUTION Clearly, repetition of digits is allowed. Since a three-digit number greater than 600 will have 6 or 7 at hundred's place. So, hundred's place can be filled in 2 ways. Each of the ten's and one's place can be filled in 5 ways.

Hence, total number of required numbers $= 2 \times 5 \times 5 = 50$.

EXAMPLE 17 *How many numbers between 3000 and 4000 can be formed from the digits 3, 4, 5, 6, 7 and 8, no digit being repeated in any number ?*

SOLUTION Clearly, a number between 3000 and 4000 must have 3 at thousand's place. So, thousand's place can be filled in only one way. Now, hundred's place can be filled in 5 ways. Since repetition of digits is not allowed so ten's and one's places can be filled in 4 and 3 ways respectively.

So, total number of required numbers $= 1 \times 5 \times 4 \times 3 = 60$.

EXAMPLE 18 *How many numbers divisible by 5 and lying between 4000 and 5000 can be formed from the digits 4, 5, 6, 7 and 8.*

SOLUTION Clearly, a number between 4000 and 5000 must have 4 at thousand's place. Since the number is divisible by 5 it must have 5 at unit's place. Now, each of the remaining places (viz. hundred's and ten's) can be filled in 5 ways.

Hence, total number of required numbers $= 1 \times 5 \times 5 \times 1 = 25$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 19 *How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5 if (i) repetition of digits is not allowed (ii) repetition of digits is allowed ?*

SOLUTION (i) In a four-digit number 0 cannot appear in the thousand's place. So, thousand's place can be filled in 5 ways. (viz. 1, 2, 3, 4, 5). Since repetition of digits is not allowed and 0 can be used at hundred's place, so hundred's place can be filled in 5 ways.

Now, any one of the remaining four digits can be used to fill up ten's place. So, ten's place can be filled in 4 ways. One's place can be filled from the remaining three digits in 3 ways.

Hence, the required number of numbers $= 5 \times 5 \times 4 \times 3 = 300$.

(ii) For a four-digit number we have to fill up four places and 0 cannot appear in the thousand's place. So, thousand's place can be filled in 5 ways. Since repetition of digits is allowed, so each of the remaining three places viz. hundred's, ten's and one's can be filled in 6 ways.

Hence, the required number of numbers $= 5 \times 6 \times 6 \times 6 = 1080$..

EXAMPLE 20 *How many numbers greater than 1000, but not greater than 4000 can be formed with the digits 0, 1, 2, 3, 4 if : (i) repetition of digits is allowed ? (ii) repetition of digits is not allowed ?*

SOLUTION (i) Every number between 1000 and 4000 is a four digit number. In thousand's place we can put either 1 or 2 or 3 but not 4. So, thousand's place can be filled in 3 ways. Since repetition of digits is allowed, so each of the hundred's, ten's and one's place can be filled in 5 ways. So, total number of numbers between 1000 and 4000, including 1000 and excluding 4000 is $3 \times 5 \times 5 \times 5 = 375$. But, we have to find the total number of numbers greater than 1000 but not greater than 4000.

Hence, required number of numbers $= 375 + 1$ (for 4000) $- 1$ (for 1000) $= 375$.

(ii) As discussed above thousand's place can be filled in 3 ways. Since repetition of digits is not allowed, so, hundred's place can be filled from the remaining digits in 4 ways. Now, three digits are left, so ten's place can be filled in 3 ways. One's place can be filled in 2 ways.

Hence, required number of numbers $= 3 \times 4 \times 3 \times 2 = 72$.

EXAMPLE 21 *How many three digit odd numbers can be formed by using the digits 1, 2, 3, 4, 5, 6 if:* [NCERT]

(i) the repetition of digits is not allowed ? (ii) the repetition of digits is allowed ?

SOLUTION For a number to be odd, we must have 1, 3 or 5 at the unit's place. So, there are 3 ways of filling the unit's place.

(i) Since the repetition of digits is not allowed, the ten's place can be filled with any of the remaining 5 digits in 5 ways. Now, four digits are left. So, hundred's place can be filled in 4 ways.

So, required number of numbers $= 3 \times 5 \times 4 = 60$

(ii) Since the repetition of digits is allowed, so each of the ten's and hundred's place can be filled in 6 ways.

Hence, required number of numbers $= 3 \times 6 \times 6 = 108$.

EXAMPLE 22 *How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?* [NCERT]

SOLUTION For a number to be even, we must have 2, 4 or 6 at the unit's place. So, there are 3 ways to fill in the unit's place. Since digits can be repeated, so each of the ten's and hundred's place can be filled in 6 ways.

Hence, required number of numbers $= 3 \times 6 \times 6 = 108$.

EXAMPLE 23 *How many numbers of 3 digits can be formed with the digits 1, 2, 3, 4, 5 when digits may be repeated ?*

SOLUTION The unit's place can be filled in 5 ways. Since, the repetition of digits is allowed, therefore ten's place can be filled in 5 ways and hundred's place can also be filled in 5 ways. Therefore, by the fundamental principle of counting, the required number of three digit numbers $= 5 \times 5 \times 5 = 125$.

EXAMPLE 24 *Find the number of numbers of 5 digits that can be formed with the digits 0, 1, 2, 3, 4 if the digits can be repeated in the same number.*

SOLUTION In a five digit number 0 cannot be put in ten thousand's place. So, the number of ways of filling up the ten thousand's place = 4.

Since the repetition of digits is allowed, therefore each of the other places can be filled in 5 ways.

So, the required number of numbers $= 4 \times 5 \times 5 \times 5 \times 5 = 2500$.

EXAMPLE 25 *How many 4-digit numbers are there, when a digit may be repeated any number of times ?*

SOLUTION In a four digit number 0 cannot be placed at thousand's place. So, thousand's place can be filled with any digit from 1 to 9. Thus, thousand's place can be filled in 9 ways.

Since repetition of digits is allowed, therefore each of the remaining 3 places can be filled in 10 ways by using the digits from 0 to 9.

Hence, the required number of numbers $= 9 \times 10 \times 10 \times 10 = 9000$.

EXAMPLE 26 *How many three-letter words can be formed using a, b, c, d, e if : (i) repetition is not allowed (ii) repetition is allowed ?*

SOLUTION (i) Clearly, the total number of three-letter words is equal to the number of ways of filling three places. First place can be filled in 5 ways. Now, four letters are left. So, the second

place can be filled in 4 ways. Since the repetition of letters is not allowed, so the third place can be filled from any one of the remaining 3 digits in 3 ways.

Hence, total number of words $=5\times 4\times 3=60$.

(ii) In this case repetition of letters is allowed, so each of the three places can be filled in 5 ways.

Hence, total number of words $=5\times 5\times 5=125$.

EXAMPLE 27 *In how many ways can the following prizes be given away to a class of 30 students, first and second in Mathematics, first and second in Physics, first in Chemistry and first in English ?*

SOLUTION Here we have to give prizes in four subjects and the process of distributing prizes can be completed by giving prizes in the four subjects.

First and second prizes can be given in Mathematics in (30×29) ways.

First and second prizes can be given in Physics in (30×29) ways.

First prize can be given in Chemistry in 30 ways.

First prize can be given in English in 30 ways.

Hence, the number of ways to give prizes in all the four subjects

$$=(30\times 29)\times(30\times 29)\times 30\times 30=6.8121\times 10^8$$

EXAMPLE 28 *In how many ways 5 rings of different types can be worn in 4 fingers ?*

SOLUTION The first ring can be worn in any of the 4 fingers. So, there are 4 ways of wearing it. Similarly, each one of the other rings can be worn in 4 ways.

Hence, the requisite number of ways $=4\times 4\times 4\times 4\times 4=4^5$.

EXAMPLE 29 *In how many ways can 5 letters be posted in 4 letter boxes ?*

SOLUTION Since each letter can be posted in any one of the four letter boxes. So, a letter can be posted in 4 ways. Since there are 5 letters and each letter can be posted in 4 ways. So, total number of ways in which all the five letters can be posted is $4\times 4\times 4\times 4\times 4=4^5$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 30 *Five persons entered the lift cabin on the ground floor of an 8-floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. Find the total number of ways in which each of the five persons can leave the cabin (i) at any one of the 7 floors (ii) at different floors.*

SOLUTION Suppose A_1, A_2, A_3, A_4, A_5 are five persons.

(i) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Similarly, each of A_2, A_3, A_4, A_5 can leave the cabin in 7 ways. Thus, the total number of ways in which each of the five persons can leave the cabin at any of the seven floors is $7\times 7\times 7\times 7\times 7=7^5$.

(ii) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Now, A_2 can leave the cabin at any of the remaining 6 floors. So, A_2 can leave the cabin in 6 ways. Similarly, A_3, A_4, and A_5 can leave the cabin in 5, 4 and 3 ways respectively. Thus, the total number of ways in which each of the five persons can leave the cabin at different floors is $7\times 6\times 5\times 4\times 3=2520$.

EXAMPLE 31 *A mint prepares metallic calendars specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of February calendars should it prepare to serve for all the possibilities in the future years ?*

SOLUTION The mint has to perform two jobs, viz.

(i) selecting the number of days in the February month (there can be 28 days or 29 days), and
(ii) selecting the first day of the February month.

The first job can be completed in 2 ways while the second can be performed in 7 ways by selecting any one of the seven days of a week.

Thus, the required number of plates $= 2 \times 7 = 14$.

EXAMPLE 32 *For a set of five true/false questions, no student has written all correct answers, and no two students have given the same sequence of answers. What is the maximum number of students in the class, for this to be possible ?*

SOLUTION Since a true/false type question can be answered in 2 ways either by marking it true or false. So, there are 2 ways of answering each of the 5 questions.

$\therefore$ Total number of different sequences of answers $= 2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$.

Out of these 32 sequences of answers there is only one sequence of answering all the five questions correctly. But no student has written all the correct answers and different students have given different sequences of answers.

$\therefore$ Maximum number of students in the class

= Number of sequences except one sequence in which all answers are correct $= 32 - 1 = 31$

EXAMPLE 33 *How many numbers are there between 100 and 1000 such that at least one of their digits is 7?*

SOLUTION Clearly, a number between 100 and 1000 has 3-digits

$\therefore$ Total number of 3-digit numbers having at least one of their digits as 7

= (Total number of three-digit numbers) – (Total number of 3-digit numbers in which 7 does not appear at all)

Total number of three-digit numbers : We have to form three-digit numbers by using the digits 0, 1, 2, 3, ..., 9. Clearly, hundred's place can be filled in 9 ways and each of the ten's and one's place can be filled in 10 ways.

So, total number of 3-digit number $= 9 \times 10 \times 10 = 900$.

Total number of three-digit number in which 7 does not appear at all : Here we have to form three-digit numbers by using the digits 0 to 9, except 7. So, hundred's place can be filled in 8 ways and each of the ten's and one's place can be filled in 9 ways. So, total number of three-digit numbers in which 7 does not appear at all is $8 \times 9 \times 9$.

Hence, total number of 3-digit numbers having at least one of their digits as 7 is

$$9 \times 10 \times 10 - 8 \times 9 \times 9 = 252.$$

EXAMPLE 34 *How many numbers are there between 100 and 1000 which have exactly one of their digits as 7?*

SOLUTION A number between 100 and 1000 contains 3-digits. So, we have to form 3-digit numbers having exactly one of their digits as 7. Such type of numbers can be divided into three types:

(i) Those numbers that have 7 in the unit's place but not in any other place.
(ii) Those numbers that have 7 in the ten's place but not in any other place.
(iii) Those numbers that have 7 in the hundred's place but not in any other place.

Required number of numbers is the total number of these three types of numbers.

We shall now count these three types of numbers separately.

(i) *Those three-digit numbers that have 7 in the unit's place but not in any other place.*

The hundred's place can have any one of the digits from 0 to 9 except 0 and 7. So, hundred's place can be filled in 8 ways. The ten's place can have any one of the digits from 0 to 9 except 7. So, the number of ways the ten's place can be filled is 9. The unit's place has 7. So, it can be filled in only one way.

Thus, there are $8 \times 9 \times 1 = 72$ numbers of the first kind.

(ii) *Those three-digit numbers that have 7 in the ten's place but not in any other place.*

The number of ways to fill the hundred's place $= 8$

(by any one of the digits from 1, 2, 3, 4, 5, 6, 8, 9)

The number of ways to fill the ten's place $= 1$ (by 7 only)

The number of ways to fill the one's place $= 9$ (by any one of the digits 0, 1, 2, 3, 4, 5, 6, 8, 9)

Thus, there are $8 \times 1 \times 9 = 72$ numbers of the second kind.

(iii) *Those three-digit numbers that have 7 in the hundred's place but not at any other place.*

In this case, the hundred's place can be filled only in one way and each of the ten's and one's place can be filled in 9 ways.

So, there are $1 \times 9 \times 9 = 81$ numbers of the third kind.

Hence, the total number of required type of numbers $= 72 + 72 + 81 = 225$.

EXAMPLE 35 *A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest. What is the total number of signals that can be made ?*

SOLUTION Since each arm can be kept in 4 positions and a signal is possible when all the 5 arms are simultaneously placed in positions.

$\therefore$ Total number of ways of placing the arms $= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

But, this includes one inadmissible case, when all the arms are in the position of rest and then no signal can be made.

Hence, required number of signals $= (4^5 - 1) = 1023$.

EXAMPLE 36 *In how many ways can 3 prizes be distributed among 4 boys, when*

(i) no boy gets more than one prize ? (ii) a boy may get any number of prizes ? (iii) no boy gets all the prizes ?

SOLUTION (i) The first prize can be given away in 4 ways as it may be given to any one of the 4 boys. The second prize can be given away in 3 ways, because the boy who got the first prize cannot receive the second prize. The third prize can be given away to anyone of the remaining 2 boys in 2 ways. So, the number of ways in which all the prizes can be given away $= 4 \times 3 \times 2 = 24$.

ALITER The total number of ways is the number of arrangements of 4 taken 3 at a time. So, the requisite number of ways $= {}^4P_3 = 4! = 24$.

(ii) The first prize can be given away in 4 ways as it may be given to anyone of the 4 boys. The second prize can also be given away in 4 ways, since it may be obtained by the boy who has already received a prize. Similarly, third prize can be given away in 4 ways.

Hence, the number of ways in which all the prizes can be given away $= 4 \times 4 \times 4 = 4^3 = 64$.

(iii) Since any one of the 4 boys may get all the prizes. So, the number of ways in which a boy gets all the 3 prizes is 4.

So, the number of ways in which a boy does not get all the prizes $= 64 - 4 = 60$.

EXAMPLE 37 *Find the total number of ways in which n distinct objects can be put into two different boxes.*

SOLUTION Let the two boxes be B_1 and B_2. We observe that there are two choices for each of the n objects. Therefore, by fundamental principle of counting

Total number of ways $= \underbrace{2 \times 2 \times \dots \times 2}_{n-\text{times}} = 2^n$

EXAMPLE 38 *Find the total number of ways in which n-distinct objects can be put into two different boxes so that no box remains empty.*

SOLUTION Each object can be put either in box B_1 (say) or in box B_2 (say). So, there are two choices for each of the n objects. Therefore, the number of choices for n distinct objects is

$\underbrace{2 \times 2 \times ... \times 2}_{n-times} = 2^n$. Two of these choices correspond to either the first or the second box being empty. Thus, there are $2^n - 2$ ways in which neither box is empty.

EXAMPLE 39 *By using the digits 0, 1, 2, 3, 4 and 5 (repetitions not allowed) numbers are formed by using any number of digits. Find the total number of non-zero numbers that can be formed.*

SOLUTION Required number of numbers

= Number of 1 digit number + No. of 2 digit numbers + ... + Number of 6 digit numbers

$= 5 + 5 \times 5 + 5 \times 5 \times 4 + 5 \times 5 \times 4 \times 3 + 5 \times 5 \times 4 \times 3 \times 2 + 5 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 5 + 25 + 100 + 300 + 600 + 600 = 1630.$

EXERCISE 15.2

BASIC

1. In a class there are 27 boys and 14 girls. The teacher wants to select 1 boy and 1 girl to represent the class in a function. In how many ways can the teacher make this selection?
2. A person wants to buy one fountain pen, one ball pen and one pencil from a stationery shop. If there are 10 fountain pen varieties, 12 ball pen varieties and 5 pencil varieties, in how many ways can he select these articles?
3. From Goa to Bombay there are two routes; air, and sea. From Bombay to Delhi there are three routes; air, rail and road. From Goa to Delhi via Bombay, how many kinds of routes are there?
4. A mint prepares metallic calenders specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of calendars should it prepare to serve for all the possibilities in future years?
5. There are four parcels and five post-offices. In how many different ways can the parcels be sent by registered post?
6. A coin is tossed five times and outcomes are recorded. How many possible outcomes are there?
7. In how many ways can an examinee answer a set of ten true/false type questions?
8. A letter lock consists of three rings each marked with 10 different letters. In how many ways it is possible to make an unsuccessful attempt to open the lock?
9. There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 2 each?
10. There are 5 books on Mathematics and 6 books on Physics in a book shop. In how many ways can a student buy : (i) a Mathematics book and a Physics book (ii) either a Mathematics book or a Physics book?
11. Given 7 flags of different colours, how many different signals can be generated if a signal requires the use of two flags, one below the other? [NCERT]
12. A team consists of 6 boys and 4 girls and other has 5 boys and 3 girls. How many single matches can be arranged between the two teams when a boy plays against a boy and a girl plays against a girl?
13. Twelve students compete in a race. In how many ways first three prizes be given?
14. How many A.P.'s with 10 terms are there whose first term is in the set {1, 2, 3} and whose common difference is in the set {1, 2, 3, 4, 5}?

15. From among the 36 teachers in a college, one principal, one vice-principal and the teacher-incharge are to be appointed. In how many ways can this be done?
16. How many three-digit numbers are there with no digit repeated?
17. How many three-digit numbers are there?
18. How many three-digit odd numbers are there?
19. How many different five-digit number licence plates can be made if
    (i) first digit cannot be zero and the repetition of digits is not allowed,
    (ii) the first-digit cannot be zero, but the repetition of digits is allowed?
20. How many four-digit numbers can be formed with the digits 3, 5, 7, 8, 9 which are greater than 7000, if repetition of digits is not allowed?
21. How many four-digit numbers can be formed with the digits 3, 5, 7, 8, 9 which are greater than 8000, if repetition of digits is not allowed?
22. In how many ways can six persons be seated in a row?
23. How many 9-digit numbers of different digits can be formed?
24. How many odd numbers less than 1000 can be formed by using the digits 0, 3, 5, 7 when repetition of digits is not allowed?
25. How many 3-digit numbers are there, with distinct digits, with each digit odd?

BASED ON LOTS

26. How many different numbers of six digits each can be formed from the digits 4, 5, 6, 7, 8, 9 when repetition of digits is not allowed?
27. How many different numbers of six digits can be formed from the digits 3, 1, 7, 0, 9, 5 when repetition of digits is not allowed?
28. How many four digit different numbers, greater than 5000 can be formed with the digits 1, 2, 5, 9, 0 when repetition of digits is not allowed?
29. Serial numbers for an item produced in a factory are to be made using two letters followed by four digits (0 to 9). If the letters are to be taken from six letters of English alphabet without repetition and the digits are also not repeated in a serial number, how many serial numbers are possible?
30. A number lock on a suitcase has 3 wheels each labelled with ten digits 0 to 9. If opening of the lock is a particular sequence of three digits with no repeats, how many such sequences will be possible? Also, find the number of unsuccessful attempts to open the lock.
31. A customer forgets a four-digit code for an Automatic Teller Machine (ATM) in a bank. However, he remembers that this code consists of digits 3, 5, 6 and 9. Find the largest possible number of trials necessary to obtain the correct code.
32. In how many ways can three jobs I, II and III be assigned to three persons A, B and C if one person is assigned only one job and all are capable of doing each job?
33. How many four digit natural numbers not exceeding 4321 can be formed with the digits 1, 2, 3 and 4, if the digits can repeat?
34. How many numbers of six digits can be formed from the digits 0, 1, 3, 5, 7 and 9 when no digit is repeated? How many of them are divisible by 10?
35. If three six faced die each marked with numbers 1 to 6 on six faces, are thrown find the total number of possible outcomes.

36. A coin is tossed three times and the outcomes are recorded. How many possible outcomes are there ? How many possible outcomes if the coin is tossed four times? Five times? n times?
37. How many numbers of four digits can be formed with the digits 1, 2, 3, 4, 5 if the digits can be repeated in the same number?
38. How many three digit numbers can be formed by using the digits 0, 1, 3, 5, 7 while each digit may be repeated any number of times?
39. How many natural numbers less than 1000 can be formed from the digits 0, 1, 2, 3, 4, 5 when a digit may be repeated any number of times?
40. How many five digit telephone numbers can be constructed using the digits 0 to 9. If each number starts with 67 and no digit appears more than once? [NCERT]

BASED ON HOTS

41. Find the number of ways in which 8 distinct toys can be distributed among 5 children.
42. Find the number of ways in which one can post 5 letters in 7 letter boxes.
43. Three dice are rolled. Find the number of possible outcomes in which at least one die shows 5.
44. Find the total number of ways in which 20 balls can be put into 5 boxes so that first box contains just one ball.
45. In how many ways can 5 different balls be distributed among three boxes?
46. In how many ways can 7 letters be posted in 4 letter boxes?
47. In how many ways can 4 prizes be distributed among 5 students, when
    (i) no student gets more than one prize?
    (ii) a student may get any number of prizes?
    (iii) no student gets all the prizes?
48. There are 10 lamps in a hall. Each one of them can be switched on independently. Find the number of ways in which the hall can be illuminated. [NCERT]

ANSWERS

1. 378 2. 600 3. 6 4. 14 5. 625 6. 32 7. 1024 8. 999
9. 512 10. (i) 30 (ii) 11 11. 42 12. 42 13. 1320 14. 15 15. 42840
16. 648 17. 900 18. 450 19. (i) 27216 (ii) 90000 20. 72 21. 48
22. 720 23. 9(9!) 24. 21 25. 60 26. 720 27. 600 28. 48 29. 151200
30. 720, 719 31. 24 32. 6 33. 229 34. 600, 120 35. 216
36. 8, 16, 2^n 37. 625 38. 100 39. 215 40. 336 41. 5^8 42. 7^5
43. 91 44. 20×4^{19} 45. 243 46. 4^7 47. (i) 5! (ii) 625 (iii) 620
48. $2^{10} - 1$.

HINTS TO SELECTED PROBLEMS

1. No. of ways $= 27 \times 14$.
2. Required number of ways $= 10 \times 12 \times 5 = 600$.
3. No of routes $= 2 \times 3 = 6$.
4. Total number of calendars $= 7 \times 2 = 14$.
5. Since a parcel can be sent to any one of the five post offices. So, required number of ways $= 5 \times 5 \times 5 \times 5 = 5^4$.

6. Since toss of each coin can result in 2 ways. So, required no. of ways $= 2 \times 2 \times 2 \times 2 \times 2 = 2^5$.
8. Required no. of ways $= 10 \times 10 \times 10 - 1$.
9. Each one of the first three questions can be answered in 4 ways and each one of the next three questions can be answered in 2 ways. So, total no. of sequences of answers $= 4 \times 4 \times 4 \times 2 \times 2 \times 2$.
11. Required no. of signals $= 7 \times 6$.
12. A boy can be selected from the first team in 6 ways, and from the second in 5 ways. So, no. of single matches between the boys of two teams $= 6 \times 5 = 30$. Similarly, the no. of single matches between the girls of two teams $= 4 \times 3 = 12$. So, total number of matches $= 30 + 12 = 42$.
13. Required no. of ways $= 12 \times 11 \times 10$.
14. There are 3 ways to choose the first term and corresponding to each such way there are 5 ways of selecting the common difference. So, required no. of A.P.'s $= 3 \times 5$.
15. Required no. of ways $= 36 \times 35 \times 34$.
16. The total no. of required numbers $= 9 \times 9 \times 8$.
17. The total no. of required numbers $= 9 \times 10 \times 10$.
18. The total no. of required number $= 9 \times 10 \times 5$.
19. (i) Required no. of licence plates $= 9 \times 9 \times 8 \times 7 \times 6$
 (ii) Required no. of licence plates $= 9 \times 10 \times 10 \times 10 \times 10$.
20. Required no. of numbers $= 3 \times 4 \times 3 \times 2$.
21. Required no. of numbers $= 2 \times 4 \times 3 \times 2$.
22. Required no. of ways $= 6 \times 5 \times 4 \times 3 \times 2 \times 1$.
23. Required no. of numbers $= 9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2$.
24. An odd number less than 1000 may be a one-digit number, two-digit number or a three-digit number. So, required no. of numbers is

 $$3\,(\text{one-digit nos.}) + 2 \times 3\,(\text{two-digit nos.}) + 2 \times 2 \times 3\,(3-\text{digit nos.}).$$
25. Required no. of numbers $= 5 \times 4 \times 3$.
26. Required no. of numbers $= 6 \times 5 \times 4 \times 3 \times 2 \times 1$.
27. Required no. of numbers $= 5 \times 5 \times 4 \times 3 \times 2 \times 1$.
28. Required no. of numbers $= 2 \times 4 \times 3 \times 2$.
29. Here we have to perform 6 jobs. So, required number of serial numbers i

 $$6 \times 5 \times 10 \times 9 \times 8 \times 7$$
30. Required number of sequences $= 10 \times 9 \times 8$.

 Also, total number of unsuccessful attempts $= 10 \times 9 \times 8 - 1$
31. Number of trials $= 4 \times 3 \times 2 \times 1$
32. Required number of ways $= 3 \times 2 \times 1$
36. Since a toss of a coin can result in a head or a tail. Therefore, if a coin is tossed n-times, then the total number of outcomes is $\underbrace{2 \times 2 \times 2 \times \ldots \times 2}_{n-times} = 2^n$
41. Each toy can be distributed in 5 ways.

 So, total number of ways $= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^8$
42. Each letter can be posted in any one of the 7 letter boxes.

 So, required number of ways $= 7 \times 7 \times 7 \times 7 \times 7 = 7^5$

43. Required number of possible outcomes

= Total number of possible outcomes – Number of possible out- comes in which 5 does not appear on any dice.

$= 6^3 - 5^3 = 216 - 125 = 91.$

44. One ball can be put in first box in 20 ways because we can put any one of the twenty balls in the first box. Now, remaining 19 balls are to be put into remaining 4 boxes. This can be done in 4^{19} ways, because there are 4 choices for each ball. Hence, the required number of ways $= 20 \times 4^{19}$.

15.3 PERMUTATIONS

Each of the arrangements which can be made by taking some or all of a number of things is called a permutation.

For example, if there are three objects, then the permutations of these objects, taking two at a time, are

$$ab, \quad ba, \quad bc, \quad cb, \quad ac, \quad ca$$

So, the number of permutations of three different things taken two at a time is 6.

NOTE It should be noted that in permutations the order of arrangement is taken into account; when the order is changed, a different permutation is obtained.

ILLUSTRATION 1 *Write down all the permutations of the set of three letters A, B, C.*

SOLUTION The permutations of three letters A, B, C taking all at a time are :

$$ABC, \quad ACB, \quad BCA, \quad BAC, \quad CBA, \quad CAB.$$

Clearly, there are 6 permutations.

ILLUSTRATION 2 *Write down all the permutations of the vowels A, E, I, O, U in English alphabets taking three at a time, and starting with A.*

SOLUTION The permutations of vowels A, E, I, O, U taking three at a time, and starting with A are:

$$AEI, \ AIE, \ AEO, \ AOE, \ AEU, \ AUE, \ AIO, \ AOI, \ AIU, \ AUI, \ AOU, \ AUO$$

Clearly, there are 12 permutations.

ILLUSTRATION 3 *Write down all the permutations of letters A, B, C, D taking three at a time.*

SOLUTION The desired permutations are :

ABC	ABD	BCD	ACD
ACB	ADB	BDC	ADC
BCA	BDA	CBD	CAD
BAC	BAD	CDB	CDA
CAB	DAB	DCB	DAC
CBA	DBA	DBC	DCA

Clearly, there are 24 permutations. These permutations are obtained by first selecting three letters out of 4 and then arranging them in all possible ways.

A NOTATION *If n and r are positive integers such that $1 \le r \le n$, then the number of all permutations of n distinct things, taken r at a time is denoted by the symbol $P(n, r)$ or ${}^nP_r$.*

Thus,

$${}^nP_r \text{ or, } P(n, r) = \text{Total number of permutations of n distinct things, taken } r \text{ at a time.}$$

In illustration 3, we have seen that there are 24 permutations, on a set of 4 letters, taken 3 at a time. Therefore, as per our notation, we have ${}^4P_3 = 24$ or, $P(4, 3) = 24$.

THEOREM 1 *Let r and n be positive integers such that $1 \le r \le n$. Then the number of all permutations of n distinct things taken r at a time is given by $n(n-1)(n-2)(n-3)\dots(n-(r-1))$.*

i.e. $P(n, r) = {}^nP_r = n(n-1)(n-2)\dots(n-(r-1))$.

PROOF The number of permutations of n distinct things, taken r at a time, is same as the number of ways in which we can fill up r-places when we have n different things at our disposal.

The first place can be filled in n ways, for any one of the n things can be used to fill up the first place. Having filled it, there are $(n-1)$ things left and any one of these $(n-1)$ things can be used to fill up the second place. So, the second place can be filled in $(n-1)$ ways. Hence, by the fundamental principle of counting, the first two places can be filled in n $(n-1)$ ways. When the first two places are filled, there are $(n-2)$ places left, so that the third place can be filled from the remaining $(n-2)$ things in $(n-2)$ ways. Therefore, the first three places can be filled in $n(n-1)(n-2)$ ways. Continuing in this manner, we find that the first $(r-1)$ places can be filled in $n(n-1)(n-2)\dots(n-(r-2))$ ways. After filling up first $(r-1)$ places, exactly $n-(r-1) = n-r+1$ things are left. So, the rth place can be filled in $(n-(r-1))$ ways. Hence, the r places can be filled in $n(n-1)(n-2)\dots(n-(r-1))$ ways.

Hence, the total number of permutations of n distinct things, taken r at a time is

$$n(n-1)(n-2)(n-3)\dots(n-(r-1)).$$

Thus, $P(n, r) = n(n-1)(n-2)(n-3)\dots(n-(r-1))$.

THEOREM 2 *Prove that:* $P(n,r) = {}^nP_r = \dfrac{n!}{(n-r)!}$.

PROOF We have,

$$P(n, r) = n(n-1)(n-2)(n-3)\dots(n-(r-1))$$

$$\Rightarrow \quad P(n, r) = \frac{n(n-1)(n-2)(n-3)\dots(n-(r-1))(n-r)(n-(r+1))\dots 3.2.1}{(n-r)(n-(r+1))\dots 3.2.1}$$

$$\Rightarrow \quad P(n, r) = \frac{n!}{(n-r)!}$$

THEOREM 3 *The number of all permutations of n distinct things, taken all at a time is n!.*

PROOF The number of all permutations of n distinct things, taken all at a time is same as the number of ways of filling n places when we have n distinct things at our disposal.

Proceeding as in theorem 1, we have

$$P(n, n) = n(n-1)(n-2)(n-3)\dots(n-(n-1)) = n(n-1)(n-2)(n-3)\dots 3.2.1 = n!$$

THEOREM 4 *Prove that* $0! = 1$.

PROOF We have,

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$\Rightarrow \quad P(n, n) = \frac{n!}{0!} \qquad \text{[Putting } r = n]$$

$$\Rightarrow \quad n! = \frac{n!}{0!} \qquad [\because P(n, n) = n! \text{ (See Theorem 3)}]$$

$$\Rightarrow \quad 0! = \frac{n!}{n!} = 1.$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I PROBLEMS BASED UPON THE VALUE OF ${}^nP_r$ OR $P(n, r)$

EXAMPLE 1 *Evaluate the following:*

(i) ${}^5P_3$ (ii) $P(15, 3)$ (iii) $P(5, 5)$

SOLUTION (i) ${}^5P_3 = \dfrac{5!}{(5-3)!}$ $\left[\because \ {}^nP_r = \dfrac{n!}{(n-r)!}\right]$

$\Rightarrow \quad {}^5P_3 = \dfrac{5!}{2!} = \dfrac{5 \times 4 \times 3 \times 2!}{2!} = 60$

(ii) $P(15, 3) = \dfrac{15!}{(15-3)!} = \dfrac{15!}{12!} = \dfrac{15 \times 14 \times 13 \times 12!}{12!} = 2730$

(iii) $P(5, 5) = \dfrac{5!}{(5-5)!} = \dfrac{5!}{0!} = 5! = 120.$

Type II ON FINDING THE VALUE OF REQUIRED UN-KNOWN WHEN A RELATION CONNECTING $P(n, r)$ IS GIVEN

EXAMPLE 2 *If* $2.\,P(5, 3) = P(n, 4)$, *find* n.

SOLUTION We have,

$2.\,P(5, 3) = P(n, 4)$

$\Rightarrow \quad P(n, 4) = 2\,.\,P(5, 3)$

$\Rightarrow \quad \dfrac{n!}{(n-4)!} = 2\left(\dfrac{5!}{(5-3)!}\right)$

$\Rightarrow \quad \dfrac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!} = \dfrac{2(5!)}{2!}$

$\Rightarrow \quad n(n-1)(n-2)(n-3) = 5!$

$\Rightarrow \quad n(n-1)(n-2)(n-3) = 5 \times 4 \times 3 \times 2 \times 1$

$\Rightarrow \quad n(n-1)(n-2)(n-3) = 5(5-1)(5-2)(5-3)$

$\Rightarrow \quad n = 5$ [By comparing two sides]

EXAMPLE 3 *If* $P(n, 4) = 20 \times P(n, 2)$, *find* n.

SOLUTION We have,

$P(n, 4) = 20 \times P(n, 2) \Rightarrow \dfrac{n!}{(n-4)!} = 20 \times \dfrac{n!}{(n-2)!} \Rightarrow (n-2)! = 20 \times (n-4)!$

$\Rightarrow \quad (n-2)(n-3)(n-4)! = 20 \times (n-4)!$

$\Rightarrow \quad (n-2)(n-3) = 20$

$\Rightarrow \quad (n-2)(n-3) = 5 \times 4$

$\Rightarrow \quad n-3 = 4 \Rightarrow n = 7$ [By comparing two sides]

EXAMPLE 4 *If* $P(5, r) = 2.\,P(6, r-1)$, *find* r. [NCERT]

SOLUTION We have,

$P(5, r) = 2\,.\,P(6, r-1)$

$\Rightarrow \quad \dfrac{5!}{(5-r)!} = 2.\,\dfrac{6!}{(6-(r-1))!} \Rightarrow \dfrac{5!}{(5-r)!} = \dfrac{2 \times 6 \times 5!}{(7-r)!} \Rightarrow \dfrac{5!}{(5-r)!} = \dfrac{12 \times 5!}{(7-r)(6-r)(5-r)!}$

$$\Rightarrow \quad 1 = \frac{12}{(7-r)(6-r)}$$

$$\Rightarrow \quad (7-r)(6-r) = 12$$

$$\Rightarrow \quad (7-2)(6-r) = 4 \times 3$$

$$\Rightarrow \quad 7-r = 4 \Rightarrow r = 3 \qquad \text{[By comparing]}$$

EXAMPLE 5 *If* ${}^{10}P_r = 5040$, *find the value of* r.

SOLUTION We have,

$${}^{10}P_r = 5040$$

$$\Rightarrow \quad \frac{10!}{(10-r)!} = 10 \times 504 \Rightarrow \frac{10!}{(10-r)!} = 10 \times 9 \times 8 \times 7 \Rightarrow \frac{10!}{(10-r)!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{6!}$$

$$\Rightarrow \quad \frac{10!}{(10-r)!} = \frac{10!}{6!} \Rightarrow (10-r)! = 6! \Rightarrow 10-r = 6 \Rightarrow r = 4.$$

EXAMPLE 6 *If* $P(n-1, 3) : P(n, 4) = 1 : 9$, find n.

SOLUTION We have,

$$P(n-1, 3) : P(n, 4) = 1 : 9$$

$$\Rightarrow \quad \frac{P(n-1, 3)}{P(n, 4)} = \frac{1}{9}$$

$$\Rightarrow \quad \frac{\dfrac{(n-1)!}{(n-1-3)!}}{\dfrac{n!}{(n-4)!}} = \frac{1}{9} \Rightarrow \frac{(n-1)!}{(n-4)!} \times \frac{(n-4)!}{n!} = \frac{1}{9} \Rightarrow \frac{(n-1)!}{n!} = \frac{1}{9} \Rightarrow \frac{(n-1)!}{n \cdot (n-1)!} = \frac{1}{9} \Rightarrow n = 9$$

EXAMPLE 7 *If* ${}^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_r$, *find* r.

SOLUTION We have,

$${}^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_r$$

$$\Rightarrow \quad \frac{9!}{(9-5)!} + 5 \cdot \frac{9!}{(9-4)!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{9!}{4!} + 5 \cdot \frac{9!}{5!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{9!}{4!} + \frac{9!}{4!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad 2 \times \frac{9!}{4!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{5 \times 2 \times 9!}{5 \times 4!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{10 \times 9!}{5!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{10!}{5!} = \frac{10!}{(10-r)!} \Rightarrow (10-r)! = 5! \Rightarrow 10-r = 5 \Rightarrow r = 5$$

EXAMPLE 8 *If* ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30\,800 : 1$, *find* r.

SOLUTION We have,

$${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30\,800 : 1$$

$$\Rightarrow \quad \frac{56!}{(56-r-6)!} : \frac{54!}{(54-r-3)!} = \frac{30800}{1}$$

$$\Rightarrow \quad \frac{56!}{(50-r)!} : \frac{54!}{(51-r)!} = 30800:1$$

$$\Rightarrow \quad \frac{56!}{(50-r)!} \times \frac{(51-r)!}{54!} = \frac{30800}{1}$$

$$\Rightarrow \quad \frac{56 \times 55 \times 54!}{(50-r)!} \times \frac{(51-r) \times (50-r)!}{54!} = \frac{30800}{1}$$

$$\Rightarrow \quad 56 \times 55 \times (51-r) = 30800 \Rightarrow (51-r) = 10 \Rightarrow r = 41.$$

EXAMPLE 9 *If* ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$, *find* n.

SOLUTION We have,

$${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$$

$$\Rightarrow \quad \frac{{}^{2n+1}P_{n-1}}{{}^{2n-1}P_n} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{(2n+1)(2n)(2n-1)!}{(n+2)(n+1)\,n\,(n-1)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{2(2n+1)}{(n+2)(n+1)} = \frac{3}{5} \Rightarrow 10(2n+1) = 3(n+2)(n+1) \Rightarrow 3n^2 + 9n + 6 = 20n + 10$$

$$\Rightarrow \quad 3n^2 - 11n - 4 = 0 \Rightarrow (n-4)(3n+1) = 0 \Rightarrow n = 4 \qquad [\because n \neq -1/3]$$

EXAMPLE 10 *If* ${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$, *find* r.

SOLUTION We have,

$${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$$

$$\Rightarrow \quad \frac{22!}{(21-r)!} : \frac{20!}{(18-r)!} = 11:52$$

$$\Rightarrow \quad \frac{22!}{(21-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22 \times 21 \times 20!}{(21-r)(20-r)(19-r)\cdot(18-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22 \times 21}{(21-r)(20-r)(19-r)} = \frac{11}{52}$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = 2 \times 21 \times 52$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = 2 \times 3 \times 7 \times 4 \times 13$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = 12 \times 13 \times 14$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = (21-7)(20-7)(19-7) \Rightarrow r = 7$$

Type III ON PROVING RESULTS RELATED TO $P(n, r)$ or ${}^nP_r$

EXAMPLE 11 *Prove the following:*

(i) $P(n, n) = 2\,P(n, n-2)$ (ii) $P(n, n) = P(n, n-1)$

(iii) $P(n, r) = P(n-1, r) + r \,.\, P(n-1, r-1)$ (iv) $P(n, r) = n \cdot P(n-1, r-1)$

SOLUTION (i) $2P(n, n-2) = 2\dfrac{n!}{(n-(n-2))!} = 2\left(\dfrac{n!}{2!}\right) = n! = P(n, n)$

(ii) $P(n, n-1) = \dfrac{n!}{(n-(n-1))!} = \dfrac{n!}{1!} = n! = P(n, n)$

(iii) $P(n-1, r) + r \cdot P(n-1, r-1) = \dfrac{(n-1)!}{(n-1-r)!} + r \cdot \dfrac{(n-1)!}{((n-1)-(r-1))!}$

$$= \frac{(n-1)!}{(n-r-1)!} + r\frac{(n-1)!}{(n-r)!} = \frac{(n-1)!}{(n-r-1)!} + r\frac{(n-1)!}{(n-r)(n-r-1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left\{1 + \frac{r}{n-r}\right\} = \frac{(n-1)!}{(n-r-1)!}\left(\frac{n-r+r}{n-r}\right)$$

$$= \frac{(n-1)!}{(n-r-1)!} \cdot \frac{n}{n-r} = \frac{n!}{(n-r)!} = P(n, r)$$

(iv) $n \cdot P(n-1, r-1) = n\dfrac{(n-1)!}{((n-1)-(r-1))!} = \dfrac{n!}{(n-r)!} = P(n, r)$

Type III PRACTICAL PROBLEMS ON PERMUTATIONS

NOTE ALITER 2 of each of the following examples should be done after studying permutations and combinations.

EXAMPLE 12 *In how many ways three different rings can be worn in four fingers with at most one in each finger?*

SOLUTION The total number of ways is same as the number of arrangements of 4 fingers, taken 3 at a time.

So, required number of ways $= {}^4P_3 = \dfrac{4!}{(4-3)!} = \dfrac{4!}{1!} = 4! = 24.$

ALITER 1 Let R_1, R_2, R_3 be three rings. Since R_1 can be put in any one of the four fingers. So, there are four ways in which R_1 can be worn. Now, R_2 can be worn in any one of the remaining three fingers in 3 ways. In the remaining 2 fingers ring R_3 can be worn in 2 ways. So, by the fundamental principle of counting the total number of ways in which three different rings can be worn in four fingers is $4 \times 3 \times 2 = 24$.

ALITER 2 Out of 4 fingers, 3 fingers can be chosen in ${}^4C_3$ ways. Now, three rings can be worn in the selected three fingers in 3! ways. Hence, three rings can be worn in four fingers in ${}^4C_3 \times 3! = 24$ ways.

EXAMPLE 13 *Seven athletes are participating in a race. In how many ways can the first three prizes be won?*

SOLUTION The total number of ways in which first three prizes can be won is the number of arrangements of seven different things taken 3 at a time.

So, required number of ways $= {}^7P_3 = \dfrac{7!}{(7-3)!} = \dfrac{7!}{4!} = \dfrac{7 \times 6 \times 5 \times 4!}{4!} = 210.$

ALITER 1 First prize can be won in seven ways. Second prize can be won by any one of the remaining six athletes in 6 ways. Now, five athletes are left. So, third prize can be won by any one of the remaining 5 athletes in 5 ways.

Hence, by the fundamental principle of counting, the required number of ways $= 7 \times 6 \times 5 = 210$.

ALITER 2 Out of 7 athletes, 3 can be chosen for prize in ${}^7C_3$ ways. Now, three prizes can be given to three chosen athletes in 3! ways.

$\therefore$ Numbers of ways in which 3 prizes can be won $= {}^7C_3 \times 3! = 210$

EXAMPLE 14 *How many different signals can be made by 5 flags from 8 flags of different colours?*

SOLUTION The total number of signals is the number of arrangements of 8 flags by taking 5 flags at a time.

Hence, required number of signals $= {}^8P_5 = \dfrac{8!}{(8-5)!} = \dfrac{8!}{3!} = \dfrac{8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!} = 6720$

EXAMPLE 15 *In how many ways can 6 persons stand in a queue?*

SOLUTION The number of ways in which 6 persons can stand in a queue is same as the number of arrangements of 6 different things taken all at a time.

Hence, the required number of ways $= {}^6P_6 = 6! = 720$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 16 *It is required to seat 8 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?*

SOLUTION In all 12 persons are to be seated in a row and in the row of 12 positions there are exactly 6 even places viz second, fourth, sixth, eighth, tenth and twelfth. It is given that four women are to occupy 4 places out of these six even places. This can be done in ${}^6P_4$ ways (ways of arranging 6 women in 4 positions). The remaining 8 positions can be filled by the 8 men in ${}^8P_8$ ways. So, by the fundamental principle of counting, the number of seating arrangements as required, is ${}^6P_4 \times {}^8P_8 = 360 \times 40320 = 14515200$.

ALITER 1 In all 12 persons are to be seated in a row and in the row of 12 positions there are exactly 6 even places viz. 2nd, 4th, 6th, 8th and 12th. It is given that 4 women are to occupy any 4 places out of these six positions. This can be done in ${}^6C_4 \times 4!$ ways. The remaining 8 positions are to be occupied by 8 men. This can be done in ${}^8C_8 \times 8!$ ways.

Hence, total number of seating arrangements $= ({}^6C_4 \times 4!) \times ({}^8C_8 \times 8!)$

$$= 360 \times 40320 = 14515200.$$

EXAMPLE 17 *Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them ?*

SOLUTION The total number of ways in which three men can wear 4 coats is the number of arrangements of 4 different coats taken 3 at a time. So, three men can wear 4 coats in ${}^4P_3$ ways. Similarly, 5 waist coats and 6 caps can be worn by three men in ${}^5P_3$ and ${}^6P_3$ ways respectively. Hence, by the fundamental principle of counting, the required number of ways as desired

$$= {}^4P_3 \times {}^5P_3 \times {}^6P_3 = (4!) \times (5 \times 4 \times 3) \times (6 \times 5 \times 4) = 172800$$

EXAMPLE 18 *How many different signals can be given using any number of flags from 5 flags of different colours ?*

SOLUTION The signals can be made by using at a time one or two or three or four or five flags. The total number of signals when r flags are used at a time from 5 flags is equal to the number of arrangements of 5, taking r at a time i.e. ${}^5P_r$. Since r can take values 1, 2, 3, 4, 5. Hence, by the fundamental principle of addition, the total number of signals

$$= {}^5P_1 + {}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5$$

$$= 5 + 5 \times 4 + 5 \times 4 \times 3 + 5 \times 4 \times 3 \times 2 + 5 \times 4 \times 3 \times 2 \times 1 = 5 + 20 + 60 + 120 + 120 = 325$$

EXAMPLE 19 *How many numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed ?*

SOLUTION Every number lying between 100 and 1000 is a three digit number. Therefore, we have to find the number of permutations of five digits 1, 2, 3, 4, 5 taken three at a time.

Hence, the required number of numbers $= {}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \times 4 \times 3 = 60$

EXAMPLE 20 *How many four digit numbers are there with distinct digits ?*

SOLUTION The total number of arrangements of ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 taking 4 at a time is ${}^{10}P_4$. But, these arrangements also include those numbers which have 0 at thousand's place. Such numbers are not four digit numbers. When 0 is fixed at thousand's place, we have to arrange remaining 9 digits by taking 3 at a time. The number of such arrangements is ${}^9P_3$.

So, the total number of numbers having 0 at thousand's place $= {}^9P_3$.

Hence, the total number of four digit numbers $= {}^{10}P_4 - {}^9P_3 = 5040 - 504 = 4536$.

EXAMPLE 21 *In how many ways 7 pictures can be hung from 5 picture nails on a wall ?*

SOLUTION The number of ways in which 7 pictures can be hung from 5 picture nails on a wall is same as the number of arrangements of 7 things, taking 5 at a time.

Hence, the required number $= {}^7P_5 = \frac{7!}{(7-5)!} = \frac{7!}{2!} = 2520$.

EXAMPLE 22 *Determine the number of natural numbers smaller than 10^4, in the decimal notation of which all the digits are distinct.*

SOLUTION The required natural numbers consist of 4 digits, 3 digits, 2 digits and one digit.

Total number of 4 digit natural numbers with distinct digits $= {}^{10}P_4 - {}^9P_3$

Total number of 3 digit natural numbers with distinct digits $= {}^{10}P_3 - {}^9P_2$

Total number of 2 digit natural numbers with distinct digits $= {}^{10}P_2 - {}^9P_1$

Total number of one digit natural numbers = 9

Hence, the required number of natural numbers $= ({}^{10}P_4 - {}^9P_3) + ({}^{10}P_3 - {}^9P_2) + ({}^{10}P_2 - {}^9P_1) + 9$

$$= 9 \times 9 \times 8 \times 7 + 9 \times 9 \times 8 + 9 \times 9 + 9 = 5274.$$

EXAMPLE 23 *How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once.*

SOLUTION There are eight letters in the word 'EQUATION'. So, the total number of words is equal to the number of arrangements of these letters, taken all at a time. The number of such arrangements is ${}^8P_8 = 8!$ Hence, the total number of words = 8!

EXAMPLE 24 *How many 4-letter words, with or without meaning, can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed?*

SOLUTION There are 10 letters in the word 'LOGARITHMS'.

So, the number of 4 - letter word = Number of arrangements of 10 letters, taken 4 at a time

$= {}^{10}P_4 = 5040$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 25 *Prove that if $r \le s \le n$, then $P(n, s)$ is divisible by $P(n, r)$.*

SOLUTION Let $s = r + k$ where $0 \le k \le s - r$. Then,

$$P(n, s) = \frac{n!}{(n-s)!} = n(n-1)(n-2)\ldots(n-(s-1))$$

$$\Rightarrow \quad P(n, s) = n(n-1)(n-2)\ldots\{n-(r+k-1)\}$$

$\Rightarrow \quad P(n, s) = n(n-1)(n-2)\ldots\{n-(r-1)\}(n-r)\{n-(r+1)\}\ldots\{n-(r+k-1)\}$

$\Rightarrow \quad P(n, s) = \{n(n-1)(n-2)\ldots n-(r-1)\}\{(n-r)(n-(r+1))\ldots(n-(r+k-1))\}$

$\Rightarrow \quad P(n, s) = P(n, r)\cdot\{(n-r)(n-(r+1))\ldots(n-(r+k-1))\}$

$$\left[\because\ P(n, r) = \frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-(r-1))\right]$$

$\Rightarrow \quad P(n, s) = P(n, r)\cdot\{(n-r)(n-(r+1))\ldots(n-(r+k-1))\}$

$\Rightarrow \quad P(n, s)$ is divisible by $P(n, r)$.

EXAMPLE 26 *If P_m stands for ${}^mP_m$, then prove that:* $1 + 1\cdot P_1 + 2\cdot P_2 + 3\cdot P_3 + \ldots + n\cdot P_n = (n+1)!$

SOLUTION We have, $P_m = {}^mP_m = m!$

$\therefore \quad 1 + 1\cdot P_1 + 2\cdot P_2 + 3\cdot P_3 + \ldots + n\cdot P_n$

$= 1 + 1 + 2\cdot 2! + 3\cdot 3! + 4\cdot 4! + \ldots + n\cdot n!$

$= 1 + \sum\limits_{r=1}^{n} r\cdot r! = 1 + \sum\limits_{r=1}^{n}\{(r+1)-1\}\,r! = 1 + \sum\limits_{r=1}^{n}[(r+1)\,r! - r!] = 1 + \sum\limits_{r=1}^{n}[(r+1)! - r!]$

$= 1 + [(2!-1!) + (3!-2!) + (4!-3!) + \ldots + ((n+1)!-n!)] = 1 + ((n+1)! - 1!) = (n+1)!$

EXAMPLE 27 *In an examination hall there are four rows of chairs. Each row has 8 chairs one behind the other. There are two classes sitting for the examination with 16 students in each class. It is desired that in each row, all students belong to the same class and that no two adjacent rows are allotted to the same class. In how many ways can these 32 students be seated?*

SOLUTION Let the two classes be C_1 and C_2 and the four rows be R_1, R_2, R_3, R_4. There are 16 students in each class. So, there are 32 students. According to the given conditions there are two different ways in which 32 students can be seated:

	R_1	R_2	R_3	R_4
I	C_1	C_2	C_1	C_2
II	C_2	C_1	C_2	C_1

Since the seating arrangement can be completed by using any one of these two ways. So, by the fundamental principle of addition,

Total number of seating arrangements = No. of arrangement in I case + No. of arrangements in II case.

In case I, 16 students of class C_1 can be seated in R_1 and R_3 in ${}^{16}P_8 \times 8! = 16!$ ways. And 16 students of class C_2 can be seated in R_2 and R_4 in ${}^{16}P_8 \times 8! = 16!$ ways

$\therefore$ Number of seating arrangements in case $I = 16! \times 16!$

Similarly, Number of seating arrangements in case $II = 16! \times 16!$

Hence, Total number of seating arrangements = $(16! \times 16!) + (16! \times 16!) = 2(16! \times 16!)$

EXAMPLE 28 *Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. Determine the number of words which have at least one letter repeated.*

SOLUTION The number of 5-letter words which can be formed from 10 letters when one or more of its letters is repeated $= 10 \times 10 \times 10 \times 10 \times 10 = 10^5$.

The number of 5-letter words which can be formed when none of their letters is repeated

= Number of arrangements of 10 letters by taking 5 at a time $= {}^{10}P_5 = 30240$

Hence, the number of 5-letter words which have at least one of their letters repeated is $10^5 - 30240 = 69760$.

EXAMPLE 29 *Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time.*

SOLUTION The total number of numbers formed with the digits 2, 3, 4, 5 taken all at a time

$= $ Number of arrangement of 4 digits, taken all at a time $= {}^4P_4 = 4! = 24$.

To find the sum of these 24 numbers, we will find the sum of digits at unit's, ten's, hundred's and thousand's places in all these numbers.

Consider the digits in the unit's places in all these numbers. Each of the digits 2, 3, 4, 5 occurs in $3! (= 6)$ times in the unit's place.

So, total for the digits in the unit's place in all the numbers $= (2+3+4+5) \times 3! = 84$.

Since each of the digits 2, 3, 4, 5 occurs $3!$ times in any one of the remaining places.

So, the sum of the digits in the ten's, hundred's and thousand's places in all the numbers

$= (2+3+4+5) \times 3! = 84$.

Hence, the sum of all the numbers $= 84\,(10^0 + 10^1 + 10^2 + 10^3) = 93324$.

EXERCISE 15.3

BASIC

1. Evaluate each of the following:
   (i) ${}^8P_3$ (ii) ${}^{10}P_4$ (iii) ${}^6P_6$ (iv) $P(6, 4)$
2. If $P(5, r) = P(6, r-1)$, find r [NCERT]
3. If $5\,P(4, n) = 6 . P(5, n-1)$, find n.
4. If $P(n, 5) = 20 . P(n, 3)$, find n
5. If ${}^nP_4 = 360$, find the value of n.
6. If $P(9, r) = 3024$, find r.
7. If $P(11, r) = P(12, r-1)$ find r.
8. If $P(n, 4) = 12 . P(n, 2)$, find n.
9. If $P(n-1, 3) : P(n, 4) = 1 : 9$, find n. [NCERT]
10. If $P(15, r-1) : P(16, r-2) = 3:4$, find r.
11. If $P(n, 5) : P(n, 3) = 2 : 1$, find n.
12. If ${}^{n+5}P_{n+1} = \frac{11(n-1)}{2}\,{}^{n+3}P_n$, find n.
13. If $P(2n-1, n) : P(2n+1, n-1) = 22 : 7$ find n.
14. Prove that: $1 . P(1, 1) + 2 . P(2, 2) + 3 . P(3, 3) + \ldots + n . P(n, n) = P(n+1, n+1) - 1$.
15. In how many ways can five children stand in a queue?
16. From among the 36 teachers in a school, one principal and one vice-principal are to be appointed. In how many ways can this be done?
17. Four letters E, K, S and V, one in each, were purchased from a plastic warehouse. How many ordered pairs of letters, to be used as initials, can be formed from them?
18. Four books, one each in Chemistry, Physics, Biology and Mathematics, are to be arranged in a shelf. In how many ways can this be done?
19. Find the number of different 4-letter words, with or without meanings, that can be formed from the letters of the word 'NUMBER'.
20. How many three-digit numbers are there, with distinct digits, with each digit odd?
21. How many words, with or without meaning, can be formed by using all the letters of the word 'DELHI', using each letter exactly once?
22. How many words, with or without meaning, can be formed by using the letters of the word 'TRIANGLE'?

BASED ON LOTS

23. There are two works each of 3 volumes and two works each of 2 volumes; In how many ways can the 10 books be placed on a shelf so that the volumes of the same work are not separated?

24. There are 6 items in column A and 6 items in column B. A student is asked to match each item in column A with an item in column B. How many possible, correct or incorrect, answers are there to this question?

25. How many three-digit numbers are there, with no digit repeated?

26. How many 6-digit telephone numbers can be constructed with digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if each number starts with 35 and no digit appears more than once?

27. In how many ways can 6 boys and 5 girls be arranged for a group photograph if the girls are to sit on chairs in a row and the boys are to stand in a row behind them?

28. If a denotes the number of permutations of $(x+2)$ things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of $x-11$ things taken all at a time such that $a=182\,bc$, find the value of x.

29. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated? [NCERT]

30. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 5, 6, 7, if no digits is repeated? [NCERT]

31. Find the numbers of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5, if no digit is repreated? Howmany of these will be even? [NCERT]

32. All the letters of the word 'EAMCOT' are arranged in different possible ways. Find the number of arrangements in which no two vowels are adjacent to each other.

ANSWERS

1. (i) 336	(ii) 5040	(iii) 720	(iv) 360
2. 4	3. 3	4. 8	5. 6
6. 4	7. 9	8. 6	9. 9
10. 14	11. 5	12. 6, 7	13. 10
15. 120	16. 1260	17. 12	18. 24
19. 360	20. 60	21. 120	22. 8!
23. 3456	24. 720	25. 648	26. 1680
27. 86400	28. 12	29. 504	30. 90
31. 120, 48	32. 144		

HINTS TO SELECTED PROBLEMS

2. We have

$$P(5, r) = P(6, r-1)$$

$$\Rightarrow \frac{5!}{(5-r)!} = \frac{6!}{\{6-(r-1)\}!} \Rightarrow \frac{5!}{(5-r)!} = \frac{6\times 5!}{(7-r)!} \Rightarrow \frac{1}{(5-r)!} = \frac{6}{(7-r)(6-r)(5-r)!}$$

$$\Rightarrow 1 = \frac{6}{(7-r)(6-r)} \Rightarrow (7-r)(6-r) = 3\times 2 \Rightarrow 7-r = 3 \Rightarrow r = 4$$

9. $P(n-1, 3) : P(n, 4) = 1 : 9 \Rightarrow \dfrac{P(n-1, 3)}{P(n, 4)} = \dfrac{1}{9} \Rightarrow \dfrac{(n-1)!}{(n-4)!} \times \dfrac{(n-4)!}{n!} = \dfrac{1}{9} \Rightarrow \dfrac{1}{n} = \dfrac{1}{9} \Rightarrow n = 9.$

15. The total no. of ways = No. of arrangements of 5 things, taken all at a time = ${}^5P_5$.

16. Total no of ways = No. of arrangements of 36 things taken two at a time = ${}^{36}P_2$.

17. The total no. of ordered pairs = No. of arrangements of 4 letters, taken two at a time = ${}^4P_2$.

18. No. of ways = No. of arrangements of 4 books, taken all at a time = ${}^4P_4$.

19. Total no. of words = No. of arrangements of 6 letters, taken 4 at a time = ${}^6P_4$.

20. Required number of numbers = Number of arrangements of digits 1, 3, 5, 7, 9 by taking 3 at a time = ${}^5P_3$.

23. Let $\frac{W_{11}, W_{12}, W_{13}}{W_1}$; $\frac{W_{21}, W_{22}, W_{23}}{W_2}$; $\frac{W_{31}, W_{32}}{W_3}$; $\frac{W_{41}, W_{42}}{W_4}$ be 4 works. These 4 works can be arranged in 4! ways. Now, volumes of each work can be arranged in the following ways : $W_1 \to 3!$ ways ; $W_2 \to 3!$ ways, $W_3 \to 2!$ ways, $W_4 \to 2!$ ways.

Hence, total no. of ways to arrange all books = $4!(3! \times 3! \times 2! \times 2!) = 3456$.

24. Each answer to the given question is an arrangement of the 6 items of column B keeping the order of items in column A fixed. Hence, the total number of answers = Number of arrangements of 6 items in column $B = {}^6P_6 = 6!$.

25. Total number of three digit numbers with distinct digits = ${}^{10}P_3 - {}^9P_2$.

26. Required number of telephone numbers = ${}^8P_4$.

27. Five girls can sit on chairs in a row in ${}^5P_5 = 5!$ ways. Also, 6 boys can stand behind them in a row in ${}^6P_6 = 6!$ ways. Hence, the total number of ways = $5! \times 6!$.

31. The total number of 4 digit numbers formed by using the digits 1, 2, 3, 4, 5 is same as the number of arrangements of 5 digits taken 4 at a time.

So, required number of numbers = ${}^5P_4 = \frac{5!}{(5-4)!} = 120$

An even number will have 2 or 4 at its unit's place. So, unit's place can be filled in 2 ways and the remaining three places (tens, hundreds and thousands) can be filled with remaining 4 digits in ${}^4P_3$ ways. Hence, total number of 4 digit even numbers formed by using the given digits is ${}^4P_3 \times 2 = 48$.

15.4 PERMUTATIONS UNDER CERTAIN CONDITIONS

In this section, we shall discuss permutations where either repetitions of items are allowed or distinction between some of the items are ignored or a particular item occurs in every arrangement etc. Such type of permutations are known as permutations under certain conditions as discussed below.

THEOREM 1 *Prove that the number of all permutations of n different objects taken r at a time, when a particular object is to be always included in each arrangement, is* $r \cdot {}^{n-1}P_{r-1}$.

PROOF Here we have to find the number of ways in which r places can be filled with n given objects such that a particular object occurs in each arrangement. Suppose the particular object is placed at the first place. Then, the remaining $(n-1)$ places can be filled with remaining $(r-1)$ objects in ${}^{n-1}P_{r-1}$ ways. Similarly, by fixing the particular object at the second, third, fourth, ..., rth places, we find that the number of permutations in each case is ${}^{n-1}P_{r-1}$.

Hence, by the fundamental principle of addition,

The required number of permutations $= {}^{n-1}P_{r-1} + {}^{n-1}P_{r-1} + \dots + {}^{n-1}P_{r-1} = r \cdot {}^{n-1}P_{r-1}$.

Q.E.D.

THEOREM 2 *Prove that the number of permutations of n distinct objects taken r at a time, when a particular object is never taken in each arrangement, is* ${}^{n-1}P_r$.

PROOF Since one particular object out of n given objects is never taken. So, we have to determine the number of ways in which r places can be filled with $(n-1)$ distinct objects.

Clearly, the number of such arrangement is ${}^{n-1}P_r$.

Q.E.D.

THEOREM 3 *Prove that the number of permutations of n different objects taken r at a time in which two specified objects always occur together is* $2!\,(r-1)\,{}^{n-2}P_{r-2}$.

PROOF First let us leave out the two specified objects. Then the number of permutations of the remaining $(n-2)$ objects, taken $(r-2)$at a time, is ${}^{n-2}P_{r-2}$. Now, we consider two specified objects temporarily as a single object and add it to each of these ${}^{n-2}P_{r-2}$ permutations which can be done in $(r-1)$ ways. Thus, the number of permutations becomes $(r-1)\,{}^{n-2}P_{r-2}$. But two specified things can be put together in 2! ways.

Hence, the required number of permutations is $2! \cdot (r-1) \cdot {}^{n-2}P_{r-2}$.

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *In how many ways can the letters of the word PENCIL be arranged so that (i) N is always next to E ? (ii) N and E are always together ?*

SOLUTION (i) Let us keep EN together and consider it as one letter. Now, we have 5 letters which can be arranged in a row in ${}^5P_5 = 5! = 120$ ways. Hence, the total number of ways in which N is always next to E is 120.

(ii) Keeping E and N together and considering it as one letter, we have 5 letters which can be arranged in ${}^5P_5 = 5!$ ways. But, E and N can be put together 2! ways (viz. EN, NE).

Hence, the total number of ways $= 5! \times 2! = 240$.

EXAMPLE 2 *How many different words can be formed with the letters of the word EQUATION so that*

(i) *the words begin with E ?* (ii) *the words begin with E and end with N ?*

(iii) *the words begin and end with a consonant ?*

SOLUTION Clearly, the given word contains 8 letters out of which 5 are vowels and 3 consonants.

(i) Since all words must begin with E. So, we fix E at the first place. Now, remaining 7 letters can be arranged in ${}^7P_7 = 7!$ ways.

So, total number of words $= 7!$

(ii) Since all words must begin with E and end with N. So, we fix E at the first place and N at the last place. Now, remaining 6 letters can be arranged in ${}^6P_6 = 6!$ ways.

Hence, the required number of words $= {}^6P_6 = 6!$

(iii) There are 3 consonants and all words should begin and end with a consonant. So, first and last places can be filled with 3 consonants in ${}^3P_2$ ways. Now, the remaining 6 places are to be filled up with the remaining 6 letters in ${}^6P_6$ ways.

Hence, the required number of words $= {}^3P_2 \times {}^6P_6 = 6 \times 720 = 4320$

EXAMPLE 3 *How many words can be formed from the letters of the word, 'TRIANGLE' ? How many of these will begin with T and end with E ?*

SOLUTION There are 8 letters in the word 'TRIANGLE'. The total number of words formed with these 8 letters is the number of arrangements of 8 items, taken all at a time, which is equal to ${}^8P_8 = 8! = 40320$. If we fix up T in the beginning and E at the end, then the remaining 6 letters can be arranged in ${}^6P_6 = 6!$ ways.

So, the total number of words which begin with T and end with $E = 6! = 720$.

EXAMPLE 4 *How many words can be formed with the letters of the word 'ORDINATE' so that vowels occupy odd places?*

SOLUTION There are 4 vowels and 4 consonants in the word 'ORDINATE'. We have to arrange 8 letters in a row such that vowels occupy odd places. There are 4 odd places viz. 1, 3, 5, 7. Four vowels can be arranged in these 4 odd places in 4! ways. Remaining 4 even places viz. 2, 4, 6, 8 are to be occupied by the 4 consonants. This can be done in 4! ways. Hence, the total number of words in which vowels occupy odd places $= 4! \times 4! = 576$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *In how many ways 5 boys and 3 girls can be seated in a row so that no two girls are together?*

SOLUTION The 5 boys can be seated in a row in ${}^5P_5 = 5!$ ways. In each of these arrangements 6 places are created, shown by the cross-marks, as given below:

$$\times B \times B \times B \times B \times B \times$$

Since no two girls are to sit together, so we may arrange 3 girls in 6 places. This can be done in ${}^6P_3$ ways i.e. 3 girls can be seated in ${}^6P_3$ ways.

Hence, the total number of seating arrangements $= {}^5P_5 \times {}^6P_3 = 5! \times 6 \times 5 \times 4 = 14400$.

EXAMPLE 6 *In how many ways can the letters of the word 'DELHI' be arranged so that the vowels occupy only even places?*

SOLUTION There are 5 distinct letters in the word 'DELHI'. We wish to find the total number of arrangements of these 5 letters so that vowels occupy only even places. There are two vowels E and I and 2 even places viz 2^{nd} and 4^{th}. These two vowels can be arranged in the two even places in 2! ways. The remaining three letters (D, L, H) can be arranged in 3 places (viz Ist 3rd, 5th) in 3! ways. Hence, by the fundamental principle of counting the total number of arrangements $= 3! \times 2! = 12$.

EXAMPLE 7 *How many words can be formed from the letters of the word 'DAUGHTER' so that*

(i) *the vowels always come together ?* (ii) *the vowels never come together ?* [NCERT]

SOLUTION There are 8 letters in the word 'DAUGHTER', including 3 vowels (A, U, E) and 5 consonants (D, G, H, T, R).

(i) Considering three vowels as one letter, we have 6 letters which can be arranged in ${}^6P_6 = 6!$ ways. But, corresponding each way of these arrangements, the vowels A, U, E can be put together in 3! ways.

Hence, required number of words $= 6! \times 3! = 720 \times 6 = 4320$

(ii) The total number of words formed by using all the eight letters of the word 'DAUGHTER' is ${}^8P_8 = 8! = 40320$.

So, the total number of words in which vowels are never together

= Total number of words – Number of words in which vowels are always together

$= 40320 - 4320 = 36000$

EXAMPLE 8 *In how many ways can 9 examination papers be arranged so that the best and the worst papers are never together?*

SOLUTION The number of arrangements in which the best and the worst papers never come together can be obtained by subtracting from the total number of arrangements, the number of arrangements in which the best and worst come together.

The total number of arrangements of 9 papers $= {}^9P_9 = 9!$

Considering the best and the worst papers as one paper, we have 8 papers which can be arranged in ${}^8P_8 = 8!$ ways. But, the best and worst papers can be put together in $2!$ ways. So, the number of permutations in which the best and the worst papers can be put together $= (2! \times 8!)$. Hence, the number of ways in which the best and the worst papers never come together $= 9! - 2! \times 8! = 9 \times 8! - 2 \times 8! = 7 \times 8! = 282240$.

EXAMPLE 9 *In how many ways can 5 children be arranged in a row such that*

(i) *two of them, Ram and Shyam, are always together?*

(ii) *two of them, Ram and Shyam, are never together?*

SOLUTION There are five children including Ram and Shyam.

(i) Considering Ram and Shyam as one child, there are four children. They can be arranged in a row in 4! ways. But Ram and Shyam can be arranged together in 2! ways.

Hence, the required number of arrangements $= 4! \times 2! = 48$.

(ii) Total number of arrangements of 5 children in a row = 5! = 120.

$\therefore$ Total number of arrangements in which Ram and Shyam are never together

= Total number of arrangements – Number of arrangements in which Ram and Shyam are together

$= 120 - 48 = 72$.

EXAMPLE 10 *A code word is to consist of two distinct English alphabets followed by two distinct numbers from 1 to 9. For example, CA 23 is a code word. How many such code words are there? How many of them end with an even integer ?*

SOLUTION There are 26 English alphabets. So, first two places in the code word can be filled in ${}^{26}P_2$ ways. In last two places we have to use two distinct numbers from 1 to 9. So, last two places can be filled in ${}^9P_2$ ways. Hence, by the fundamental principle of counting, the total number of code words $= {}^{26}P_2 \times {}^9P_2 = 650 \times 72 = 46800$.

Number of code words ending with an even integer.

In this case, the code word can have any of the numbers 2, 4, 6, 8 at the extreme right position. So, the extreme right position can be filled in 4 ways. Now, next left position can be filled with any one of the remaining 8 digits in 8 ways and the two extreme left positions can be filled by two English alphabets in ${}^{26}P_2$ ways.

Hence, the total number of code words which end with an even integer $= 4 \times 8 \times {}^{26}P_2$

$= 4 \times 8 \times 650 = 20800$.

EXAMPLE 11 *The Principal wants to arrange 5 students on the platform such that the boy 'SALIM' occupies the second position and such that the girl, 'SITA' is always adjacent to the girl 'RITA'. How many such arrangements are possible ?*

SOLUTION Since SALIM occupies the second position and the two girls RITA and SITA are always adjacent to each other. So, none of these two girls can occupy the first seat. Thus, first seat can be occupied by any one of the remaining two students in 2 ways. Second seat can be occupied by SALIM in only one way.

Now, in the remaining three seats SITA and RITA can be seated in the following four ways:

	I	II	III	IV	V
1.	×	SALIM	SITA	RITA	×
2.	×	SALIM	RITA	SITA	×
3.	×	SALIM	×	SITA	RITA
4.	×	SALIM	×	RITA	SITA

Now, only one seat is left which can be occupied by the 5th student in one way.

Hence, the number of required type of arrangements $= 2 \times 4 \times 1 = 8$.

EXAMPLE 12 *How many numbers between 400 and 1000 can be formed with the digits 0, 2, 3, 4, 5, 6 if no digit is repeated in the same number ?*

SOLUTION Number between 400 and 1000 consist of three digits with digit at hundred's place greater than or equal to 4. Hundred's place can be filled, by using the digits 4, 5, 6 in 3 ways. Now, ten's and unit's places can be filled by the remaining 5 digits in ${}^5P_2$ ways.

Hence, the required number of numbers $= 3 \times {}^5P_2 = 3 \times \frac{5!}{3!} = 3 \times 20 = 60.$

EXAMPLE 13 *In a class of 10 students there are 3 girls A, B, C. In how many different ways can they be arranged in a row such that no two of the three girls are consecutive.*

SOLUTION There are 7 boys and 3 girls. Seven boys can be arranged in a row in ${}^7P_7 = 7!$ ways.

Now, we have 8 places in which we can arrange 3 girls in ${}^8P_3$ ways.

Hence, by the fundamental principle of counting, the number of arrangements $= 7! \times {}^8P_3$

$= 7! \times 336.$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 14 *When a group photograph is taken, all the seven teachers should be in the first row and all the twenty students should be in the second row. If the two corners of the second row are reserved for the two tallest students, interchangeable only between them, and if the middle seat of the front row is reserved for the Principal, how many arrangements are possible?*

SOLUTION Since the middle seat of the front row is reserved for the Principal, the remaining 6 teachers can be arranged in the front row in ${}^6P_6 = 6!$ ways.

The two corners of the second row are reserved for the two tallest students. They can occupy these two places in 2! ways. The remaining 18 seats may be occupied by the remaining 18 students in 18! ways.

Hence, by the fundamental principle of counting, the total number of arrangements

$= 6! \times (18! \times 2!) = 18! \times 1440.$

EXAMPLE 15 *How many even numbers are there with three digits such that if 5 is one of the digits, then 7 is the next digit?*

SOLUTION We have to determine the total number of even numbers formed by using the given condition. So, at unit's place we can use one of the digits 0, 2, 4, 6, 8. If 5 is at ten's place then, as per the given condition, 7 should be at unit's place. In such a case the number will not be an even number. So, 5 cannot be at ten's and one's places. Hence, 5 can be only at hundred's place.

Now two cases arise.

Case I *When 5 is at hundred's place:*

If 5 is at hundred's place, then 7 will be at ten's place. So, unit's place can be filled in 5 ways by using the digits 0, 2, 4, 6, 8.

So, total number of even numbers $= 1 \times 1 \times 5 = 5$.

Case II *When 5 is not at hundred's place:*

In this case, hundred's place can be filled in 8 ways (0 and 5 cannot be used at hundred's place). In ten's place we can use any one of the ten digits except 5. So, ten s place can be filled in 9 ways. At unit's place we have to use one of the even digits 0, 2, 4, 6, 8. So, units place can be filled in 5 ways.

So, total number of even numbers $= 8 \times 9 \times 5 = 360$

Hence, the total number of required even numbers $= 360 + 5 = 365$.

EXAMPLE 16 *How many four digit numbers divisible by 4 can be made with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed ?*

SOLUTION Recall that a number is divisible by 4 if the number formed by the last two digits is divisible by 4. The digits at unit's and ten's places can be arranged as follows:

Th	H	T	O
×	×	1	2
×	×	2	4
×	×	3	2
×	×	5	2

Now, corresponding each such way the remaining three digits at thousand's and hundred's places can be arranged in ${}^3P_2$ ways.

Hence, the required number of numbers $= {}^3P_2 \times 4 = 3! \times 4 = 24$.

EXAMPLE 17 *Find the number of ways in which 5 boys and 5 girls be seated in a row so that*

(i) *No two girls may sit together.* (ii) *All the girls sit together and all the boys sit together.*
(iii) *All the girls are never together.*

SOLUTION (i) 5 boys can be seated in a row in ${}^5P_5 = 5!$ ways. Now, in the 6 gaps 5 girls can be arranged in ${}^6P_5$ ways.

Hence, the number of ways in which no two girls sit together $= 5! \times {}^6P_5 = 5! \times 6!$

(ii) The two groups of girls and boys can be arranged in 2! ways. 5 girls can be arranged among themselves in 5! ways. Similarly, 5 boys can be arranged among themselves in 5! ways. Hence, by the fundamental principle of counting, the total number of requisite seating arrangements $= 2!(5! \times 5!) = 2(5!)^2$.

(iii) The total number of ways in which all the girls are never together
= Total number of arrangements – Total number of arrangements in which all the girls are always together
$= 10! - 5! \times 6!$

EXAMPLE 18 *Five boys and five girls form a line with the boys and girls alternating. Find the number of ways of making the line.*

SOLUTION 5 boys can be arranged in a line in ${}^5P_5 = 5!$ ways. Since the boys and girls are alternating. So, corresponding each of the $5!$ ways of arrangements of 5 boys we obtain 5 places marked by cross as shown below:

(i) $B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times$ (ii) $\times B_1 \times B_2 \times B_3 \times B_4 \times B_5$.

Clearly, 5 girls can be arranged in 5 places marked by cross in (5! + 5!) ways.

Hence, the total number of ways of making the line $= 5! \times (5! + 5!) = 2\,(5!)^2$

EXAMPLE 19 *In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls sit together in a back row on adjacent seats ?*

SOLUTION Total number of persons = 3 girls + 9 boys = 12.

Total number of numbered seats $= 2 \times 3 + 4 \times 2 = 14$

So, total number of ways in which 12 persons can be seated on 14 seats

= Number of arrangements or 14 seats by taking 12 at a time = ${}^{14}P_{12}$.

Three girls can be seated together in a back row on adjacent seats in the following ways:

1, 2, 3 or 2, 3, 4 of first van

and, 1, 2, 3 or 2, 3, 4 of second one.

In each way the three girls can interchange among themselves in 3! ways. So, the total number of ways in which three girls can be seated together in a back row on adjacent seats $= 4 \times 3!$

Now, 9 boys are to be seated on remaining 11 seats, which can be done in ${}^{11}P_9$ ways.

Hence, by the fundamental principle of counting, the total number of seating arrangements is ${}^{11}P_9 \times 4 \times 3!$.

EXAMPLE 20 *A tea party is arranged for 16 persons along two sides of a long table with 8 chairs on each side. Four persons wish to sit on one particular and two on the other side. In how many ways can they be seated?*

SOLUTION Let the two sides be A and B. Assume that four persons wish to sit on side A. Four persons who wish to sit on side A can be accommodated on eight chairs in ${}^8P_4$ ways and two persons who wish to sit on side B can be accommodated on 8 chairs in ${}^8P_2$ ways. Now, 10 persons are left, who can sit on 10 chairs on both the sides of the table in 10! ways.

Hence, the total number of ways in which 16 persons can be seated $= {}^8P_4 \times {}^8P_2 \times 10!$

EXERCISE 15.4

BASIC

1. In how many ways can the letters of the word 'FAILURE' be arranged so that the consonants may occupy only odd positions?
2. In how many ways can the letters of the word 'STRANGE' be arranged so that
   (i) the vowels come together ? (ii) the vowels never come together? and
   (iii) the vowels occupy only the odd places?
3. How many words can be formed from the letters of the word 'SUNDAY'? How many of these begin with D?

4. How many words can be formed out of the letters of the word, 'ORIENTAL', so that the vowels always occupy the odd places ?
5. How many different words can be formed with the letters of word 'SUNDAY'? How many of the words begin with N? How many begin with N and end in Y?
6. How many different words can be formed from the letters of the word 'GANESHPURI' ? In how many of these words:
   (i) the letter G always occupies the first place?
   (ii) the letters P and I respectively occupy first and last place?
   (iii) the vowels are always together?
   (iv) the vowels always occupy even places?

BASED ON LOTS

7. How many permutations can be formed by the letters of the word, 'VOWELS', when
   (i) there is no restriction on letters? (ii) each word begins with E?
   (iii) each word begins with O and ends with L? (iv) all vowels come together?
   (v) all consonants come together?
8. How many words can be formed out of the letters of the word 'ARTICLE', so that vowels occupy even places?
9. In how many ways can a lawn tennis mixed double be made up from seven married couples if no husband and wife play in the same set?
10. m men and n women are to be seated in a row so that no two women sit together. If $m > n$ then show that the number of ways in which they can be seated as $\dfrac{m!\,(m+1)!}{(m-n+1)\,!}$.
11. How many words (with or without dictionary meaning) can be made from the letters in the word MONDAY, assuming that no letter is repeated, if
   (i) 4 letters are used at a time? (ii) all letters are used at a time?
   (iii) all letters are used but first is vowel?
12. How many three letter words can be made using the letters of the word 'ORIENTAL'?

ANSWERS

1. 576 2. (i) 1440 (ii) 3600 (iii) 1440 3. 720, 120 4. 576
5. 720, 120, 24 6. 10! (i) 9! (ii) 8! (iii) 7! × 4! (iv) 5! × 6!
7. (i) 720 (ii) 120 (iii) 24 (iv) 240 (v) 144 8. 144 9. 840
11. (i) 360 (ii) 720 (iii) 240 12. 336

15.5 PERMUTATIONS OF OBJECTS NOT ALL DISTINCT

So far we were discussing permutations of distinct objects (things) by taking some or all at a time. In this section, we intend to discuss the permutations of a given number of objects when objects are not all different. For example, the number of arrangements of the letters of the word MISSISSIPPI, the number of six digit numbers formed by using the digits 1, 1, 2, 3, 3, 4 etc. The following theorem is very helpful to determine the number of such arrangements.

THEOREM *The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second such that $p + q = n$ is $\dfrac{n!}{p!\,q!}$.*

PROOF Let the required number of permutations be x. Consider one of these x permutations.

Now, replace p alike things in this permutation by p distinct things which are also different from others. These p different things may be permuted among themselves in $p!$ ways without changing the positions of other things. Similarly, if we replace q alike things by q distinct things, which are also different from others, then they can be permuted among themselves in $q!$ ways.

Thus, if both the replacements are done simultaneously, then we find that each one of the x permutations give rise to $p! \times q!$ permutations. Therefore, x permutations give rise to $x \times p! \times q!$ permutations. Now, each of these $x \times p! \times q!$ permutations, is a permutation of n different things, taken all at a time.

$\therefore$ $x \times p! \times q! =$ Number of permutations of n different things taken all at a time $= n!$

Hence, $$x = \frac{n!}{p!\,q!}$$

Q.E.D.

REMARK 1 *The number of permutations of n things, of which p_1 are alike of one kind; p_2 are alike of second kind; p_3 are alike of third kind; ...; p_r are alike of rth kind such that $p_1 + p_2 + ... + p_r = n$, is*

$$\frac{n!}{p_1!\,p_2!\,p_3!\dots p_r!}$$

REMARK 2 *The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining all are distinct, is* $\dfrac{n!}{p!\,q!}$.

REMARK 3 *Suppose there are r things to be arranged, allowing repetitions. Let further $p_1, p_2, \dots p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then the total number of permutations of these r objects to the above condition is* $\dfrac{(p_1 + p_2 + \dots + p_r)!}{p_1!\,p_2!\,p_3!\dots p_r!}$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *How many different words can be formed with the letters of the word 'MISSISSIPPI' ? In how many of these permutations four I's do not come together?* [NCERT]

SOLUTION There are 11 letters in the given word, of which 4 are S's, 4 are I's and 2 are P's. So, total number of words is the number of arrangements of 11 things, of which 4 are similar of one kind, 4 are similar of second kind and 2 are similar of third kind i.e. $\dfrac{11!}{4!\,4!\,2!}$.

Hence, the total number of words $= \dfrac{11!}{4!\,4!\,2!} = 34650$.

Considering 4 I's as one letter, we have 8 letters of which 4 are S's and 2 are P's. These 8 letters can be arranged in $\dfrac{8!}{4!\,2!}$ ways.

$\therefore$ Number of words in which 4 I's come together $= \dfrac{8!}{4!\,2!} = 840$.

Hence, number of words in which 4 I's do not come together $= 34650 - 840 = 33810$.

EXAMPLE 2 *How many permutations of the letters of the word 'APPLE' are there?*

SOLUTION Here there are 5 letters, two of which are of the same kind. The others are each of its own kind. So, the required number of permutations is $\dfrac{5!}{2!1!1!1!} = \dfrac{120}{2} = 60$.

EXAMPLE 3 *How many words can be formed using the letter A thrice, the letter B twice and the letter C thrice ?*

SOLUTION We are given 8 letters viz. A, A, A, B, B, C, C, C. Clearly, there are 8 letters of which three are of one kind, two are of second kind and three are of third kind.

So, the total number of permutations $= \dfrac{8!}{3!\,2!\,3!} = 560$.

Hence, the requisite number of words = 560.

EXAMPLE 4 *Find the number of different permutations of the letters of the word BANANA ?*

SOLUTION Clearly, there are six letters in the word 'BANANA' of which three are alike of one kind (3 A's), two are alike of second kind (2 N's) and one of its own kind.

$\therefore$ Total number of their permutations $= \dfrac{6!}{3!\,2!\,1!} = 60$.

Hence, the requisite number of words = 60

EXAMPLE 5 (i) *How many different words can be formed with the letters of the word HARYANA?*

(ii) *How many of these begin with H and end with N?*

(iii) *In how many of these H and N are together?*

SOLUTION (i) There are 7 letters in the word 'HARYANA' of which 3 are A's and remaining all are each of its own kind.

So, total number of words $= \dfrac{7!}{3!1!1!1!1!} = \dfrac{7!}{3!} = 840$.

(ii) After fixing H in first place and N in last place, we have 5 letters out of which three are alike i.e. A's and remaining all are each of its own kind.

So, total number of words $= \dfrac{5!}{3!} = 20$.

(iii) Considering H and N together we have $7 - 2 + 1 = 6$ letters out of which three are alike i.e. A's and others are each of its own kind. These six letters can be arranged in $\dfrac{6!}{3!}$ ways. But H and N can be arranged amongst themselves in 2! ways.

Hence, the requisite number of words $= \dfrac{6!}{3!} \times 2! = 120 \times 2 = 240$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 6 *How many different words can be formed by using all the letters of the word 'ALLAHABAD'?* [NCERT]

(i) *In how many of them vowels occupy the even positions ?*

(ii) *In how many of them both L do not come together ?*

SOLUTION There are 9 letters in the word 'ALLAHABAD' out of which 4 are A's, 2 are L's and the rest are all distinct.

So, the requisite number of words $= \dfrac{9!}{4!\,2!} = 7560$.

(i) There are 4 vowels and all are alike i.e. 4 A's. Also, there are 4 even places viz 2nd, 4th, 6th and 8th. So, these 4 even places can be occupied by 4 vowels in $\dfrac{4!}{4!} = 1$ way. Now, we are left with 5 places in which 5 letters, of which two are alike (2 L's) and other distinct, can be arranged in $\dfrac{5!}{2!}$ ways.

Hence, the total number of words in which vowels occupy the even places $= \dfrac{5!}{2!} \times \dfrac{4!}{4!} = \dfrac{5!}{2!} = 60$.

(ii) Considering both L together and treating them as one letter we have 8 letters out of which A repeats 4 times and others are distinct. These 8 letters can be arranged in $\frac{8!}{4!}$ ways.

So, the number of words in which both L come together $= \frac{8!}{4!} = 1680$.

Hence, the number of words in which both L do not come together

= Total no. of words – No. of words in which both L come together = 7560 – 1680 = 5880.

EXAMPLE 7 *Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements* **[NCERT]**

(i) *do the words start with P?* (ii) *do all the vowels always occur together?*

(iii) *do all the vowels never occur together?* (iv) *do the words begin with I and end in P?*

SOLUTION In the word 'INDEPENDENCE' there are 12 letters of which 3 are N's, 4 are E's and 2 are D's. Therefore,

Total number of arrangements $= \frac{12!}{3!\,4!\,2!} = 1663200$

(i) After fixing the letter P at the extreme left position, there are 11 letters consisting of 3 N's, 4E's and 2D's. These 11 letters can be arranged in $\frac{11!}{3!\,4!\,2!} = 138600$

$\therefore$ Number of words beginning with P $= \frac{11!}{3!\,4!\,2!} = 138600$

(ii) There are 5 vowels in the given word of which 4 are E's and one I. These vowels can be put together in $\frac{5!}{4!\,1!}$ ways. Considering these 5 vowels as one letter there are 8 letters (taking 7 remaining letters) which can be arranged in $\frac{8!}{3!\,2!}$ ways (as there are 3 N's and 2D's). Since corresponding to each arrangement of 5 vowels there are $\frac{8!}{3!\,2!}$ ways of arranging remaining 7 letters and one letter formed by 5 vowels.

Hence, by fundamental principle of multiplication, the required number of arrangements is $\frac{8!}{3!\,2!} \times \frac{5!}{4!\,1!} = 16800$

(iii) The required number of arrangements

= The total number of arrangements – The number of arrangements in which all the vowels occur together

$= 1663200 - 16800 = 1646400$

(iv) Let us fix I at the extreme left end and P at the extreme right end. Now, we are left with 10 letters of which 3 are N's, 4 are E's and 2 are D's. These ten letters can be arranged in $\frac{10!}{4!\,3!\,2!}$ ways.

Hence, required number of arrangements $= \frac{10!}{4!\,3!\,2!} = 12600$.

EXAMPLE 8 *In how many ways can the letters of the word PERMUTATIONS be arranged if (i) the words start with P end with S (ii) vowels are all together.*

SOLUTION (i) There are 12 letters in the given word of which 2 are T's and the remaining are distinct. Remaining 10 letters between P and S can be arranged in $\frac{10!}{2!}$ ways.

$\therefore$ Total number of words starting with P and ending in S $= \frac{10!}{2!} = 1814400$

(ii) There are 5 vowels in the given word. These vowels can be put together in 5! ways. Considering these 5 vowels as one letter, we have 8 letters (7 remaining letters and one letter formed by 5 vowels) of which 2 are T's. These 8 letters can be arranged in $\frac{8!}{2!}$ ways.

Hence, by the fundamental principle of multiplication, required number of words is $5! \times \frac{8!}{2!} = 2419200$.

EXAMPLE 9 *How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3 ?*

SOLUTION Any number greater than a million will contain all the seven digits.

Now, we have to arrange these seven digits, out of which 2 occur twice, 3 occurs twice and the rest are distinct.

The number of such arrangements $= \frac{7!}{2! \times 3!} = 420.$

These arrangements also include those numbers which contain 0 at the million's place. Keeping 0 fixed at the millionth place, we have 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are distinct. These 6 digits can be arranged in $\frac{6!}{2! \times 3!} = 60$ ways.

Hence, the number of required numbers = 420 – 60 = 360.

EXAMPLE 10 *There are six periods in each working day of a school. In how many ways can one arrange 5 subjects such that each subject is allowed at least one period?*

SOLUTION Since each subject is allowed at least one period. So, we first select one subject for the left out period. This can be done in ${}^5C_1$ ways. Now, six subject can be arranged in $\frac{6!}{2!}$ ways.

Hence, the total number of arrangements $= {}^5C_1 \times \frac{6!}{2!} = 1800$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 11 *How many arrangements can be made with the letters of the word 'MATHEMATICS'? In how many of them vowels are together ?*

SOLUTION There are 11 letters in the word 'MATHEMATICS' of which two are M's, two are A's, two are T's and all other are distinct. So,

Required number of arrangements $= \frac{11!}{2! \times 2! \times 2!} = 4989600$

There are 4 vowels viz. A, E, A, I. Considering these four vowels as one letter we have 8 letters (M, T, H, M, T, C, S and one letter obtained by combining all vowels), out of which M occurs twice, T occurs twice and the rest all different. These 8 letters can be arranged in $\frac{8!}{2! \times 2!}$ ways.

But, the four vowels (A, E, A, I) can be put together in $\frac{4!}{2!}$ ways.

Hence, the total number of arrangements in which vowels are always together $= \frac{8!}{2! \times 2!} \times \frac{4!}{2!}$

$$= 10080 \times 12 = 120960.$$

EXAMPLE 12 *If all the letters of the word 'AGAIN' be arranged as in a dictionary, what is the fiftieth word?* [NCERT]

SOLUTION In dictionary the words at each stage are arranged in alphabetical order. Starting with the letter A, and arranging the other four letters GAIN, we obtain 4! = 24 words.

Thus, there are 24 words which start with A. These are the first 24 words.

Then, starting with G, and arranging the other four letters A, A, I, N in different ways, we obtain $\frac{4!}{2!}=\frac{24}{2}=12$ words. Thus, there are 12 words, which start with G.

Now, we start with I. The remaining 4 letters A, G, A, N can be arranged in $\frac{4!}{2!}=12$ ways. So, there are 12 words, which start with I.

Thus, we have so far constructed 48 words. The 49th word is NAAGI and hence the 50th word is NAAIG.

EXAMPLE 13 *The letters of the word 'RANDOM' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'RANDOM'.*

SOLUTION In a dictionary the words at each stage are arranged in alphabetical order. In the given problem we must therefore consider the words beginning with A, D, M, N, O, R in order. A will occur in the first place as often as there are ways of arranging the remaining 5 letters all at a time i.e. A will occur 5! times. Similarly, D, M, N, O will occur in the first place the same number of times.

$\therefore$ Number of words starting with A = 5! = 120
Number of words starting with D = 5! = 120
Number of words starting with M = 5! = 120
Number of words starting with N = 5! = 120
Number of words starting with O = 5! = 120

Number of words beginning with R is 5!, but one of these words is the word RANDOM. So, we first find the number of words beginning with RAD and RAM.

No. of words starting with RAD = 3! = 6
No. of words starting with RAM = 3! = 6

Now, the words beginning with 'RAN' must follow. There are 3! words beginning with RAN. One of these words is the word RANDOM itself.

The first word beginning with RAN is the word RANDMO and the next word is RANDOM.

$\therefore$ Rank of RANDOM $= 5 \times 120 + 2 \times 6 + 2 = 614$.

EXAMPLE 14 *If the different permutations of the word, 'EXAMINATION' are listed as in a dictionary, how many items are there in the list before the first word starting with E?* [NCERT]

SOLUTION In a dictionary the words at each stage are arranged in alphabetical order. In the given problem we have to find the total number of words starting with A, because the very next word will start with E.

For finding the number of words starting with A, we have to find the number of arrangements of the remaining 10 letters, EXMINATION, of which there are 2 I's, 2 N's and the others each of its own kind.

The number of such arrangements $=\frac{10!}{2!\,2!}=907200$.

Hence, the required number of items = 907200.

EXERCISE 15.5

BASIC

1. Find the number of words formed by permuting all the letters of the following words:
   (i) INDEPENDENCE (ii) INTERMEDIATE (iii) ARRANGE
   (iv) INDIA (v) PAKISTAN (vi) RUSSIA
   (vii) SERIES (viii) EXERCISES (ix) CONSTANTINOPLE

2. In how many ways can the letters of the word 'ALGEBRA' be arranged without changing the relative order of the vowels and consonants?
3. How many words can be formed with the letters of the word 'UNIVERSITY', the vowels remaining together?
4. Find the total number of arrangements of the letters in the expression $a^3 b^2 c^4$ when written at full length.
5. How many words can be formed with the letters of the word 'PARALLEL' so that all L's do not come together?
6. How many words can be formed by arranging the letters of the word 'MUMBAI' so that all M's come together?
7. How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?
8. How many different signals can be made from 4 red, 2 white and 3 green flags by arranging all of them vertically on a flagstaff?
9. How many number of four digits can be formed with the digits 1, 3, 3, 0?

BASED ON LOTS

10. In how many ways can the letters of the word 'ARRANGE' be arranged so that the two R's are never together?
11. How many different numbers, greater than 50000 can be formed with the digits 0, 1, 1, 5, 9.
12. How many words can be formed from the letters of the word 'SERIES' which start with S and end with S?
13. How many permutations of the letters of the word 'MADHUBANI' do not begin with M but end with I?
14. Find the number of numbers, greater than a million, that can be formed with the digits 2, 3, 0, 3, 4, 2, 3.
15. There are three copies each of 4 different books. In how many ways can they be arranged in a shelf?
16. How many different arrangements can be made by using all the letters in the word 'MATHEMATICS'. How many of them begin with C? How many of them begin with T?
17. A biologist studying the genetic code is interested to know the number of possible arrangements of 12 molecules in a chain. The chain contains 4 different molecules represented by the initials A (for Adenine), C (for Cytosine), G (for Guanine) and T (for Thymine) and 3 molecules of each kind. How many different such arrangements are possible?
18. In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable? [NCERT]
19. How many numbers greater than 1000000 can be formed by using the digits 1, 2, 0, 2, 4, 2, 4? [NCERT]
20. In how many ways can the letters of the word ASSASSINATION be arranged so that all the S's are together? [NCERT]
21. Find the total number of permutations of the letters of the word 'INSTITUTE'. [NCERT]

BASED ON HOTS

22. The letters of the word 'SURITI' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'SURITI'.

23. If the letters of the word 'LATE' be permuted and the words so formed be arranged as in a dictionary, find the rank of the word LATE.
24. If the letters of the word 'MOTHER' are written in all possible orders and these words are written out as in a dictionary, find the rank of the word 'MOTHER'.
25. If the permutations of a, b, c, d, e taken all together be written down in alphabetical order as in dictionary and numbered, find the rank of the permutation debac.
26. Find the total number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together.
27. In how many ways can the letters of the word "INTERMEDIATE" be arranged so that:
    (i) the vowels always occupy even places?
    (ii) the relative order of vowels and consonants do not alter?
28. The letters of the word 'ZENITH' are written in all possible orders. How many words are possible if all these words are written out as in a dictionary ? What is the rank of the word 'ZENITH'?

ANSWERS

1. (i) 1663200 (ii) 19958400 (iii) 1260 (iv) 60
   (v) 20160 (vi) 360 (vii) 180 (viii) 30240
   (ix) $\frac{14!}{24}$

2. 72
3. 60480
4. 1260
5. 3000
6. 120
7. 18
8. 1260
9. 9
10. 900
11. 24
12. 12
13. 17640
14. 360
15. $12!/(3!)^4$
16. $\frac{11!}{2!\,2!\,2!}, \frac{10!}{2!\,2!\,2!}, \frac{10!}{2!\,2!}$
17. 369600
18. 1260
19. 360
20. 151200
21. $\frac{9!}{2!\,3!}$
22. 236
23. 14
24. 309
25. 93
26. 35
27. (i) 21600 (ii) 21600
28. 616

HINTS TO SELECTED PROBLEMS

2. The consonants can be arranged among themselves in 4! ways and the vowels among themselves in $\frac{3!}{2!}$ ways. Hence, the required number of arrangements $= 4! \times \frac{3!}{2!} = 72$.

4. There are 3 a's, 2 b's and 4 c's. So, the total number of arrangements $= \frac{9!}{3!\,2!\,4!} = 1260$.

7. There are 4 odd digits 1, 1, 3, 3 and 4 odd places. So, odd digits can be arranged in odd places in $\frac{4!}{2!\,2!}$ ways. The remaining 3 even digits 2, 2, 4 can be arranged in 3 even places in $\frac{3!}{2!}$ ways. Hence, the requisite number of numbers $= \frac{4!}{2!\,2!} \times \frac{3!}{2!} = 18$.

8. We have to arrange 9 flags, out of which 4 are of one kind, 2 are of another kind and 3 are of third kind. So, total number of signals $= \frac{9!}{4!\,2!\,3!}$.

9. Required number of numbers $= \frac{4!}{2!} - \frac{3!}{2!}$.

11. Numbers greater than 50000 will have either 5 or 9 in the first place and will consist of 5 digits.

Number of numbers of with digit 5 at first place $= \frac{4!}{2!}$

Number of numbers with digit 9 at first place $= \frac{4!}{2!}$

Hence, the required number of numbers $= \frac{4!}{2!} + \frac{4!}{2!} = 24$.

18. Required number of ways $= \frac{(4+3+2)!}{4!\,3!\,2!} = \frac{9!}{4!\,3!\,2!} = 1260$

19. Number of numbers greater than 1000000 that can be formed by using the digits 1, 2, 0, 2, 4, 2, 4.

= Number of numbers formed by given digits – Number of numbers having 0 as left most digit

$$= \frac{7!}{3!\,2!} - \frac{6!}{3!\,2!} = \frac{7!-6!}{3!\,2!} = \frac{6\times 6!}{3!\,2!} = 360$$

20. Considering all S as one letter there are 10 letters containing 3A's, 2I's, 2N's, 1T, 1O which can be arranged in $\frac{10!}{3!\,2!\,2!} = 151200$ ways.

21. There are 9 letters in the word INSTITUTE containing 2I's, 3T's, 1N, 1S, 1U and 1E. These letters can be arranged in $\frac{9!}{2!\,3!} = 30240$ ways.

26. Six '+' signs can be arranged in a row in $\frac{6!}{6!} = 1$ way. Now, we are left with seven places in which four different things can be arranged in ${}^7P_4$ ways but as all the four '–' signs are identical, therefore, four '–' signs can be arranged in $\frac{{}^7P_4}{4!} = 35$ ways.

Hence, the required number of ways $= 1 \times 35 = 35$.

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The value of ${}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$ is ....
2. If ${}^nP_4 : {}^nP_5 = 1:2$, then $n =$ ....
3. If ${}^{12}P_r = 1320$, then $r =$ ....
4. The number of permutations of n distinct object, taken r at a time, when repetitions are not allowed, is ....
5. The number of permutations of n distinct objects, taken r at a time, when repetitions are allowed, is ....
6. The number of ways 'm' men and 'n' women $(m > n)$ can be seated in arow so that no two women sit together is ....
7. In an examination there are three multiple choice questions and each question has four choice. The number of ways in which a student can fail to get all answers correct, is ....

8. The number of ways in which three letters can be posted in five letter boxes, is ....
9. The number of six digit numbers, all digits of which are odd, is ....
10. The number of different words that can be made from the letters of the word INTERMEDIATE, such that two vowels never come together, is ....

ANSWERS

1. ${}^nP_r$ 2. 6 3. 3 4. ${}^nP_r$ 5. n^r 6. $\dfrac{m!(m+1)!}{(m-n+1)!}$ 7. 63

8. 5^3 9. 5^6 10. $\dfrac{6!}{2!}\times\dfrac{{}^7P_6}{3!2!}=151200$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. In how many ways can 4 letters be posted in 5 letter boxes?
2. Write the number of 5 digit numbers that can be formed using digits 0, 1 and 2.
3. In how many ways 4 women draw water from 4 taps, if no tap remains unused?
4. Write the total number of possible outcomes in a throw of 3 dice in which at least one of the dice shows an even number.
5. Write the number of arrangements of the letters of the word BANANA in which two N's come together.
6. Write the number of ways in which 7 men and 7 women can sit on a round table such that no two women sit together.
7. Write the number of words that can be formed out of the letters of the word 'COMMITTEE'.
8. Write the number of all possible words that can be formed using the letters of the word 'MATHEMATICS'.
9. Write the number of ways in which 6 men and 5 women can dine at a round table if no two women sit together.
10. Write the number of ways in which 5 boys and 3 girls can be seated in a row so that each girl is between 2 boys.
11. Write the remainder obtained when $1!+2!+3!+\ldots+200!$ is divided by 14.
12. Write the number of numbers that can be formed using all for digits 1, 2, 3, 4.

ANSWERS

1. 5^4 2. 2×3^4 3. $4!$ 4. 189 5. 20 6. $7!\times 6!$ 7. $\dfrac{9!}{(2!)^3}$ 8. $\dfrac{11!}{2!\,2!\,2!}$

9. $6!\times 5!$ 10. 2880 11. 5 12. 24

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The number of permutations of n different things taking r at a time when 3 particular things are to be included is

(a) ${}^{n-3}P_{r-3}$ (b) ${}^{n-3}P_r$ (c) ${}^nP_{r-3}$ (d) $r!\,{}^{n-3}C_{r-3}$

2. The number of five-digit telephone numbers having at least one of their digits repeated is
   (a) 90000 (b) 100000 (c) 30240 (d) 69760
3. The number of words that can be formed out of the letters of the word "ARTICLE" so that vowels occupy even places is
   (a) 574 (b) 36 (c) 754 (d) 144
4. How many numbers greater than 10 lacs be formed from 2, 3, 0, 3, 4, 2, 3 ?
   (a) 420 (b) 360 (c) 400 (d) 300
5. The number of different signals which can be given from 6 flags of different colours taking one or more at a time, is
   (a) 1958 (b) 1956 (c) 16 (d) 64
6. The number of words from the letters of the word 'BHARAT' in which B and H will never come together, is
   (a) 360 (b) 240 (c) 120 (d) none of these
7. The number of six letter words that can be formed using the letters of the word "ASSIST" in which S's alternate with other letters is
   (a) 12 (b) 24 (c) 18 (d) none of these
8. The number of arrangements of the word "DELHI" in which E precedes I is
   (a) 30 (b) 60 (c) 120 (d) 59
9. The number of ways in which the letters of the word 'CONSTANT' can be arranged without changing the relative positions of the vowels and consonants is
   (a) 360 (b) 256 (c) 444 (d) none of these
10. The number of ways to arrange the letters of the word CHEESE are
    (a) 120 (b) 240 (c) 720 (d) 6
11. Number of all four digit numbers having different digits formed of the digits 1, 2, 3, 4 and 5 and divisible by 4 is
    (a) 24 (b) 30 (c) 125 (d) 100
12. If the letters of the word KRISNA are arranged in all possible ways and these words are written out as in a dictionary, then the rank of the word KRISNA is
    (a) 324 (b) 341 (c) 359 (d) none of these
13. If in a group of n distinct objects, the number of arrangements of 4 objects is 12 times the number of arrangements of 2 objects, then the number of objects is
    (a) 10 (b) 8 (c) 6 (d) none of these
14. The number of ways in which 6 men can be arranged in a row so that three particular men are consecutive, is
    (a) $4! \times 3!$ (b) $4!$ (c) $3! \times 3!$ (d) none of these
15. A 5-digit number divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways in which this can be done is
    (a) 216 (b) 600 (c) 240 (d) 3125
16. The product of r consecutive positive integers is divisible by
    (a) $r!$ (b) $r!+1$ (c) $(r+1)!$ (d) none of these
17. If ${}^{k+5}P_{k+1} = \dfrac{11(k-1)}{2} \cdot {}^{k+3}P_k$, then the values of k are

(a) 7 and 11 (b) 6 and 7 (c) 2 and 11 (d) 2 and 6

18. The number of arrangements of the letters of the word BHARAT taking 3 at a time is
(a) 72 (b) 120 (c) 14 (d) none of these.

19. The number of words that can be made by re-arranging the letters of the word APURBA so that vowels and consonants are alternate is
(a) 18 (b) 35 (c) 36 (d) none of these

20. The number of different ways in which 8 persons can stand in a row so that between two particular persons A and B there are always two persons, is
(a) $60 \times 5!$ (b) $15 \times 4! \times 5!$ (c) $4! \times 5!$ (d) none of these

21. The number of ways in which the letters of the word ARTICLE can be arranged so that even places are always occupied by consonants is
(a) 576 (b) ${}^4C_3 \times 4!$ (c) $2 \times 4!$ (d) none of these

22. In a room there are 12 bulbs of the same wattage, each having a separate switch. The number of ways to light the room with different amounts of illumination is
(a) $12^2 - 1$ (b) 2^{12} (c) $2^{12} - 1$ (d) none of these

23. There are four bus routes between A and B; and three bus routes between B and C. A man can travel round trip in number of ways by bus from A to C via B. If he does not want to use a bus route more than once, the number of ways he can make round trip, is
(a) 72 (b) 144 (c) 14 (d) 19

[NCERT EXEMPLAR]

24. All the letters of the word 'EAMCET' are arranged in different possible ways. The number of such arrangements in which two vowels are adjacent to each other, is
(a) 360 (b) 144 (c) 72 (d) 54

[NCERT EXEMPLAR]

25. The number of possible outcomes when a coin is tossed 6 times, is
(a) 36 (b) 64 (c) 12 (d) 32

[NCERT EXEMPLAR]

26. The number of different four digit numbers that can be formed with the digits 2, 3, 4, 7 and using each digit exactly once, is
(a) 120 (b) 96 (c) 24 (d) 100

[NCERT EXEMPLAR]

27. The sum of the digits in unit place of all the numbers formed with the help of 3, 4, 5 and 6 taken all at a time, is
(a) 432 (b) 108 (c) 36 (d) 18

[NCERT EXEMPLAR]

ANSWERS

1. (d)	2. (d)	3. (d)	4. (b)	5. (b)	6. (b)	7. (a)	8. (b)	9. (a)
10. (a)	11. (a)	12. (a)	13. (c)	14. (a)	15. (a)	16. (a)	17. (b)	18. (a)
19. (c)	20. (a)	21. (a)	22. (c)	23. (a)	24. (b)	25. (b)	26. (c)	27. (b)

SUMMARY

1. The continued product of first n natural numbers is called the "n factorial" and is denoted by $\lfloor n$ or $n!$.

   Thus, $n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n$

   Factorials of proper fractions and negative integers are not defined.

2. $\dfrac{(2n)!}{n!} = 1 \cdot 3 \cdot 5 \ldots (2n-1)\, 2^n$

3. $n! + 1$ is not divisible by any natural number between 2 and n.

4. Let p be a prime number and n be a natural number, if $E_p(n)$ denotes the exponent of p in n, then

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots + \left[\frac{n}{p^s}\right]$$

   where s is the largest positive integer such that $p^s \le n < p^{s+1}$ and $[x]$ denotes the greatest integer less than or equal to x.

5. If n is a natural number and r is a positive integer such that $0 \le r \le n$, then ${}^nP_r = \dfrac{n!}{(n-r)!}$.

6. (i) (*Fundamental Principle of Multiplication*): If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, second job can be completed in n ways; then the two jobs in succession can be completed in $m \times n$ ways.

   (ii) (*Fundamental Principle of Addition*) If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in (ways.

7. (i) Let r and n be positive integers such that $1 \le r \le n$. Then, the number of all permutations of n distinct items or objects taken r at a time is

$$n(n-1)(n-2)(n-3)\ldots(n-(r-1))$$

   (ii) The number of all permutations (arrangements) of n distinct objects taken all at a time is $n!$.

   (iii) The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second such that $p + q = n$, is $\dfrac{n!}{p!\,q!}$.

   (iv) The number of permutations of n things, of which p_1 are alike of one kind; p_2 are alike of second kind; p_3 are alike of third kind; ...; p_r are alike of rth kind such that $p_1 + p_2 + \ldots + p_r = n$, is $\dfrac{n!}{p_1!\,p_2!\,p_3!\ldots p_r!}$.

   (v) The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining all are distinct, is $\dfrac{n!}{p!\,q!}$.

   (vi) Suppose there are r things to be arranged, allowing repetitions. Let further $p_1, p_2, \ldots p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then the total number of permutations of these r objects to the above condition is $\dfrac{(p_1 + p_2 + \ldots + p_r)!}{p_1!\,p_2!\,p_3!\ldots p_r!}$

CHAPTER 16

COMBINATIONS

16.1 INTRODUCTION

In the previous chapter, we have studied arrangements of a certain number of objects by taking some of them or all at a time. Most of the times we are not interested in arranging the objects, but we are more concerned in selecting a number of objects from given number of objects. In other words, we do not want to specify the ordering of selected objects. For example, a company may want to select 3 persons out of 10 applicants, a student may want to choose three books from his library at a time etc.

Suppose we want to select three persons out of 4 persons A, B, C and D. We may choose A, B, C or A, B, D or A, C, D or B, C, D. Note that we have not listed A, B, C ; B, C, A; C, A, B; B, A, C; C, B, A and A, C, B separately here, because they represent the same selection A, B, C. But, they give rise to different arrangements. It is evident from the above discussion that in a selection the order in which objects are arranged is immaterial.

16.2 COMBINATIONS

COMBINATIONS *Each of the different selections made by taking some or all of a number of objects, irrespective of their arrangements is called a combination.*

ILLUSTRATION 1 *List the different combinations formed of three letters A, B, C taken two at a time.*

SOLUTION The different combinations formed of three letters A, B, C are: $AB,\ AC,\ BC$.

ILLUSTRATION 2 *Write all combinations of four letters A, B, C, D taken two at a time.*

SOLUTION Various combinations of two letters out of four letters A, B, C, D are:

$$AB,\ AC,\ AD,\ BC,\ BD,\ CD.$$

DIFFERENCE BETWEEN A PERMUTATION AND COMBINATION

(i) In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.

(ii) In a combination, the ordering of the selected objects is immaterial whereas in a permutation, the ordering is essential. For example, A, B and B, A are same as combinations but different as permutations.

(iii) Practically to find the permutations of n different items, taken r at a time, we first select r items from n items and then arrange them. So, usually the number of permutations exceeds the number of combinations.

(iv) Each combination corresponds to many permutations. For example, the six permutations ABC, ACB, BCA, BAC, CBA and CAB correspond to the same combination ABC.

REMARK *Generally we use the word 'arrangements' for permutations and the word 'selections' for combinations.*

NOTATION *The number of all combinations of n objects, taken r at a time is generally denoted by $C(n, r)$ or, ${}^nC_r$ or, $\binom{n}{r}$.*

Thus, ${}^nC_r$ or $C(n, r)$ = Number of ways of selecting r objects from n objects.

Clearly, ${}^nC_r$ is defined only when n and r are non-negative integers such that $0 \le r \le n$.

THEOREM *The number of all combinations of n distinct objects, taken r at a time is given by*

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

PROOF Let the number of combinations of n distinct objects taken r at a time be x. Consider one of these x ways. There are r objects in this selection which can be arranged in $r!$ ways. Thus, each of the x combinations gives rise to $r!$ permutations. So, x combinations will give rise to $x \times (r!)$ permutations. Consequently, the number of permutations of n things, taken r at a time is $x \times (r!)$. But, this number is also equal to ${}^nP_r$.

$$\therefore \quad x\,(r!) = {}^nP_r$$

$$\Rightarrow \quad x = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\,r!} \qquad \left[\because\ {}^nP_r = \frac{n!}{(n-r)!}\right]$$

$$\Rightarrow \quad {}^nC_r = \frac{n!}{(n-r)!\,r!}$$

Q.E.D.

REMARK 1 *We have,*

$${}^nC_r = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)(n-2)\dots(n-r+1)(n-r)(n-r-1)\dots 3.2.1}{\{(n-r)(n-r-1)\dots 3.2.1\}\{1.2.3\dots.r\}}$$

$$\Rightarrow \quad {}^nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots.r}$$

Sometimes this form of ${}^nC_r$ is also very convenient to use.

REMARK 2 *We have,* ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$. Putting $r = n$ and $r = 0$ successively, we obtain

$${}^nC_n = \frac{n!}{(n-n)!\,n!} = \frac{n!}{n!\,0!} = 1 \text{ and, } {}^nC_0 = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!} = 1 \Rightarrow {}^nC_n = {}^nC_0 = 1. \qquad [\because\ 0! = 1]$$

REMARK 3 ${}^nC_r = \dfrac{n!}{(n-r)!\,r!} = \dfrac{1}{r!}\left(\dfrac{n!}{(n-r)!}\right) = \dfrac{{}^nP_r}{r!}$ i.e. ${}^nC_r \times r! = {}^nP_r$.

16.3 PROPERTIES OF ${}^nC_r$ OR, $C(n, r)$

In this section, we shall discuss some important properties of ${}^nC_r$.

PROPERTY 1 *For $0 \le r \le n$, we have ${}^nC_r = {}^nC_{n-r}$.*

PROOF We find that

$${}^nC_{n-r} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\,r!} = {}^nC_r$$

REMARK 1 *The use of this property simplifies the calculation of ${}^nC_r$ when r is large.*

For example, if we want to calculate ${}^{20}C_{19}$, by using this property, we get

$${}^{20}C_{19} = {}^{20}C_{20-19} = {}^{20}C_1 = 20.$$

REMARK 2 *The above property can be restated as follows:*

If x and y are non-negative integers such that $x + y = n$, then ${}^nC_x = {}^nC_y$

This can also be stated as : ${}^nC_x = {}^nC_y \Rightarrow x = y$, or $x + y = n$

ILLUSTRATION 1 *If ${}^nC_7 = {}^nC_4$, find the value of n.*

SOLUTION We know that : ${}^nC_x = {}^nC_y \Leftrightarrow x + y = n$ or $x = y$.

$$\therefore \quad {}^nC_7 = {}^nC_4 \Rightarrow n = 7 + 4 = 11$$

PROPERTY 2 *Let n and r be non-negative integers such that $r \le n$. Then, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.*

PROOF We have,

$$
\begin{aligned}
{}^nC_r + {}^nC_{r-1} &= \frac{n!}{(n-r)!\,r!} + \frac{n!}{\{n-(r-1)\}!\,(r-1)!} = \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!} \\
&= \frac{n!}{(n-r)!\,r\,(r-1)!} + \frac{n!}{(n-r+1)\,(n-r)!\,(r-1)!} = \frac{n!}{(n-r)!\,(r-1)!}\left\{\frac{1}{r} + \frac{1}{n-r+1}\right\} \\
&= \frac{n!}{(n-r)!\,(r-1)!}\left\{\frac{n-r+1+r}{r\,(n-r+1)}\right\} = \frac{n!\,(n+1)}{(n-r)!\,(r-1)!\,r\,(n-r+1)} \\
&= \frac{(n+1)\,n!}{(n-r+1)\,(n-r)!\,r\,(r-1)!} = \frac{(n+1)!}{(n-r+1)!\,r!} = \frac{(n+1)!}{((n+1)-r)!\,r!} = {}^{n+1}C_r.
\end{aligned}
$$

REMARK 3 *This property is known as Pascal's rule and it can also be proved by giving combinatorial arguments.*

ILLUSTRATION 2 *Find the value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$.*

SOLUTION ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$

$$
\begin{aligned}
&= {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3 \\
&= {}^{47}C_3 + {}^{47}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 \\
&= ({}^{47}C_3 + {}^{47}C_4) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 \\
&= {}^{48}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 && [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r] \\
&= ({}^{48}C_3 + {}^{48}C_4) + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 && [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r] \\
&= {}^{49}C_4 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 \\
&= ({}^{49}C_3 + {}^{49}C_4) + {}^{50}C_3 + {}^{51}C_3 && [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r] \\
&= {}^{50}C_4 + {}^{50}C_3 + {}^{51}C_4 \\
&= ({}^{50}C_3 + {}^{50}C_4) + {}^{51}C_3 && [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r] \\
&= {}^{51}C_4 + {}^{51}C_3 = {}^{52}C_4 && [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]
\end{aligned}
$$

PROPERTY 3 *Let n and r be non-negative integers such that $1 \le r \le n$. Then, ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$.*

PROOF We find that

$$
{}^nC_r = \frac{n!}{(n-r)!\,r!} = \frac{n\,(n-1)!}{\{(n-1)-(r-1)\}!\,r\,(r-1)!} = \frac{n}{r}\,\frac{(n-1)!}{\{(n-1)-(r-1)\}!\,(r-1)!} = \frac{n}{r}\;{}^{n-1}C_{r-1}
$$

REMARK 4 *This property is very useful to find the value of ${}^nC_r$.*

For example, ${}^{10}C_3 = \frac{10}{3} \times {}^9C_2 = \frac{10}{3} \times \frac{9}{2} \times {}^8C_1 = \frac{10}{3} \times \frac{9}{2} \times \frac{8}{1}\,{}^7C_0 = \frac{10}{3} \times \frac{9}{2} \times \frac{8}{1} \times 1 = 120$

REMARK 5 *By using the above property, we find that* ${}^nC_r = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot \frac{n-2}{r-2} \cdots \frac{n-(r-2)}{2} \cdot \frac{n-(r-1)}{1}$

For example, ${}^9C_4 = \frac{9}{4} \times \frac{8}{3} \times \frac{7}{2} \times \frac{6}{1} = 126.$

PROPERTY 4 If $1 \le r \le n$, then $n\,{}^{n-1}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$.

PROOF We have,

$$n\,.\,{}^{n-1}C_{r-1} = n\cdot\frac{(n-1)!}{\{(n-1)-(r-1)\}!\,(r-1)!} = \frac{n!}{(n-r)!\,(r-1)!} = \frac{(n-r+1)\cdot n!}{(n-r+1)\,(n-r)!\,(r-1)!}$$

$$= (n-r+1)\left\{\frac{n!}{(n-r+1)!\,(r-1)!}\right\} = (n-r+1)\left\{\frac{n!}{\{n-(r-1)\}!\,(r-1)!}\right\}$$

$$= (n-r+1)\,{}^nC_{r-1}$$

PROPERTY 5 ${}^nC_x = {}^nC_y \Rightarrow x = y$ *or*, $x + y = n$.

PROOF We have,

$${}^nC_x = {}^nC_y$$

$$\Rightarrow \quad {}^nC_x = {}^nC_y = {}^nC_{n-y} \qquad [\because\ {}^nC_y = {}^nC_{n-y}]$$

$$\Rightarrow \quad x = y \quad \text{or} \quad x = n - y \Rightarrow x = y \quad \text{or} \quad x + y = n.$$

REMARK 6 *If ${}^nC_x = {}^nC_y$ and $x \ne y$, then $x + y = n$.*

ILLUSTRATION 3 *If ${}^nC_{15} = {}^nC_8$, find the value of ${}^nC_{21}$.*

SOLUTION We have,

$${}^nC_{15} = {}^nC_8 \Rightarrow n = (15+8) = 23 \qquad [{}^nC_x = {}^nC_y \Rightarrow x + y = n]$$

$$\therefore \quad {}^nC_{21} = {}^{23}C_{21} = {}^{23}C_{23-21} \qquad [\because\ {}^nC_r = {}^nC_{n-r}]$$

$$= {}^{23}C_2 = \frac{23}{2}\times\frac{22}{1}\times{}^{21}C_0 \qquad \left[\because\ {}^nC_r = \frac{n}{r}\times\frac{n-1}{r-1}\times{}^{n-2}C_{r-2}\right]$$

$$= \frac{23}{2}\times\frac{22}{1}\times 1 = 23\times 11 = 253 \qquad [\because\ {}^nC_0 = 1]$$

ILLUSTRATION 4 *If ${}^{10}C_x = {}^{10}C_{x+4}$, find the value of x.*

SOLUTION We have, ${}^{10}C_x = {}^{10}C_{x+4} \Rightarrow x + x + 4 = 10 \Rightarrow 2x = 6 \Rightarrow x = 3$.

PROPERTY 6 *If n is an even natural number, then the greatest of the values ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ is ${}^nC_{n/2}$*

If n is an odd natural number, then the greatest of the values ${}^nC_0, {}^nC_1, {}^nC_2, \dots {}^nC_n$ is ${}^nC_{n-1/2} = {}^nC_{n+1/2}$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate the following: (i) ${}^{10}C_8$ (ii) ${}^{100}C_{98}$ (iii) ${}^{52}C_{52}$*

SOLUTION (i) ${}^{10}C_8 = {}^{10}C_{10-8}$ $\qquad [\because\ {}^nC_r = {}^nC_{n-r}]$

$$= {}^{10}C_2 = \frac{10}{2}\times\frac{9}{1}\times{}^8C_0 \qquad \left[\because\ {}^nC_r = \frac{n}{r}\,.\,{}^{n-1}C_{r-1}\right]$$

$$= \frac{10}{2}\times\frac{9}{1}\times 1 = 45 \qquad [\because\ {}^nC_0 = 1]$$

(ii) ${}^{100}C_{98} = {}^{100}C_{100-98}$ $\qquad [\because\ {}^nC_r = {}^nC_{n-r}]$

$$= {}^{100}C_2 = \frac{100}{2} \times \frac{99}{1} \times {}^{98}C_0 \qquad \left[\because {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right]$$

$$= \frac{100}{2} \times \frac{99}{1} \times 1 = 4950 \qquad [\because {}^nC_0 = 1]$$

(iii) $\quad {}^{52}C_{52} = 1 \qquad [\because {}^nC_n = 1]$

EXAMPLE 2 *If* ${}^nC_8 = {}^nC_6$, *find* ${}^nC_2$

SOLUTION If ${}^nC_x = {}^nC_y$ and $x \neq y$, then $x + y = n$. Therefore, ${}^nC_8 = {}^nC_6 \Rightarrow n = (8+6) = 14$

$$\therefore \quad {}^nC_2 = {}^{14}C_2 = \frac{14}{2} \times \frac{13}{1} \times {}^{12}C_0 \qquad \left[\because {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right]$$

$$= \frac{14}{2} \times \frac{13}{1} \times 1 = 91 \qquad [\because {}^nC_0 = 1]$$

EXAMPLE 3 *If* ${}^nP_r = 720$ and ${}^nC_r = 120$, find r.

SOLUTION We know that

$${}^nC_r = \frac{{}^nP_r}{r!} \Rightarrow 120 = \frac{720}{r!} \Rightarrow r! = 6 \Rightarrow r! = 3! \Rightarrow r = 3.$$

EXAMPLE 4 *If the ratio* ${}^{2n}C_3 : {}^nC_3$ *is equal to* $11:1$, *find* n.

SOLUTION We have,

$${}^{2n}C_3 : {}^nC_3 = 11:1$$

$$\Rightarrow \quad \frac{{}^{2n}C_3}{{}^nC_3} = \frac{11}{1} \Rightarrow \frac{\dfrac{(2n)!}{(2n-3)!\,3!}}{\dfrac{n!}{(n-3)!\,(3!)}} = \frac{11}{1} \Rightarrow \frac{(2n)!}{(2n-3)!} \times \frac{(n-3)!}{n!} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{(2n)(2n-1)(2n-2)(2n-3)!}{(2n-3)!} \times \frac{(n-3)!}{n(n-1)(n-2)(n-3)!} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{(2n)(2n-1)(2n-2)}{n(n-1)(n-2)} = \frac{11}{1} \Rightarrow \frac{4(2n-1)}{n-2} = \frac{11}{1} \Rightarrow 8n - 4 = 11n - 22 \Rightarrow 3n = 18 \Rightarrow n = 6$$

EXAMPLE 5 *Prove that:* ${}^{2n}C_n = \dfrac{2^n \{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)\}}{n!}$.

SOLUTION We have,

$${}^{2n}C_n = \frac{2n!}{(2n-n)!\,n!} = \frac{(2n)!}{n!\,n!} = \frac{(2n)(2n-1)(2n-2)\ldots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{n!\,n!}$$

$$= \frac{\{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)\}\{2 \cdot 4 \cdot 6 \cdot \ldots 2n\}}{n!\,n!} = \frac{\{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)\} \times 2^n \{1 \cdot 2 \cdot 3 \cdot \ldots n\}}{n!\,n!}$$

$$= \frac{\{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)\} \times 2^n \times n!}{n!\,n!} = 2^n \frac{\{1 \cdot 3 \cdot 5 \ldots (2n-1)\}}{n!}$$

EXAMPLE 6 *If* ${}^{n+2}C_8 : {}^{n-2}P_4 = 57:16$, *find* n.

SOLUTION We have,

$${}^{n+2}C_8 : {}^{n-2}P_4 = 57:16$$

$$\Rightarrow \quad \frac{{}^{n+2}C_8}{{}^{n-2}P_4} = \frac{57}{16}$$

$$\Rightarrow \quad \frac{(n+2)!}{8!\,(n-6)!} \times \frac{(n-6)!}{(n-2)!} = \frac{57}{16}$$

$$\Rightarrow \quad \frac{(n+2)(n+1)\,n\,(n-1)(n-2)!}{8!} \times \frac{1}{(n-2)!} = \frac{57}{16}$$

$$\Rightarrow \quad (n+2)(n+1)\,n\,(n-1) = \frac{57}{16} \times 8! = \frac{19 \times 3}{16} \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$\Rightarrow \quad (n+2)(n+1)(n-1)\,n = 143640$$

$$\Rightarrow \quad (n-1)\,n\,(n+1)(n+2) = 19 \times 3 \times 7 \times 6 \times 5 \times 4 \times 3$$

$$\Rightarrow \quad (n-1)\,n\,(n+1)(n+2) = 19 \times (3 \times 7) \times (6 \times 3) \times (4 \times 5)$$

$$\Rightarrow \quad (n-1)\,n\,(n+1)(n+2) = 18 \times 19 \times 20 \times 21 \Rightarrow n-1 = 18 \Rightarrow n = 19$$

EXAMPLE 7 *If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, then find ${}^rC_2$.*

SOLUTION It is given that ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$

$$\therefore \quad \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{84}{126}$$

$$\Rightarrow \quad \frac{r+1}{n-r} = \frac{2}{3} \qquad \left[\because \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r}\right]$$

$$\Rightarrow \quad 2n - 5r = 3 \qquad \text{...(i)}$$

Replacing r by $(r-1)$ in $\dfrac{{}^nC_r}{{}^nC_{r+1}} = \dfrac{r+1}{n-r}$, we get

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{r}{n-(r-1)}$$

$$\Rightarrow \quad \frac{36}{84} = \frac{r}{n-r+1} \qquad \left[\because {}^nC_{r-1} = 36 \text{ and } {}^nC_r = 84\right]$$

$$\Rightarrow \quad \frac{r}{n-r+1} = \frac{3}{7} \Rightarrow 3n - 10r = -3 \qquad \text{...(ii)}$$

Solving (i) and (ii), we get $r = 3$.

$$\therefore \quad {}^rC_2 = {}^3C_2 = \frac{3!}{(3-2)!\,2!} = 3.$$

NOTE *Students are advised to learn that* $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$ *as it is a very useful result.*

EXAMPLE 8 *If ${}^nP_r = {}^nP_{r+1}$ and ${}^nC_r = {}^nC_{r-1}$, find the values of n and r.*

SOLUTION We have,

$${}^nP_r = {}^nP_{r+1}$$

$$\Rightarrow \quad \frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!} \Rightarrow \frac{1}{(n-r)(n-r-1)!} = \frac{1}{(n-r-1)!} \Rightarrow n-r = 1 \qquad \text{...(i)}$$

and, $\quad {}^nC_r = {}^nC_{r-1}$

$$\Rightarrow \quad \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-r+1)!\,(r-1)!} \Rightarrow \frac{n!}{(n-r)!\,r\,(r-1)!} = \frac{n!}{(n-r+1)(n-r)!\,(r-1)!}$$

$$\Rightarrow \quad \frac{1}{r} = \frac{1}{n-r+1} \Rightarrow n-r+1 = r \Rightarrow n-2r = -1 \qquad \text{...(ii)}$$

Solving (i) and (ii), we obtain $n = 3$ and $r = 2$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 9 *Prove that the product of r consecutive positive integers is divisible by r!.*

SOLUTION Let the r consecutive positive integers be $(n+1), (n+2), (n+3), ..., (n+r)$. Then,

$$\text{Product} = (n+1)(n+2)(n+3)\dots(n+r) = \frac{n!(n+1)(n+2)(n+3)\dots(n+r)}{n!}$$

$$= \frac{1\cdot 2\cdot 3\cdot \dots n\cdot(n+1)(n+2)\dots(n+r)}{n!} = \frac{(n+r)!}{n!} = \frac{(n+r)!}{r!\{(n+r)-r\}!}(r!)$$

$$= \left({}^{n+r}C_r\right) r!\text{, which is divisible by } r! \qquad [\because {}^{n+r}C_r \text{ is an integer}]$$

EXERCISE 16.1

BASIC

1. Evaluate the following:

   (i) ${}^{14}C_3$ (ii) ${}^{12}C_{10}$ (iii) ${}^{35}C_{35}$ (iv) ${}^{n+1}C_n$ (v) $\sum_{r=1}^{5} {}^5C_r$

2. If ${}^nC_{12} = {}^nC_5$, find the value of n.
3. If ${}^nC_4 = {}^nC_6$, find ${}^{12}C_n$.
4. If ${}^nC_{10} = {}^nC_{12}$, find ${}^{23}C_n$.
5. If ${}^{24}C_x = {}^{24}C_{2x+3}$, find x.
6. If ${}^{18}C_x = {}^{18}C_{x+2}$, find x.
7. If ${}^{15}C_{3r} = {}^{15}C_{r+3}$, find r.
8. If ${}^8C_r - {}^7C_3 = {}^7C_2$, find r.

BASED ON LOTS

9. If ${}^{15}C_r : {}^{15}C_{r-1} = 11:5$, find r.
10. If ${}^{n+2}C_8 : {}^{n-2}P_4 = 57:16$, find n.
11. If ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225:11$, find r.
12. If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P., then find n.
13. If ${}^{2n}C_3 : {}^nC_2 = 44:3$, find n.
14. If ${}^{16}C_r = {}^{16}C_{r+2}$, find ${}^rC_4$.
15. If $\alpha = {}^mC_2$, then find the value of ${}^\alpha C_2$.

BASED ON HOTS

16. Prove that the product of $2n$ consecutive negative integers is divisible by $(2n)!$
17. For all positive integers n, show that ${}^{2n}C_n + {}^{2n}C_{n-1} = \frac{1}{2}({}^{2n+2}C_{n+1})$.
18. Prove that: ${}^{4n}C_{2n} : {}^{2n}C_n = [1\cdot 3\cdot 5\dots(4n-1)] : [1\cdot 3\cdot 5\dots(2n-1)]^2$.
19. Evaluate ${}^{20}C_5 + \sum_{r=2}^{5} {}^{25-r}C_4$.
20. Let r and n be positive integers such that $1 \le r \le n$. Then prove the following:

(i) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$ (ii) $n\,{}^{n-1}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$

(iii) $\dfrac{{}^nC_r}{{}^{n-1}C_{r-1}} = \dfrac{n}{r}$ (iv) ${}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$.

ANSWERS

1. (i) 364 (ii) 66 (iii) 1 (iv) $(n+1)$ (v) 31 2. 17 3. 66
4. 23 5. 7 6. 8 7. 3 8. 3, 5 9. 5 10. 19 11. 7
12. 14, 7 13. 6 14. 35 15. $\dfrac{(m+1)(m)(m-1)(m-2)}{8}$ 19. 42504

HINTS TO SELECTED PROBLEMS

16. Let $(-r), (-r-1), (-r-2), \dots, (-r-2n+1)$ be $2n$ consecutive negative integers. Then, their product P is given by

$$P = (-1)^{2n}\, r(r+1)(r+2)\dots(r+2n-1) = \frac{(r-1)!(r)(r+1)\dots(r+2n-1)}{(r-1)!}$$

$$\Rightarrow\ P = \frac{(r+2n-1)!}{(r-1)!} = \frac{(r+2n-1)!}{(r-1)!(2n)!}(2n)! = \left({}^{r+2n-1}C_{2n}\right)(2n)!$$

Clearly, P is divisible by $(2n)!$

16.4 PRACTICAL PROBLEMS ON COMBINATIONS

In this section, we intend to discuss some problems in real life where the formula for ${}^nC_r$ and its meaning can be applied.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *From a class of 32 students, 4 are to be chosen for a competition. In how many ways can this be done ?*

SOLUTION Out of 32 students, 4 students can be selected in ${}^{32}C_4$ ways.

$\therefore$ Required number of ways ${}^{32}C_4 = \dfrac{32!}{28!\,4!}$.

EXAMPLE 2 *Three gentlemen and three ladies are candidates for two vacancies. A voter has to vote for two candidates. In how many ways can one cast his vote ?*

SOLUTION Clearly, there are 6 candidates and a voter has to vote for any two of them. So, the required number of ways is the number of ways of selecting 2 out of 6 i.e. ${}^6C_2$.

Hence, the required number of ways $= {}^6C_2 = \dfrac{6!}{2!\,4!} = 15$.

EXAMPLE 3 *If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party?*

SOLUTION It is to note here that, when two persons shake hands, it is counted as one handshake, not two. So, this is a problem on combinations.

The total number of handshakes is same as the number of ways of selecting 2 persons among 12 persons i.e. ${}^{12}C_2 = \dfrac{12!}{10! \times 2!} = 66$.

EXAMPLE 4 *A question paper has two parts, Part A and Part B, each containing 10 questions. If a student has to choose 8 from Part A and 5 from Part B, in how many ways can he choose the questions ?*

SOLUTION There are 10 questions in Part A out of which 8 questions can be chosen in ${}^{10}C_8$ ways. Similarly, 5 questions can be chosen from part B containing 10 questions in ${}^{10}C_5$ ways.

Hence, the total number of ways of selecting 8 questions from part A and 5 from part B

$$= {}^{10}C_8 \times {}^{10}C_5 = \frac{10!}{8!\,2!} \times \frac{10!}{5! \times 5!} = 11340.$$

EXAMPLE 5 *In how many ways a committee of 5 members can be selected from 6 men and 5 women, consisting of 3 men and 2 women ?*

SOLUTION Three men out of 6 men can be selected in ${}^6C_3$ ways. Two women out of 5 women can be selected in ${}^5C_2$ ways. Therefore, by the fundamental principle of counting, 3 men out of 6 men and 2 women out of 5 women can be selected in

$${}^6C_3 \times {}^5C_2 = \left(\frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = 200 \text{ ways.}$$

EXAMPLE 6 *In how many ways can a cricket eleven be chosen out of a batch of 15 players if*

(i) there is no restriction on the selection? *(ii) a particular player is always chosen?*

(iii) a particular player is never chosen?

SOLUTION (i) The total number of ways of selecting 11 players out of 15 is

$${}^{15}C_{11} = {}^{15}C_{15-11} = {}^{15}C_4 = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = 1365$$

(ii) If a particular player is always chosen. This means that 10 players are selected out of the remaining 14 players.

$\therefore$ Required number of ways $= {}^{14}C_{10} = {}^{14}C_{14-10} = {}^{14}C_4 = 1001$

(iii) If a particular player is never chosen. This means that 11 players are selected out of the remaining 14 players.

$\therefore$ Required number of ways $= {}^{14}C_{11} = {}^{14}C_{14-11} = {}^{14}C_3 = 364$

EXAMPLE 7 *A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if at least five women have to be included in a committee ? In how many of these committees (i) the women are in majority (ii) the men are in majority ?*

SOLUTION There are 9 women and 8 men. A committee of 12, consisting of at least 5 women, can be formed by choosing :

(i) 5 women and 7 men (ii) 6 women and 6 men (iii) 7 women and 5 men
(iv) 8 women and 4 men (v) 9 women and 3 men

$\therefore$ Total number of ways of forming the committee

$$= {}^9C_5 \times {}^8C_7 + {}^9C_6 \times {}^8C_6 + {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3$$

$$= 126 \times 8 + 84 \times 28 + 36 \times 56 + 9 \times 70 + 1 \times 56 = 6062$$

Clearly, women are in majority in (iii), (iv) and (v) cases as discussed above.

So, total number of committees in which women are in majority

$$= {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3 = 36 \times 56 + 9 \times 70 + 1 \times 56 = 2702$$

Clearly, men are in majority in only (i) case as discussed above.

So, total number of committees in which men are in majority = ${}^9C_5 \times {}^8C_7 = 126 \times 8 = 1008$.

EXAMPLE 8 *A committee of three persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?* [NCERT]

SOLUTION There are 5 persons (2 men and 3 women). In order consitute a committee of 3 persons we need to select three persons out of given 5 persons. This can be done in ${}^5C_3$ ways.

So, the committee can be formed in ${}^5C_3 = \dfrac{5!}{3!\,2!} = 10$ ways.

Now, 1 man can be selected from 2 men in ${}^2C_1$ ways and 2 women can be selected from 3 women in ${}^3C_2$ ways.

Therefore, required number of committees is ${}^2C_1 \times {}^3C_2 = 2 \times 3 = 6$

EXAMPLE 9 *What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these*

(i) *four cards are of the same suit?*
(ii) *four cards belong to four different suits?*
(iii) *four cards are face cards?*
(iv) *two are red cards and two are black cards?*
(v) *cards are of the same colour?* [NCERT]

SOLUTION Four cards can be chosen from 52 playing cards in ${}^{52}C_4$ ways.

Now, $${}^{52}C_4 = \frac{52!}{48!\,4!} = \frac{49 \times 50 \times 51 \times 52}{2 \times 3 \times 4} = 270725$$

Hence, required number of ways = 270725

(i) There are four suits (diamond, spade, club and heart) of 13 cards each. Therefore, there are ${}^{13}C_4$ ways of choosing 4 diamond cards, ${}^{13}C_4$ ways of choosing 4 club cards, ${}^{13}C_4$ ways of choosing 4 spade cards and ${}^{13}C_4$ ways of choosing heart cards.

$\therefore$ Required number of ways $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 = 4 \times {}^{13}C_4 = 4 \times \dfrac{13!}{9!\,4!} = 2860$

(ii) There are 13 cards in each suit. Four cards drawn belong to four different suits means one card is drawn from each suit. Out of 13 diamond cards one card can be drawn in ${}^{13}C_1$ ways. Similarly, there are ${}^{13}C_1$ ways of choosing one club card, ${}^{13}C_1$ ways of choosing one spade card and ${}^{13}C_1$ ways of choosing one heart card.

$\therefore$ Number of ways of selecting one card from each suit $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$

(iii) There are 12 face cards out of which 4 cards can be chosen in ${}^{12}C_4$ ways.

$\therefore$ Required number of ways $= {}^{12}C_4 = \dfrac{12!}{4!\,8!} = 495$

(iv) There are 26 red cards and 26 black cards. Therefore, 2 red cards can be chosen in ${}^{26}C_2$ ways and 2 black cards can be chosen in ${}^{26}C_2$ ways. Hence, 2 red and 2 black cards can be chosen in

$${}^{26}C_2 \times {}^{26}C_2 = \left(\frac{26!}{24!\,2!}\right)^2 = (325)^2 = 105625 \text{ ways.}$$

(v) Out of 26 red cards, 4 red cards can be chosen in ${}^{26}C_4$ ways. Similarly, 4 black cards can be chosen in $26C_4$ ways.

Hence, 4 red or 4 black cards can be chosen in ${}^{26}C_4 + {}^{26}C_4 = 2 \times {}^{26}C_4 = 2 \times \dfrac{26!}{4!\,22!} = 29900$ ways.

EXAMPLE 4 *A question paper has two parts, Part A and Part B, each containing 10 questions. If a student has to choose 8 from Part A and 5 from Part B, in how many ways can he choose the questions ?*

SOLUTION There are 10 questions in Part A out of which 8 questions can be chosen in $^{10}C_8$ ways. Similarly, 5 questions can be chosen from part B containing 10 questions in $^{10}C_5$ ways.

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(ii) There are 13 cards in each suit. Four cards drawn belong to four different suits means one card is drawn from each suit. Out of 13 diamond cards one card can be drawn in ${}^{13}C_1$ ways. Similarly, there are ${}^{13}C_1$ ways of choosing one club card, ${}^{13}C_1$ ways of choosing one spade card and ${}^{13}C_1$ ways of choosing one heart card.

$\therefore$ Number of ways of selecting one card from each suit $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$

(iii) There are 12 face cards out of which 4 cards can be chosen in ${}^{12}C_4$ ways.

$\therefore$ Required number of ways $= {}^{12}C_4 = \dfrac{12!}{4!\,8!} = 495$

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Hence, 4 red or 4 black cards can be chosen in ${}^{26}C_4 + {}^{26}C_4 = 2 \times {}^{26}C_4 = 2 \times \dfrac{26!}{4!\,22!} = 29900$ ways.

EXAMPLE 10 *Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is to be included ?*

SOLUTION The committee can be formed in the following ways:

(i) By selecting 2 men and 1 woman (ii) By selecting 1 man and 2 women

Now, 2 men out of 5 men and 1 woman out of 2 woman can be chosen in ${}^5C_2 \times {}^2C_1$ ways.

And, 1 man out of 5 men and 2 women out of 2 women can be chosen in ${}^5C_1 \times {}^2C_2$ ways.

$\therefore$ Total number of ways of forming the committee = ${}^5C_2 \times {}^2C_1 + {}^5C_1 \times {}^2C_2 = 20 + 5 = 25.$

EXAMPLE 11 *In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicket keepers ? Assume that the team of 11 players requires 5 batsmen, 3 all-rounder, 2 bowlers and 1 wicket keeper.*

SOLUTION The selection of team is divided into four phases:

(i) Selection of 5 batsmen out of 10. This can be done in ${}^{10}C_5$ ways.

(ii) Selection of 3 all-rounders out of 5. This can be done in ${}^5C_3$ ways.

(iii) Selection of 2 bowlers out of 8. This can be done in ${}^8C_2$ ways.

(iv) Selection of one wicket keeper out of 2. This can be done in ${}^2C_1$ ways.

The selection of team is completed by completing all the four phases.

$\therefore$ The team can be selected in ${}^{10}C_5 \times {}^5C_3 \times {}^8C_2 \times {}^2C_1 = 141120$ ways.

EXAMPLE 12 *A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be done, when*

(i) at least two ladies are included? *(ii) at most two ladies are included ?*

SOLUTION (i) A committee of 5 persons, consisting of at least two ladies, can be formed in the following ways:

I Selecting 2 ladies out of 4 and 3 gents out of 6. This can be done in ${}^4C_2 \times {}^6C_3$ ways.

II Selecting 3 ladies out of 4 and 2 gents out of 6. This can be done in ${}^4C_3 \times {}^6C_2$ ways.

III Selecting 4 ladies out of 4 and 1 gent out of 6. This can be done in ${}^4C_4 \times {}^6C_1$ ways.

Since the committee is formed in each case. Therefore, by the fundamental principle of addition,

$$\text{The total number of ways of forming the committee} = {}^4C_2 \times {}^6C_3 + {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1$$
$$= 120 + 60 + 6 = 186$$

(ii) A committee of 5 persons, consisting of at most two ladies, can be constituted in the following ways :

I Selecting 5 gents only out of 6. This can be done in ${}^6C_5$ ways.

II Selecting 4 gents only out of 6 and one lady out of 4. This can be done in ${}^6C_4 \times {}^4C_1$ ways.

III Selecting 3 gents only out of 6 and two ladies out of 4. This can be done is ${}^6C_3 \times {}^4C_2$ ways.

Since the committee is formed in each case. So, the total number of ways of forming the committee $= {}^6C_5 + {}^6C_4 \times {}^4C_1 + {}^6C_3 \times {}^4C_2 = 6 + 60 + 120 = 186.$

EXAMPLE 13 *A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour ?*

SOLUTION The selection of 6 balls, consisting of at least two balls of each colour from 5 red and 6 white balls, can be made in the following ways :

(i) By selecting 2 red balls out of 5 and 4 white balls out of 6. This can be done in ${}^5C_2 \times {}^6C_4$ ways.

(ii) By selecting 3 red balls out of 5 and 3 white balls out of 6. This can be done in ${}^5C_3 \times {}^6C_3$ ways.

(iii) By selecting 4 red balls out of 5 and 2 white balls out of 6. This can be done in ${}^5C_4 \times {}^6C_2$ ways.

Since the selection of 6 balls can be completed in any one of the above ways.

Hence, by the fundamental principle of addition, the total number of ways to select the balls

$$= {}^5C_2 \times {}^6C_4 + {}^5C_3 \times {}^6C_3 + {}^5C_4 \times {}^6C_2 = 10 \times 15 + 10 \times 20 + 5 \times 15 = 425.$$

EXAMPLE 14 *For the post of 5 teachers, there are 23 applicants, 2 posts are reserved for SC candidates and there are 7 SC candidates among the applicants. In how many ways can the selection be made ?*

SOLUTION Clearly, there are 7 SC candidates and 16 other candidates. We have to select 2 out of 7 SC candidates and 3 out of remaining 21 candidates. This can be done in ${}^7C_2 \times {}^{21}C_3$ ways.

$\therefore$ The number of ways of making the selection $= {}^7C_2 \times {}^{21}C_3 = 27930.$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 15 *How many triangles can be formed by joining the vertices of a hexagon ?*

SOLUTION There are 6 vertices of a hexagon. One triangle is formed by selecting a group of 3 vertices from given 6 vertices. This can be done in ${}^6C_3$ ways.

$\therefore$ Number of triangles $= {}^6C_3 = \dfrac{6!}{3!\,3!} = 20.$

EXAMPLE 16 *How many diagonals are there in a polygon with n sides ?*

SOLUTION A polygon of n sides has n vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon. Number of line segments obtained by joining the vertices of an n sided polygon taken two at a time

$$= \text{Number of ways of selecting 2 out of } n = {}^nC_2 = \frac{n(n-1)}{2}$$

Out of these lines, n lines are the sides of the polygon.

$\therefore$ Number of diagonals of the polygon $= \dfrac{n(n-1)}{2} - n = \dfrac{n(n-3)}{2}.$

EXAMPLE 17 *A polygon has 44 diagonals. Find the number of its sides.*

SOLUTION Let there be n sides of the polygon. We know that the number of diagonals of n sided polygon is $\dfrac{n(n-3)}{2}$.

$\therefore$ $\dfrac{n(n-3)}{2} = 44 \Rightarrow n^2 - 3n - 88 = 0 \Rightarrow (n-11)(n+8) = 0 \Rightarrow n = 11$ $(\because n > 0)$

Hence, there are 11 sides of the polygon.

EXAMPLE 18 *How many chords can be drawn through 21 points on a circle?*

SOLUTION A chord is obtained by joining any two points on a circle. Therefore, total number of chords drawn through 21 points is same as the number of ways of selecting 2 points out of 21 points. This can be done in ${}^{21}C_2$ ways.

Hence, total number of chords $= {}^{21}C_2 = \dfrac{21!}{19!\,2!} = 21 \times 10 = 210.$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 19 *A person wishes to make up as many different parties as he can out of his 20 friends such that each party consists of the same number of persons. How many friends should he invite ?*

SOLUTION Suppose he invites r friends at a time. Then the total number of parties is ${}^{20}C_r$. We have to find the maximum value of ${}^{20}C_r$ which is for $r = 10$, because ${}^{n}C_r$ is maximum for $r = n/2$, when n is even.

Hence, he should invite 10 friends at a time in order to form the maximum number of parties.

EXAMPLE 20 *If m parallel lines in plane are intersected by a family of n parallel lines. Find the number of parallelograms formed.*

SOLUTION A parallelogram is formed by choosing two straight lines from the set of m parallel lines and two straight lines from the set of n parallel lines.

Two straight lines from the set of m parallel lines can be chosen in ${}^{m}C_2$ ways and two straight lines from the set of n parallel lines can be chosen in ${}^{n}C_2$ ways.

Hence, the number of parallelograms formed $= {}^{m}C_2 \times {}^{n}C_2$

$$= \frac{m(m-1)}{2} \times \frac{n(n-1)}{2} = \frac{mn(m-1)(n-1)}{4}$$

EXAMPLE 21 *There are 10 points in a plane, no three of which are in the same straight line, excepting 4 points, which are collinear. Find the (i) number of straight lines obtained from the pairs of these points; (ii) number of triangles that can be formed with the vertices as these points.*

SOLUTION (i) Number of straight lines formed joining the 10 points, taking 2 at a time $= {}^{10}C_2$

$$= \frac{10!}{2!\,8!} = 45.$$

Number of straight lines formed by joining the four points, taking 2 at a time $= {}^{4}C_2 = \frac{4!}{2!\,2!} = 6$

But, 4 collinear points, when joined pairwise give only one line.

$\therefore$ Required number of straight lines $= 45 - 6 + 1 = 40$.

(ii) Number of triangles formed by joining the points, taking 3 at a time $= {}^{10}C_3 = \frac{10!}{3!\,7!} = 120$.

Number of triangles formed by joining the 4 points, taken 3 at a time $= {}^{4}C_3 = {}^{4}C_1 = 4$.

But, 4 collinear points cannot form a triangle when taken 3 at a time.

So, Required number of triangles $= 120 - 4 = 116$.

EXAMPLE 22 *In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. Find the number of points of intersection of the straight lines.*

SOLUTION The number of points of intersection of 37 straight lines is ${}^{37}C_2$. But 13 straight lines out of the given 37 straight lines pass through the same point A. Therefore instead of getting ${}^{13}C_2$ points, we get merely one point A. Similarly, 11 straight lines out of the given 37 straight lines intersect at point B. Therefore instead of getting ${}^{11}C_2$ points, we get only one point B. Hence, the number of intersection points of the lines is ${}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2 = 535$.

EXAMPLE 23 *From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen ?* **[NCERT]**

SOLUTION We have the following possibilities:

(i) *Three particular students join the excursion party:* In this case, we have to choose 7 students from the remaining 22 students. This can be done in ${}^{22}C_7$ ways.

(ii) *Three particular students do not join the excursion party:* In this case, we have to choose 10 students from the remaining 22 students. This can be done in ${}^{22}C_{10}$ ways.

Hence, the required number of ways $= {}^{22}C_7 + {}^{22}C_{10} = 817190$.

EXAMPLE 24 *A boy has 3 library tickets and 8 books of his interest in the library. Of these 8, he does not want to borrow Chemistry Part II, unless Chemistry Part I is also borrowed. In how many ways can he choose the three books to be borrowed ?*

SOLUTION We have the following two possibilities:

(i) *When Chemistry part I is borrowed*: In this case the boy may borrow Chemistry Part II. So, he has to select now two books out of the remaining 7 books of his interest. This can be done in ${}^7C_2$ ways.

(ii) *When Chemistry part I is not borrowed* : In this case the boy does not want to borrow Chemistry Part II. So, he has to select three books from the remaining 6 books. This can be done in ${}^6C_3$ ways.

Hence, the required number of ways $= {}^7C_2 + {}^6C_3 = 21 + 20 = 41$.

EXAMPLE 25 *In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two minus signs are together ?*

SOLUTION The plus signs can be arranged in only one way, because all are identical, as shown below:

	+		+		+		+		+		+		+	

A blank box in the above arrangement shows available space for the minus signs. Since there are 7 plus signs, the number of blank boxes is therefore 8. The five minus signs are now to be arranged in the 8 boxes so that no two of them are together. Now, 5 boxes out of 8 can be chosen in ${}^8C_5$ ways. Since all minus signs are identical, so 5 minus signs can be arranged in 5 chosen boxes in only one way. Hence, the number of possible arrangements $= 1 \times {}^8C_5 \times 1 = 56$.

EXAMPLE 26 *In how many ways can 21 identical books on English and 19 identical books on Hindi be placed in a row on a shelf so that two books on Hindi may not be together?*

SOLUTION In order that no two books on Hindi are together, we must first arrange all books in English in a row. Since all English books are identical, so they can be arranged in a row in only one way as shown below:

$$\times E \times E \times E \times E \times \ldots \times E \times E$$

Here, E denotes the position of an English book and $\times$ that of a Hindi book.

Since there are 21 books on English, the number places mark $\times$ are therefore 22. Now, 19 books on Hindi are to be arranged in these 22 places so that no two of them are together. Out of 22 places 19 places for Hindi books can be chosen in ${}^{22}C_{19}$ ways. Since all books on Hindi are identical, so 19 books on Hindi can be arranged in 19 chosen places in only one way. Hence, the required number of ways $= 1 \times {}^{22}C_{19} \times 1 = 1540$.

EXERCISE 16.2

BASIC

1. From a group of 15 cricket players, a team of 11 players is to be chosen. In how many ways can this be done?

2. How many different boat parties of 8, consisting of 5 boys and 3 girls, can be made from 25 boys and 10 girls?
3. In how many ways can a student choose 5 courses out of 9 courses if 2 courses are compulsory for every student?
4. In how many ways can a football team of 11 players be selected from 16 players? How many of these will (i) include 2 particular players? (ii) exclude 2 particular players?
5. There are 10 professors and 20 students out of whom a committee of 2 professors and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees:
   (i) a particular professor is included. (ii) a particular student is included.
   (iii) a particular student is excluded.
6. How many different products can be obtained by multiplying two or more of the numbers 3, 5, 7, 11 (without repetition)?
7. From a class of 12 boys and 10 girls, 10 students are to be chosen for a competition; at least including 4 boys and 4 girls. The 2 girls who won the prizes last year should be included. In how many ways can the selection be made?
8. How many different selections of 4 books can be made from 10 different books, if
   (i) there is no restriction; (ii) two particular books are always selected;
   (iii) two particular books are never selected?
9. From 4 officers and 8 jawans in how many ways can 6 be chosen (i) to include exactly one officer (ii) to include at least one officer?
10. A sports team of 11 students is to be constituted, choosing at least 5 from class XI and at least 5 from class XII. If there are 20 students in each of these classes, in how many ways can the teams be constituted?
11. A student has to answer 10 questions, choosing at least 4 from each of part A and part B. If there are 6 questions in part A and 7 in part B, in how many ways can the student choose 10 questions?
12. In an examination, a student has to answer 4 questions out of 5 questions; questions 1 and 2 are however compulsory. Determine the number of ways in which the student can make the choice.
13. A candidate is required to answer 7 questions out of 12 questions which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. In how many ways can he choose the 7 questions?

BASED ON LOTS

14. There are 10 points in a plane of which 4 are collinear. How many different straight lines can be drawn by joining these points.
15. Find the number of diagonals of (i) a hexagon (ii) a polygon of 16 sides.
16. How many triangles can be obtained by joining 12 points, five of which are collinear ?
17. In how many ways can a committee of 5 persons be formed out of 6 men and 4 women when at least one woman has to be necessarily selected ?
18. In a village, there are 87 families of which 52 families have at most 2 children. In a rural development programme, 20 families are to be helped chosen for assistance, of which at least 18 families must have at most 2 children. In how many ways can the choice be made ?
19. A group consists of 4 girls and 7 boys. In how many ways can a tem of 5 members be selected if the team has (i) no girl? (ii) at least one boy and one girl? (iii) at least 3 girls? [NCERT]

20. A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women? [NCERT]
21. Find the number of (i) diagonals (ii) triangles formed in a decagon.
22. Determine the number of 5 cards combinations out of a deck of 52 cards if at least one of the 5 cards has to be a king ? [NCERT]
23. We wish to select 6 persons from 8, but if the person A is chosen, then B must be chosen. In how many ways can the selection be made ?
24. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls? [NCERT]
25. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour. [NCERT]
26. Determine the number of 5 cards combinations out of a deck of 52 cards if there is exactly one ace in each combination. [NCERT]
27. In how many ways can one select a cricket team of eleven from 17 players in which only 5 persons can bowl if each cricket team of 11 must include exactly 4 bowlers?
28. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected. [NCERT]
29. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student? [NCERT]
30. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consits of :

    (i) exactly 3 girls? (ii) at least 3 girls? (iii) at most 3 girls? [NCERT]
31. In an examination, a question paper consists of 12 questions divided into two parts i.e., Part I and Part II, containing 5 and 7 questions, respectively. A student is required to attempt 8 questions in all, selecting at least 3 from each part. In how many ways can a student select the questions? [NCERT]

BASED ON HOTS

32. A parallelogram is cut by two sets of m lines parallel to its sides. Find the number of parallelograms thus formed.
33. Out of 18 points in a plane, no three are in the same straight line except five points which are collinear. How many (i) straight lines (ii) triangles can be formed by joining them ?

ANSWERS

1. 1365 2. 6375600 3. 35 4. 4368 (i) 2002 (ii) 364
5. 51300 (i) 10260 (ii) 7695 (iii) 43605 6. 11 7. 104874
8. (i) 210 (ii) 28 (iii) 70 9. (i) 224 (ii) 896 10. $2\,({}^{20}C_5 \times {}^{20}C_6)$
11. 266 12. 3 13. 780 14. 40 15. (i) 9 (ii) 104 16. 210 17. 246
18. ${}^{52}C_{18} \times {}^{35}C_2 + {}^{52}C_{19} \times {}^{35}C_1 + {}^{52}C_{20} \times {}^{35}C_0$ 19. (i) 21(ii) 441(iii) 91
20. 10, 6 21. (i) 35 (ii) 120 22. 886656 23. 22 24. 40 25. 2000
26. 778320 27. 3960 28. 200
29. 35 30. (i) 504 (ii) 588 (iii) 1630 31. 420 32. $({}^{m+2}C_2)^2$
33. (i) 144 (ii) 806

HINTS TO SELECTED PROBLEMS

2. Required no. of boat parties = ${}^{25}C_5 \times {}^{10}C_3$.

3. Since 2 courses are compulsory. So, the student is to choose 3 courses out of the remaining 7 courses. This can be done in ${}^7C_3$ ways.

4. We have to select 11 players out of 16. So, required number of ways = ${}^{16}C_{11}$.

   (i) Since 2 particular players are always included, so, we have to select 9 players out of the remaining 14 players. This can be done in ${}^{14}C_9$ ways.

   (ii) Since 2 particular players are excluded from every selection, so, we have to select 11 players from the remaining 14 players. This can be done in ${}^{14}C_{11}$ ways.

6. Total number of products = Number of ways of selecting 2 or 3 or all out of 4 numbers 3, 5, 7, 11

$$= {}^4C_2 + {}^4C_3 + {}^4C_4 = 6 + 4 + 1 = 11.$$

7. Since two girls who won the prizes last year are to be included in every selection. So, we have to select 8 students out of 12 boys and 8 girls, choosing at least 4 boys and at least two girls. This can be done in ${}^{12}C_6 \times {}^8C_2 + {}^{12}C_5 \times {}^8C_3 + {}^{12}C_4 \times {}^8C_4 = 104874$ ways.

9. (i) Required number of ways = ${}^4C_1 \times {}^8C_5$

   (ii) Required number of ways = Total no. of ways – No. of ways of selecting no officer

$$= {}^{12}C_6 - {}^8C_6.$$

10. Required number of ways = ${}^{20}C_5 \times {}^{20}C_6 + {}^{20}C_6 \times {}^{20}C_5$.

11. The various possibilities are : (i) 4 from part A and 6 from part B (ii) 5 from part A and 5 from part B (iii) 6 from part A and 4 from part B.

    So, the required number of ways = ${}^6C_4 \times {}^7C_6 + {}^6C_5 \times {}^7C_5 + {}^6C_6 \times {}^7C_4 = 266$.

12. Required number of ways = ${}^3C_2$.

13. Required number of ways = ${}^6C_5 \times {}^6C_2 + {}^6C_4 \times {}^6C_3 + {}^6C_3 \times {}^6C_4 + {}^6C_2 \times {}^6C_5 = 780$.

14. Number of straight lines = ${}^{10}C_2 - {}^4C_2 + 1$.

16. Number of triangles = ${}^{12}C_3 - {}^5C_3$.

18. 52 families have at most 2 children, while 35 families have more than 2 children. The selection of 20 families of which at least 18 families must have at most 2 children can be made as under:

    (i) 18 families out of 52 and 2 families out of 35

    or, (ii) 19 families out of 52 and 1 family out of 35

    or, (iii) 20 families out of 52.

19. (i) From a group of 4 girls and 7 boys, a team of 5 consisting of no girls can be chosen in ${}^7C_5 = 21$ ways.

    (ii) A team of 5 consisting of at least one boy and one girl can be chosen in

$${}^7C_1 \times {}^4C_4 + {}^7C_2 \times {}^4C_3 + {}^7C_3 \times {}^4C_2 + {}^7C_4 \times {}^4C_1 = 441 \text{ ways.}$$

    (iii) A team of 5 consisting of at least 3 girls can be chosen in

$${}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 = 91 \text{ ways.}$$

21. A committee of 3 persons out of 2 men and 3 women can be constituted in ${}^5C_3 = 10$ ways.

    A committee of 1 man and 2 women can be constituted in ${}^2C_1 \times {}^3C_2 = 6$ ways.

22. Required number of combinations = Total number of 5 card combinations – Number of 5 car combinations having no king.

$$= {}^{52}C_5 - {}^{48}C_5 = 886656.$$

24. Number of ways of selecting team $= {}^5C_3 \times {}^4C_3 = 40$.

25. Number of ways of selecting 9 balls $= {}^6C_3 \times {}^5C_3 \times {}^5C_3 = 2000$.

26. Out of 4 aces one ace can be selected in ${}^4C_1$ ways and from the remaining 48 cards, four cards can be selected in ${}^{48}C_4$ ways. So, number of 5 cards combinations consisting of exactly one ace $= {}^4C_1 \times {}^{48}C_4 = 778320$.

27. Required number of ways $= {}^5C_4 \times {}^{12}C_7$.

28. Out of 5 black and 6 red balls, 2 black and 3 red balls can be chosen in ${}^5C_2 \times {}^6C_3 = 200$ ways.

29. Required number of ways = Number of ways of selecting 3 courses out of 7 courses.
$= {}^7C_3$ ways $= 35$.

30. (i) A committee consisting of 3 girls and 4 boys can be formed in ${}^9C_4 \times {}^4C_3 = 504$ ways.
(ii) A committee consisting of at least 3 girls can be formed in ${}^9C_4 \times {}^4C_3 + {}^9C_3 \times {}^4C_4$ $= 588$ ways.
(iii) A committee of at most 3 girls can be formed in
$${}^9C_7 \times {}^4C_0 + {}^9C_6 \times {}^4C_1 + {}^9C_5 \times {}^4C_2 + {}^9C_4 \times {}^4C_3 = 1632 \text{ ways.}$$

31. At least 3 questions can be selected in the following ways:

Part I	Part II
3	5
4	4
5	3

So, required number of ways $= {}^5C_3 \times {}^7C_5 + {}^5C_4 \times {}^7C_4 + {}^5C_5 \times {}^7C_3 = 420$.

32. Each set of parallel lines consists of $(m + 2)$ lines and each parallelogram is formed by choosing two lines from the first set and two straight lines from the second set.
Hence, the total number of parallelograms $= {}^{m+2}C_2 \times {}^{m+2}C_2$.

16.5 MIXED PROBLEMS ON PERMUTATIONS AND COMBINATIONS

In this section, we intend to discuss some practical problems where both permutations and combinations are used as is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed ?*

SOLUTION Three consonants out of 7 and 2 vowels out of 4 can be chosen in ${}^7C_3 \times {}^4C_2$ ways. Thus, there are ${}^7C_3 \times {}^4C_2$ groups each containing 3 consonants and 2 vowels. Since each group contains 5 letters, which can be arranged among themselves in 5! ways.

Hence, the required number of words $= ({}^7C_3 \times {}^4C_2) \times 5! = 25200$.

EXAMPLE 2 *How many four-letter words can be formed using the letters of the word 'FAILURE', so that*
(i) *F is included in each word ?* (ii) *F is not included in any word ?*

SOLUTION There are 7 letters in the word 'FAILURE'.

(i) To include F in every 4 letter word, we first select four letters from the 7 letters of the word 'FAILURE' such that F is included in every selection. This can be done by selecting three letters from the remaining 6 letters i.e. A, I, L, U, R, E in ${}^6C_3$ ways. Now, there are 4 letters in each of

${}^6C_3$ selections. Consider one of these ${}^6C_3$ selections. This selection contains 4 letters which can be arranged in 4! ways. Thus, each of ${}^6C_3$ selections provides 4! words.

Hence, the total number of words $= {}^6C_3 \times 4! = 480$.

(ii) If F is not to be included in any word, then we first select 4 letters from the remaining 6 letters. This can be done in ${}^6C_4$ ways. Now, every selection has 4 letters which can be arranged in a row in 4! ways.

Hence, the total number of words $= {}^6C_4 \times 4! = 360$.

EXAMPLE 3 *How many words with or without meaning, can be formed using all the letters of the word EQUATION at a time so that vowels and consonants occur together?* [NCERT]

SOLUTION There are 5 vowels and 3 consonants in the word EQUATION. All vowels can be put together in 5! ways and all consonants can be put together in 3! ways. Considering all vowels as one letter and all consonants as at letter, vowels and consonants can be arranged in 2! ways. Therefore, vowels and consonants can be put together in $5! \times 3! \times 2!$ ways i.e. 1440 ways.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *How many five-letter words containing 3 vowels and 2 consonants can be formed using the letters of the word 'EQUATION' so that the two consonants occur together ?*

SOLUTION There are 5 vowels and 3 consonants in the word 'EQUATION'. Three vowels out of 5 and 2 consonants out of 3 can be chosen in ${}^5C_3 \times {}^3C_2$ ways. So, there are ${}^5C_3 \times {}^3C_2$ groups each containing 3 consonants and two vowels. Now, each group contains 5 letters which are to be arranged in such a way that 2 consonants occur together. Considering 2 consonants as one letter, we have 4 letters which can be arranged in 4! ways. But two consonants can be put together in 2! ways. Therefore, 5 letters in each group can be arranged in $4! \times 2!$ ways.

Hence, the required number of words $= ({}^5C_3 \times {}^3C_2) \times 4! \times 2! = 1440$.

EXAMPLE 5 *How many words with or without meaning, each 2 of vowels and 3 consonants can be formed from the letters of the word DAUGHTER?* [NCERT]

SOLUTION There are 3 vowels and 5 consonants in the word DAUGHTER out of which 2 vowels and 3 consonants can be chosen in ${}^3C_2 \times {}^5C_3$ ways. These selected five letters can now be arranged in 5! ways.

Hence, required number of words $= {}^3C_2 \times {}^5C_3 \times 5! = 3 \times 10 \times 120 = 3600$

EXAMPLE 6 *The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and 2 different consonants can be formed from the alphabet ?* [NCERT]

SOLUTION Out of 5 vowels and 21 consonants, 2 vowels and 2 consonants can be chosen in ${}^5C_2 \times {}^{21}C_2$ ways. These selected 4 letters can now be arranged in 4! ways. Therefore, by the fundamental principle of counting, required number of words is

$${}^5C_2 \times {}^{21}C_2 \times 4! = 10 \times 210 \times 24 = 50400.$$

EXAMPLE 7 *In how many ways can 5 girls and 3 boys be seated in a row so that no two boys are together?* [NCERT]

SOLUTION Since boys are to be separated. Therefore, let us first seat 5 girls. This can be done in 5! ways. For each such arrangement, three boys can be seated only at the cross marked places.

$$\times G \times G \times G \times G \times G \times$$

There are 6 crossed marked places and three boys can be seated in ${}^6C_3 \times 3!$ ways. Hence, by the fundamental principle of counting, the total number of ways is $5! \times {}^6C_3 \times 3! = 14400$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 8 *How many words can be formed by taking 4 letters at a time out of the letters of the word 'MATHEMATICS'.*

SOLUTION There are 11 letters viz. MM, AA, TT, H, E, I, C, S. All these letters are not distinct, so we cannot use ${}^nP_r$. We can choose 4 letters from the following ways:

(i) All the four distinct letters: There are 8 distinct letters viz. M, A, T, H, E, I, C, S out of which 4 can be chosen in ${}^8C_4$ ways. So, the total number of groups of 4 letters = ${}^8C_4$. Each such group has 4 letters which can be arranged in 4! ways.

Hence, the total number of words = ${}^8C_4 \times 4! = {}^8P_4 = 1680$.

(ii) Two distinct and two alike letters: There are 3 pairs of alike letters viz MM, AA, TT, out of which one pair can be chosen in ${}^3C_1$ ways. Now we have to choose two letters out of the remaining 7 different types of letters which can be done in ${}^7C_2$ ways. So, the total number of groups of 4 letters in which two are different and 2 are alike is ${}^3C_1 \times {}^7C_2$. Each such group has 4 letters of which 2 are alike and remaining two distinct and they can be arranged in $\frac{4!}{2!}$ ways.

Hence, the total number of words in which two letters are alike = ${}^3C_1 \times {}^7C_2 \times \frac{4!}{2!} = 756$.

(iii) Two alike of one kind and two alike of other kind: There are 3 pairs of 2 alike letters out of which 2 pairs can be chosen in ${}^3C_2$ ways. So, there are ${}^3C_2$ groups of 4 letters each. In each group there are 4 letters of which 2 are alike of one kind and two alike of other kind. These 4 letters can be arranged in $\frac{4!}{2!\,2!}$ ways. Hence, the total number of words in which two letters are alike of one kind and two alike of other kind = ${}^3C_2 \times \frac{4!}{2!\,2!} = 18$.

From (i), (ii) and (iii) the total number of 4 letter words $= 1680 + 756 + 18 = 2454$.

EXAMPLE 9 *Eighteen guests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three others on the other side. Determine the number of ways in which the seating arrangement can be made.*

SOLUTION Since four particular guests want to sit on a particular side A (say) and three others on the other side B (say). So, we are left with 11 guests out of which we choose 5 for side A in ${}^{11}C_5$ ways and the remaining 6 for side B in ${}^6C_6$ ways. Hence, the number of selections for the two sides is ${}^{11}C_5 \times {}^6C_6$.

Now 9 persons on each side of the table can be arranged among themselves in 9! ways.

Hence, the total number of arrangements = ${}^{11}C_5 \times {}^6C_6 \times 9! \times 9! = \frac{11!}{6!5!} \times 9! \times 9!$

EXAMPLE 10 *How many four-letter words can be formed using the letter of the word 'INEFFECTIVE'?*

SOLUTION There are 11 letters in the word 'INEFFECTIVE'. viz. EEE, FF, II, C, T, N, V.

The four-letter words may consist of:

(i) 3 alike letters and 1 distinct letter

(ii) 2 alike letters of one kind and 2 alike letters of the second kind
(iii) 2 alike letters and 2 distinct letters
(iv) all different letters

Now we, shall discuss these four cases one by one:

(i) *3 alike letters and 1 distinct letter*: There is one set of three alike letters viz. EEE. So, three alike letters can be selected in one way. Out of the 6 different letters F, I, T, N, V, C one letter can be selected in ${}^6C_1$ ways. Thus, three alike and one different letter can be selected in $1 \times {}^6C_1 = {}^6C_1$ ways. So, there are ${}^6C_1$ groups each of which contains 3 alike letters and one different letter. These 4 letters can be arranged in $\frac{4!}{3!1!}$ ways.

Hence, the total number of words consisting of three alike and one distinct letters

$$= {}^6C_1 \times \frac{4!}{3!1!} = {}^6C_1 \times 4 = 24.$$

(ii) *2 alike letters of one kind and 2 alike letters of second kind:* There are three sets of two alike letters viz EE, FF, II. Out of these three sets two can be selected in ${}^3C_2$ ways. So, there are ${}^3C_2$ groups each of which contains 4 letters out of which 2 are alike of one type and two are alike of second type. Now, 4 letters in each group can be arranged in $\frac{4!}{2!\,2!}$ ways.

Hence, the total number of words consisting of two alike letters of one type and 2 alike letters of second type $= {}^3C_2 \times \frac{4!}{2!\,2!} = 18.$

(iii) *2 alike and 2 different letters:* Out of 3 sets of two alike letters one set can be chosen in ${}^3C_1$ ways. Now, from the remaining 6 distinct letters, 2 letters can be chosen in ${}^6C_2$ ways. Thus, 2 alike letters and 2 distinct letters can be selected in $({}^3C_1 \times {}^6C_2)$ ways. So, there are $({}^3C_1 \times {}^6C_2)$ groups of 4 letters each. Now, letters of each group can be arranged among themselves in $\frac{4!}{2!}$ ways.

Hence, the total number of words consisting of two alike letters and 2 distinct

$$= {}^3C_1 \times {}^6C_2 \times \frac{4!}{2!} = 540.$$

(iv) *All different letters:* There are 7 distinct letters E, F, I, T, N, V, C out of which 4 can be selected in ${}^7C_4$ ways. So, there are ${}^7C_4$ groups of 4 letters each. The letters in each of ${}^7C_4$ groups can be arranged in 4! ways.

So, the total number of 4 letter words in which all letters are distinct $= {}^7C_4 \times 4! = 840.$

Hence, the total number of 4-letter words $= 24 + 18 + 540 + 840 = 1422.$

EXAMPLE 11 *In how many ways can the letters of the word PERMUTATIONS be arranged if there are always 4 letters between P and S?* **[NCERT]**

SOLUTION There 12 letters in the given word of which 2 are T's. There can be 4 letters between P and S in one of the following ways:

(i) There are 2T's and 2 other letters from the remaining 8 letters (excluding 2T's and *P* and *S*).
(ii) One *T* and 3 other letters from the remaining 8 letters.
(iii) There is no *T* and 4 other letters.

Let us now find the number of words in each case.

(i) In the first case, 2 letters can be chosen from remaining 8 letters in ${}^8C_2$ ways. Now, 2T's and 2 other letters can be arranged between P and S in $\frac{4!}{2!}$ ways. Also, P and S can interchange their positions. So, 2T's and 2 other letters can be arranged between P and S in ${}^8C_2 \times \frac{4!}{2!} \times 2!$ ways. Considering these six letters as one letter and the remaining 6 letters can be arranged in 7! ways.

$\therefore$ Total number of words, in this case $= {}^8C_2 \times \frac{4!}{2!} \times 2! \times 7!$

(ii) In this case, 3 letters can be chosen from the remaining 8 letters in ${}^8C_3$ ways. Now, one T and 3 other letters from the remaining 8 letters can be arranged between P and S in 4! ways. Also, P and S can interchange their positions. So, one T and 3 other letters can be arranged between P and S in ${}^8C_3 \times 4! \times 2!$ ways. Considering these six letters as one letter and the remaining 6 letters can be arranged in 7! ways.

$\therefore$ Total number of words formed $= {}^8C_3 \times 4! \times 2! \times 7!$

(iii) In this case, 4 letters other than 2T's can be chosen from the remaining 8 letters in ${}^8C_4$ ways. These 4 letters can be arranged between P and S in 4! ways. Also, P and S can interchange their positions in 2! ways. Thus, 4 letters between P and S can be arranged in ${}^8C_4 \times 4! \times 2!$ ways. Taking these 6 letters as one letter with the remaining 6 letters (including 2T's), we have 7 letters which can be arranged in $\frac{7!}{2!}$ ways.

$\therefore$ Number of words formed $= {}^8C_4 \times 4! \times 2! \times \frac{7!}{2!}$

Hence, total number of words $= {}^8C_2 \times \frac{4!}{2!} \times 2! \times 7! + {}^8C_3 \times 4! \times 2! \times 7! + {}^8C_4 \times 4! \times 2! \times \frac{7!}{2!}$

$= 25401600$

EXERCISE 16.3

BASIC

1. How many different words, each containing 2 vowels and 3 consonants can be formed with 5 vowels and 17 consonants?
2. There are 10 persons named $P_1, P_2, P_3 \ldots, P_{10}$. Out of 10 persons, 5 persons are to be arranged in a line such that is each arrangement P_1 must occur whereas P_4 and P_5 do not occur. Find the number of such possible arrangements.
3. How many words, with or without meaning can be formed from the letters of the word 'MONDAY', assuming that no letter is repeated, if (i) 4 letters are used at a time (ii) all letters are used at a time (iii) all letters are used but first letter is a vowel? [NCERT]
4. Find the number of permutations of n distinct things taken r together, in which 3 particular things must occur together.
5. How many words each of 3 vowels and 2 consonants canbe formed from the letters of the word INVOLUTE? [NCERT]
6. Find the number of permutations of n different things taken r at a time such that two specified things occur together? [NCERT]

BASED ON HOTS

7. Find the number of ways in which : (a) a selection (b) an arrangement, of four letters can be made from the letters of the word 'PROPORTION'.

8. How many words can be formed by taking 4 letters at a time from the letters of the word 'MORADABAD'?
9. A business man hosts a dinner to 21 guests. He is having 2 round tables which can accommodate 15 and 6 persons each. In how many ways can he arrange the guests?
10. Find the number of combinations and permutations of 4 letters taken from the word 'EXAMINATION'.
11. A tea party is arranged for 16 persons along two sides of a long table with 8 chairs on each side. Four persons wish to sit on one particular side and two on the other side. In how many ways can they be seated?

ANSWERS

1. 816000 2. ${}^7C_4 \times 5!$ 3. (i) 360 (ii) 720 (iii) 240 4. ${}^{n-3}C_{r-3}\,(r-2)!\,3!$
5. 2880 6. $2(r-1)\,{}^{n-2}P_{r-2}$ 7. (a) 53 (b) 758 8. 626
9. ${}^{21}C_{15} \times 14! \times 5!$ 10. 2454 11. ${}^{10}C_4 \times (8!)^2$

HINTS TO SELECTED PROBLEMS

1. 2 vowels out of 5 and 3 consonants out of 17 can be chosen in ${}^5C_2 \times {}^{17}C_3$ ways.
   Now, 5 letters in each selection can be arranged in 5! ways.
   So, total number of words $= {}^5C_2 \times {}^{17}C_3 \times 5! = 816000$
3. (i) Total number of 4 letter words formed from the letters of the word 'MONDAY" $= {}^6C_4 \times 4! = 360$.
   (ii) Total number of words formed by using all letters of the word 'MONDAY' $= 6! = 720$
   (iii) There are two vowels A and O. So, first place can be filled in 2 ways and the remaining 5 places can be filled in 5! ways.
   So, total number of words beginning with a vowel $= 2 \times 5! = 240$.
5. Required number of words $= {}^4C_3 \times {}^4C_2 \times 5!$
6. Out of $(n-2)$ remaining things select $(r-2)$ things in ${}^{n-2}C_{r-2}$ ways. Consider two specified things as one and mix it with $(r-2)$ selected things. Now we have $(r-1)$ things which can be arranged in $(r-1)!$ ways, but two specified things can be put together in 2! ways. Hence, required number of ways $= {}^{n-2}C_{r-2} \times (r-1)! \times 2!$.
9. Total number of ways $= {}^{21}C_{15} \times {}^6C_6 \times 14! \times 5!$
11. 4 persons wish to sit on side A(say) and two on the other side B(say). So, 10 persons are left, out of which 4 persons for side A can be selected in ${}^{10}C_4$ ways and 6 persons for side B from the remaining 6 persons in ${}^6C_6$ ways. Hence, the number of selections for two sides $= {}^{10}C_4 \times {}^6C_6$. Now, 8 persons on each side can be arranged amongst themselves in 8! ways. Hence, the total number of seating arrangements $= {}^{10}C_4 \times {}^6C_6 \times 8! \times 8!$

REVISION EXERCISE

1. Eight chairs are numbered 1 to 8. Two women and 3 men wish to occupy one chair each. First the women choose the chairs from amongst the chairs 1 to 4 and then men selects from the remaining chairs. Find the total number of possible arrangements.

2. A candidate is required to answer 7 questions out of 12 questions, which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. Find the number of different ways of doing questions.
3. Out of 18 points in a plane, no three are in the same line except five points which are collinear. Find the number of lines that can be formed joining the points.
4. We wish to select 6 persons from 8, but if the person A is chosen, then B must be chosen. In how many ways can selections be made?
5. How many automobile license plates can be made if each plate contains two different letters followed by three different digits?
6. Find the number of permutations of n distinct things taken r together, in which 3 particular things must occur together.
7. Find the number of positive integers greater than 6000 and less than 7000 which are divisible by 5, provided that no digit is to be repeated.
8. There are 10 persons named $P_1, P_2, P_3, \ldots P_{10}$. Out of 10 persons, 5 persons are to be arranged in line such that in each arrangement P_1 must occur whereas P_4 and P_5 do not occur. Find the number of such possible arrangements.
9. There are 10 lamps in a hall. Each one of them can be switched on independently. Find the number of ways in which the hall can be illuminated.
10. A box contains two white, three black and four red balls. In how many ways can three balls we drawn from the box, if atleast one black ball is to be included in the draw.
11. Find the number of integers greater than 7000 then can be formed with the digits 3, 5, 7, 8 and 9 where no digit is repeated.
12. If 20 lines are drawn in plane such that no two of them are parallel and no three are concurrent, in how many points will they intersect each other?
13. In a certain city, all telephone numbers have six digits, the first two digits always being 41 or 42 or 46 or 62 or 64. How many telephone numbers have all six digits distinct?
14. In an examination, a student has to answer 4 questions out of 5 questions; questions 1 and 2 are however compulsory. Determine the number of ways in which the student can make the choice.
15. 18 mice were placed in two experimental groups and one control group, with all groups equally large. In how many ways can the mice be placed into three groups?
16. A bag contains six white marbles and five red marbles. Find the number of ways in which four marbles can be drawn from the bag if (i) they can be of any colour (ii) two must be white and two red and (iii) they must all be of the same colour.
17. In how many ways can a football team of 11 players be selected from 16 players? How many of them will (i) include 2 particular players? (ii) exclude 2 particular players?
18. A sports team of 11 students is to be constituted, choosing at least 5 from Class XI and at least 5 from Class XII. If there are 20 students in each of these classes, in how many ways can be the team be constituted?
19. A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has (i) no girls (ii) at least one boys and one girl (iii) at least three girls.

ANSWERS

1. 1440 2. 780 3. 144 4. 22 5. 468000 6. ${}^{n-3}C_{r-3} \times (r-2)! \times 3!$

7. 112 8. 4200 9. 1023 10. 91 11. 192 12. 190 13. 8400 14. 3

15. $\dfrac{18!}{(6!)^3}$ 16. (i) ${}^{11}C_4$ (ii) ${}^{6}C_2 \times {}^{5}C_2$ (iii) ${}^{6}C_4 + {}^{5}C_4$ 17. (i) ${}^{14}C_9$ (ii) ${}^{14}C_{11}$

18. $2\,({}^{20}C_5 \times {}^{20}C_6)$ 19. (i) 21 (ii) 441 (iii) 91

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If ${}^{n}P_r = 840$ and ${}^{n}C_r = 35$, then $r =$ ....
2. The value of ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$ is ....
3. The value ${}^{n}P_r \div {}^{n}C_r$ is ....
4. If n is even, then ${}^{n}C_r$ is maximum when ....
5. If $2 \times {}^{n}C_5 = 9 \times {}^{n-2}C_5$, then $n =$ ....
6. If ${}^{n}C_{r-1} = 36$, ${}^{n}C_r = 84$ and ${}^{n}C_{r+1} = 126$, then $r =$ ....
7. If ${}^{18}C_{15} + 2\,({}^{18}C_{16}) + {}^{17}C_{16} + 1 = {}^{n}C_3$, then $n =$ ....
8. If ${}^{n}C_{12} = {}^{n}C_6$, then ${}^{n}C_2 =$ ....
9. If ${}^{189}C_{35} + {}^{189}C_r = {}^{190}C_r$, then $r =$ ....
10. If ${}^{n}P_4 = 24\,.\,{}^{n}C_5$, then the value of n is ....
11. The value of ${}^{n}C_r + 2\,{}^{n}C_{r-1} + {}^{n}C_{r-2}$, $2 \le r \le n$, is ....
12. A box contain 2 white balls, 3 black balls and 4 red balls. The number of ways three balls be drawn from the box if at least one black ball is to be included in the draw is......................... .
13. Three balls are drawn from a bag containing 5 red, 4 white and 3 black balls. The number of ways in which this can be done if at least 2 are red is ....
14. The total number of ways in which six '+' and four '–' signs can be arranged in aline such that no two '–' signs occur together is ....
15. A committee of 6 is to be chosen from 10 men and 7 women so as to contain at least 3 men and 2 women. The number of different ways this can be done, if two particular women refuse to serve on the same committee is ....
16. The number of committees of five persons with a chair person can be selected from 12 persons, is ....
17. The number of automobile license plates that can be made if each plate contains two different letters of English alphabet followed by three distinct digits, is ....
18. The number of permutations of n distinct objects taken r at a time in which three particular objects occurs together is ....
19. Out of 10 persons $P_1, P_2, \ldots, P_{10}$, 5 persons are to be arranged in a line such that in each arrangement P_1 must occur whereas P_4 and P_5 do not occur. The number of such arrangements is ....

ANSWERS

1. 4 2. 0 3. $r!$ 4. $r = \dfrac{n}{2}$ 5. 10 6. 3 7. 20 8. 153

9. 36 10. 9 11. ${}^{n+2}C_r$ 12. 64 13. 80 14. 35 15. 7800 16. 3960
17. 4,6800 18. ${}^{n-3}C_{r-3}\,(r-2)!\,3!$ 19. ${}^{7}C_4 \times 5!$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write $\sum_{r=0}^{m} {}^{n+r}C_r$ in the simplified form.
2. If ${}^{35}C_{n+7} = {}^{35}C_{4n-2}$, then write the values of n.
3. Write the number of diagonals of an n-sided polygon.
4. Write the expression ${}^{n}C_{r+1} + {}^{n}C_{r-1} + 2 \times {}^{n}C_r$ in the simplest form.
5. Write the value of $\sum_{r=1}^{6} {}^{56-r}C_3 + {}^{50}C_4$.
6. There are 3 letters and 3 directed envelopes. Write the number of ways in which no letter is put in the correct envelope.
7. Write the maximum number of points of intersection of 8 straight lines in a plane.
8. Write the number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines.
9. Write the number of ways in which 5 red and 4 white balls can be drawn from a bag containing 10 red and 8 white balls.
10. Write the number of ways in which 12 boys may be divided into three groups of 4 boys each.
11. Write the total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants.

ANSWERS

1. ${}^{n+m+1}C_{n+1}$ 2. 3, 6 3. $\dfrac{n(n-3)}{2}$ 4. ${}^{n+2}C_{r+1}$ 5. ${}^{56}C_4$ 6. 2
7. 28 8. 18 9. ${}^{10}C_5 \times {}^{8}C_4$ 10. $\dfrac{12!}{(4!)^3\, 3!}$ 11. ${}^{4}C_2 \times 5C_3 \times 5!$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If ${}^{20}C_r = {}^{20}C_{r-10}$, then ${}^{18}C_r$ is equal to
   (a) 4896 (b) 816 (c) 1632 (d) none of these
2. If ${}^{20}C_r = {}^{20}C_{r+4}$, then ${}^{r}C_3$ is equal to
   (a) 54 (b) 56 (c) 58 (d) none of these
3. If ${}^{15}C_{3r} = {}^{15}C_{r+3}$, then r is equal to
   (a) 5 (b) 4 (c) 3 (d) 2
4. If ${}^{20}C_{r+1} = {}^{20}C_{r-1}$, then r is equal to
   (a) 10 (b) 11 (c) 19 (d) 12

5. If $C(n, 12) = C(n, 8)$, then $C(22, n)$ is equal to
(a) 231 (b) 210 (c) 252 (d) 303

6. If ${}^mC_1 = {}^nC_2$, then
(a) $2m = n$ (b) $2m = n(n+1)$ (c) $2m = n(n-1)$ (d) $2n = m(m-1)$

7. If ${}^nC_{12} = {}^nC_8$, then $n =$
(a) 20 (b) 12 (c) 6 (d) 30

8. If ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_x$, then $x =$
(a) r (b) $r-1$ (c) n (d) $r+1$

9. If ${}^{(a^2-a)}C_2 = {}^{(a^2-a)}C_4$, then $a =$
(a) 2 (b) 3 (c) 4 (d) none of these

10. ${}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$ is equal to
(a) 30 (b) 31 (c) 32 (d) 33

11. Total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants is equal to
(a) 60 (b) 120 (c) 7200 (d) none of these

12. There are 12 points in a plane. The number of the straight lines joining any two of them when 3 of them are collinear, is
(a) 62 (b) 63 (c) 64 (d) 65

13. Three persons enter a railway compartment. If there are 5 seats vacant, in how many ways can they take these seats ?
(a) 60 (b) 20 (c) 15 (d) 125

14. In how many ways can a committee of 5 be made out of 6 men and 4 women containing at least one women ?
(a) 246 (b) 222 (c) 186 (d) none of these

15. There are 10 points in a plane and 4 of them are collinear. The number of straight lines joining any two of them is
(a) 45 (b) 40 (c) 39 (d) 38

16. There are 13 players of cricket, out of which 4 are bowlers. In how many ways a team of eleven be selected from them so as to include at least two bowlers ?
(a) 72 (b) 78 (c) 42 (d) none of these

17. If $C_0 + C_1 + C_2 + \ldots + C_n = 256$, then ${}^{2n}C_2$ is equal to
(a) 56 (b) 120 (c) 28 (d) 91

18. The number of ways in which a host lady can invite for a party of 8 out of 12 people of whom two do not want to attend the party together is
(a) $2 \times {}^{11}C_7 + {}^{10}C_8$ (b) ${}^{10}C_8 + {}^{11}C_7$
(c) ${}^{12}C_8 - {}^{10}C_6$ (d) none of these

19. Given 11 points, of which 5 lie on one circle, other than these 5, no 4 lie on one circle. Then the number of circles that can be drawn so that each contains at least 3 of the given points is

(a) 216 (b) 156 (c) 172 (d) none of these

20. How many different committees of 5 can be formed from 6 men and 4 women on which exact 3 men and 2 women serve ?
(a) 6 (b) 20 (c) 60 (d) 120

21. If ${}^{43}C_{r-6} = {}^{43}C_{3r+1}$, then the value of r is
(a) 12 (b) 8 (c) 6 (d) 10 (e) 14

22. The number of diagonals that can be drawn by joining the vertices of an octagon is
(a) 20 (b) 28 (c) 8 (d) 16

23. The value of $\left({}^7C_0 + {}^7C_1\right) + \left({}^7C_1 + {}^7C_2\right) + \ldots + \left({}^7C_6 + {}^7C_7\right)$ is
(a) $2^7 - 1$ (b) $2^8 - 2$ (c) $2^8 - 1$ (d) 2^8

24. Among 14 players, 5 are bowlers. In how many ways a team of 11 may be formed with at least 4 bowlers?
(a) 265 (b) 263 (c) 264 (d) 275

25. A lady gives a dinner party for six guests. The number of ways in which they may be selected from among ten friends if two of the friends will not attend the party together is
(a) 112 (b) 140 (c) 164 (d) none of these

26. If ${}^{n+1}C_3 = 2 \cdot {}^nC_2$, then $n =$
(a) 3 (b) 4 (c) 5 (d) 6

27. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is
(a) 6 (b) 9 (c) 12 (d) 18

28. The number of ways in which a committee consisting of 3 men and 2 women, can be chosen from 7 men and 5 women, is
(a) 45 (b) 350 (c) 4200 (d) 230

[NCERT EXEMPLAR]

29. The number of signals that can be sent by 6 flags of different colours taking one or more at a time is
(a) 63 (b) 1956 (c) 720 (d) 21

[NCERT EXEMPLAR]

30. The straight lines l_1, l_2 and l_3 are parallel and lie in the same plane. A total number of m points are taken on l_1, n points on l_2, k points on l_3. The maximum number of triangles formed with vertices at these points are
(a) ${}^{m+n+k}C_3$ (b) ${}^{m+n+k}C_3 - {}^mC_3 - {}^nC_3 - {}^kC_3$
(c) ${}^mC_3 + {}^nC_3 + {}^kC_3$ (d) ${}^mC_3 \times {}^nC_3 \times {}^kC_3$ [NCERT EXEMPLAR]

31. The number of committees of five persons with a chairperson that can be formed from 12 persons, is
(a) ${}^{12}C_5$ (b) ${}^{12}C_4$ (c) $12 \times {}^{11}C_4$ (d) ${}^{11}C_4$

[NCERT EXEMPLAR]

32. Total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants is equal to
(a) 60 (b) 120 (c) 7200 (d) 720
[NCERT EXEMPLAR]

33. A five digit number divisible by 3 is to be formed using the numbers 0, 1, 2, 3, 4 and 5 without repetitions. The total number of ways this can be done is
(a) 216 (b) 600 (c) 240 (d) 3125
[NCERT EXEMPLAR]

34. Everybody in a room shakes hands with everybody else. The total number of hand shakes is 66. The total number of persons in the room is
(a) 11 (b) 12 (c) 13 (d) 14
[NCERT EXEMPLAR]

35. The number of triangles that are formed by choosing the vertices from a set of 12 points, seven of which lie on the same line is
(a) 105 (b) 15 (c) 175 (d) 185
[NCERT EXEMPLAR]

36. Given 5 different green dyes, four different blue dyes and three different red dyes, the number of combinations of dyes which can be chosen taking at least one green and one blue dye is
(a) 3600 (b) 3720 (c) 3800 (d) 3600
[NCERT EXEMPLAR]

37. The total number of 9 digit numbers which have all different digits is
(a) 10! (b) 9! (c) $9 \times 9!$ (d) $10 \times 10!$
[NCERT EXEMPLAR]

38. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is
(a) 6 (b) 18 (c) 12 (d) 9
[NCERT EXEMPLAR]

39. The number of 5-digit telephone numbers having at least one of their digits repeated is
(a) 90,000 (b) 10,000 (c) 30240 (d) 69760
[NCERT EXEMPLAR]

40. The number of ways in which we can choose a committee from four men and six women so that the committee includes at least two man and exactly twice as many women as men, is
(a) 94 (b) 126 (c) 128 (d) none of these
[NCERT EXEMPLAR]

41. The number of ways in which a team of eleven players can be selected from 22 players always including 2 of them and excluding 4 of them, is
(a) ${}^{16}C_{11}$ (b) ${}^{16}C_5$ (c) ${}^{16}C_9$ (d) ${}^{20}C_9$
[NCERT EXEMPLAR]

ANSWERS

1. (b)	2. (b)	3. (c)	4. (a)	5. (a)	6. (c)	7. (a)	8. (d)
9. (b)	10. (b)	11. (c)	12. (c)	13. (a)	14. (a)	15. (b)	16. (b)
17. (b)	18. (c)	19. (b)	20. (d)	21. (a)	22. (a)	23. (b)	24. (c)
25. (b)	26. (c)	27. (d)	28. (b)	29. (b)	30. (b)	31. (c)	32. (c)
33. (a)	34. (b)	35. (d)	36. (b)	37. (c)	38. (b)	39. (d)	40. (a)
41. (c)							

SUMMARY

1. If n is a natural number and r is a non-negative integer such that $0 \le r \le n$, then

   (i) ${}^nC_r = \frac{n!}{(n-r)!\,r!}$ (ii) ${}^nC_r \times r! = {}^nP_r$

   (iii) ${}^nC_r = {}^nC_{n-r}$ (iv) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

   (v) ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r} \times \frac{n-1}{r-1} \cdot {}^{n-2}C_{r-2} = \dots = \frac{n}{r} \times \frac{n-1}{r-1} \times \frac{n-2}{r-2} \times \dots \times \frac{n-(r-1)}{1}$

   (vi) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or, $x + y = n$

   (vii) If n is an even natural number, then the greatest among ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ is ${}^nC_{\frac{n}{2}}$.

   If n is an odd natural number, then the greatest among ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ is ${}^nC_{\frac{n-1}{2}}$ or, ${}^nC_{\frac{n+1}{2}}$.

2. The number of ways of selecting r items or objects from a group of n distinct items or objects is $\frac{n!}{(n-r)!\,r!} = {}^nC_r$.

CHAPTER 17

BINOMIAL THEOREM

17.1 INTRODUCTION

An algebraic expression containing two terms is called a binomial expression.

For example, $(a+b)$, $(2x-3y)$, $\left(x+\frac{1}{y}\right)$, $\left(x+\frac{3}{x}\right)$, $\left(\frac{2}{x}-\frac{1}{x^2}\right)$ etc. are binomial expressions.

Similarly, an algebraic expression containing three terms is called a *trinomial*. In general, expressions containing more than two terms are known as multinomial expression.

The general form of the binomial expression is $(x+a)$ and the expansion of $(x+a)^n$, $n \in N$ is called the *binomial theorem*. This theorem was first given by **Sir Issac Newton.** It gives a formula for the expansion of the powers of a binomial expression.

In earlier classes, we have learnt that:

$$(x+a)^0 = 1$$

$$(x+a)^1 = x+a$$

$$(x+a)^2 = x^2+2ax+a^2$$

$$(x+a)^3 = x^3+3x^2a+3xa^2+a^3$$

$$(x+a)^4 = x^4+4x^3a+6x^2a^2+4xa^3+a^4$$

We observe that the coefficients in the above expansions follow a particular pattern as given below:

Index of the binomial	*Coefficients of various terms*
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1

We also observe that each row is bounded by 1 on both sides. Any entry, except the first and last, in a row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right. The above pattern is known as ***Pascal's triangle.*** It has been checked that the above pattern also holds good for the coefficients in the expansions of the binomial expressions having index (exponent) greater than 4 as given below.

Index of the binomial	*Coefficients of various terms*
0	1
1	1 ∇ 1
2	1 ∇ 2 ∇ 1
3	1 ∇ 3 ∇ 3 ∇ 1
4	1 ∇ 4 ∇ 6 ∇ 4 ∇ 1
5	1 ∇ 5 ∇ 10 ∇ 10 ∇ 5 ∇ 1
6	1 ∇ 6 ∇ 15 ∇ 20 ∇ 15 ∇ 6 ∇ 1
....	....
....	....

Pascal's Triangle

Using the above Pascal's triangle, we obtain

$$(x+a)^1 = x + a$$

or, $$(x+a)^1 = {}^1C_0\, x^1\, a^0 + {}^1C_1\, x^0\, a^1 \qquad \left[\because\ {}^1C_0 = 1 = {}^1C_1\right]$$

$$(x+a)^2 = x^2 + 2ax + a^2$$

or, $$(x+a)^2 = {}^2C_0\, x^2\, a^0 + {}^2C_1\, x^{2-1}\, a^1 + {}^2C_2\, x^{2-2}\, a^2 \qquad \left[\because\ {}^2C_0 = 1,\ {}^2C_1 = 2,\ {}^2C_2 = 1\right]$$

$$(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$$

or, $$(x+a)^3 = {}^3C_0\, x^3\, a^0 + {}^3C_1\, x^{3-1}\, a^1 + {}^3C_2\, x^{3-2}\, a^2 + {}^3C_3\, x^{3-3}\, a^3$$

$$(x+a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$$

or, $$(x+a)^4 = {}^4C_0\, x^4\, a^0 + {}^4C_1\, x^{4-1}\, a^1 + {}^4C_2\, x^{4-2}\, a^2 + {}^4C_3\, x^{4-3}\, a^3 + {}^4C_4\, x^{4-4}\, a^4$$

$$(x+a)^5 = {}^5C_0\, x^5\, a^0 + {}^5C_1\, x^{5-1}\, a^1 + {}^5C_2\, x^{5-2}\, a^2 + {}^5C_3\, x^{5-3}\, a^3 + {}^5C_4\, x^{5-4}\, a^4 + {}^5C_5\, x^{5-5}\, a^5$$

By looking at the above expansions we can easily guess that the general formula would be of the form as given in the following theorem.

17.2 BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

THEOREM *If x and a are real numbers, then for all $n \in N$,*

$$(x+a)^n = {}^nC_0\, x^n\, a^0 + {}^nC_1\, x^{n-1}\, a^1 + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots + {}^nC_{n-1}\, x^1\, a^{n-1} + {}^nC_n\, x^0\, a^n$$

i.e., $$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, a^r$$

PROOF We shall prove the theorem by using the principle of mathematical induction on n.

Let $P(n)$ be the statement:

$$(x+a)^n = {}^nC_0\, x^n\, a^0 + {}^nC_1\, x^{n-1}\, a^1 + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots + {}^nC_{n-1}\, x^1\, a^{n-1} + {}^nC_n\, x^0\, a^n$$

Step I We have, $P(1)$: $(x+a)^1 = {}^1C_0\, x^1\, a^0 + {}^1C_1\, x^0\, a^1$

We know that: $(x+a)^1 = x + a = {}^1C_0\, x^1\, a^0 + {}^1C_1\, x^0\, a^1$

$\therefore$ $P(1)$ is true.

Step II Let $P(m)$ be true. Then,

$$(x+a)^m = {}^mC_0\, x^m a^0 + {}^mC_1\, x^{m-1} a^1 + {}^mC_2\, x^{m-2} a^2 + \ldots + {}^mC_{m-1}\, x^1 a^{m-1} + {}^mC_m\, x^0 a^m \quad \ldots\text{(i)}$$

We shall now show that $P(m+1)$ is true. For this we have to show that

$$(x+a)^{m+1} = {}^{m+1}C_0\, x^{m+1} a^0 + {}^{m+1}C_1\, x^m a^1 + {}^{m+1}C_2\, x^{m-1} a^2 + \ldots + {}^{m+1}C_m\, x^1 a^m + {}^{m+1}C_{m+1}\, x^0 a^{m+1}$$

Now, $(x+a)^{m+1}$

$$= (x+a)\,.\,(x+a)^m = (x+a)\left[{}^mC_0\, x^m a^0 + {}^mC_1\, x^{m-1} a^1 + \ldots + {}^mC_r\, x^{m-r} a^r + \ldots + {}^mC_{m-1}\, x^1 a^{m-1} + {}^mC_m\, x^0 a^m\right]$$

$$= {}^mC_0\, x^{m+1} a^0 + ({}^mC_1 + {}^mC_0)\, x^m a^1 + ({}^mC_2 + {}^mC_1)\, x^{m-1} a^2 + \ldots + ({}^mC_r + {}^mC_{r-1})\, x^{m-r+1} a^r + \ldots + ({}^mC_{m-1} + {}^mC_m)\, x^1 a^m + {}^mC_m\, a^{m+1}$$

$$= {}^{m+1}C_0\, x^{m+1} a^0 + {}^{m+1}C_1\, x^m a^1 + {}^{m+1}C_2\, x^{m-1} a^2 + \ldots + {}^{m+1}C_r\, x^{(m+1)-r} a^r + \ldots + {}^{m+1}C_m\, x^1 a^m + {}^{m+1}C_{m+1}\, a^{m+1} \quad \left[\because\ {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r,\ r = 1, 2, 3, \ldots, m\right]$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, the theorem is true for all $n \in N$.

Q.E.D.

17.3 SOME IMPORTANT CONCLUSIONS FROM THE BINOMIAL THEOREM

In this section, we shall draw some useful conclusions from the binomial theorem.

(i) We have,

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$$

or,
$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots + {}^nC_r\, x^{n-r} a^r + \ldots + {}^nC_n\, x^0 a^n$$

Since r can have values from 0 to n, the total number of terms in the expansion is $(n+1)$.

(ii) The sum of the indices of x and a in each term is n.

(iii) Since ${}^nC_r = {}^nC_{n-r}$, for $r = 0, 1, 2, \ldots, n$

$$\therefore\ {}^nC_0 = {}^nC_n,\ {}^nC_1 = {}^nC_{n-1},\ {}^nC_2 = {}^nC_{n-2} = \ldots$$

So, the coefficients of terms equidistant from the beginning and end are equal. These coefficients are known as the binomial coefficients.

(iv) Replacing a by $-a$, we get

$$(x-a)^n = {}^nC_0\, x^n a^0 - {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 - {}^nC_3\, x^{n-3} a^3 + \ldots + (-1)^r\, {}^nC_r\, x^{n-r} a^r + \ldots + (-1)^n\, {}^nC_n\, x^0 a^n.$$

i.e.
$$(x-a)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^{n-r} a^r$$

Thus, the terms in the expansion of $(x-a)^n$ are alternatively positive and negative, the last term is positive or negative according as n is even or odd.

(v) Putting $x = 1$ and $a = x$ in the expansion of $(x + a)^n$, we get

$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r x^r + \ldots + {}^nC_n x^n$$

i.e. $$(1 + x)^n = \sum_{r=0}^{n} {}^nC_r x^r$$

This is the expansion of $(1 + x)^n$ in ascending powers of x.

(vi) Putting $a = 1$ in the expansion of $(x + a)^n$, we get

$$(1 + x)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \ldots + {}^nC_r x^{n-r} + \ldots + {}^nC_{n-1} x + {}^nC_n$$

i.e. $$(1 + x)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r}$$

This is the expansion of $(1 + x)^n$ in descending powers of x.

(vii) Putting $x = 1$ and $a = -x$ in the expansion of $(x + a)^n$, we get

$$(1 - x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \ldots + (-1)^r\, {}^nC_r x^r + \ldots + (-1)^n\, {}^nC_n x^n.$$

i.e. $$(1 - x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r x^r$$

(viii) The coefficient of $(r + 1)$th term in the expansion of $(1 + x)^n$ is ${}^nC_r$.

(ix) The coefficient of x^r in the expansion of $(1 + x)^n$ is ${}^nC_r$.

(x) $$(x + a)^n + (x - a)^n = 2\left\{{}^nC_0 x^n a^0 + {}^nC_2 x^{n-2} a^2 + {}^nC_4 x^{n-4} a^4 + \ldots\right\}$$

and, $$(x + a)^n - (x - a)^n = 2\left\{{}^nC_1 x^{n-1} a^1 + {}^nC_3 x^{n-3} a^3 + \ldots\right\}$$

NOTE: *If n is odd than $\{(x + a)^n + (x - a)^n\}$ and $\{(x + a)^n - (x - a)^n\}$ both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$ whereas if n is even, then $\{(x + a)^n + (x - a)^n\}$ has $\left(\frac{n}{2} + 1\right)$ terms and $\left\{(x + a)^n - (x - a)^n\right\}$ has $\left(\frac{n}{2}\right)$ terms.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I DETERMINING THE NUMBER OF TERMS IN THE EXPANSIONS OF BINOMIAL AND TRINOMIAL EXPRESSIONS

EXAMPLE 1 *Find the number of terms in the expansions of the following:*

(i) $(2x - 3y)^9$ (ii) $(1 + 5\sqrt{2}\,x)^9 + (1 - 5\sqrt{2}\,x)^9$

(iii) $(\sqrt{x} + \sqrt{y})^{10} + (\sqrt{x} - \sqrt{y})^{10}$ (iv) $(2x + 3y - 4z)^n$

(v) $[(3x + y)^8 - (3x - y)^8]$ (vi) $(1 + 2x + x^2)^{20}$

SOLUTION (i) The expansion of $(x + a)^n$ has $(n + 1)$ terms. So, the expansion of $(2x - 3y)^9$ has 10 terms.

(i) If n is odd, then the expansion of $(x + a)^n + (x - a)^n$ contains $\left(\frac{n+1}{2}\right)$ terms. So, the expansion of $(1 + 5\sqrt{2}x)^9 + (1 - 5\sqrt{2}\,x)^9$ has $\left(\frac{9+1}{2}\right) = 5$ terms.

(iii) If n is even, then the expansion of $\left\{(x+a)^n+(x-a)^n\right\}$ has $\left(\frac{n}{2}+1\right)$ terms.

So, $(\sqrt{x}+\sqrt{y})^{10}+(\sqrt{x}-\sqrt{y})^{10}$ has 6 terms.

(iv) We have,

$$(2x+3y-4z)^n=\left\{2x \;+(3y-4z)\right\}^n$$

$$={}^nC_0\,(2x)^n\,(3y-4z)^0+{}^nC_1\,(2x)^{n-1}\,(3y-4z)^1+{}^nC_2\,(2x)^{n-2}\,(3y-4z)^2+\ldots+{}^nC_{n-1}\,(2x)^1\,(3y-4z)^{n-1}+{}^nC_n\,(3y-4z)^n.$$

Clearly, the first term in the above expansion gives one term, second term gives two terms, third term gives three terms and so on.

So, total number of terms $=1+2+3+\ldots+n+(n+1)=\dfrac{(n+1)(n+2)}{2}$

(v) If n is even, then $\left\{(x+a)^n-(x-a)^n\right\}$ has $\frac{n}{2}$ terms. So, $(3x+y)^8-(3x-y)^8$ has 4 terms.

(vi) We have,

$$(1+2x+x^2)^{20}=\left\{(1+x)^2\right\}^{20}=(1+x)^{40}$$

So, there are 41 terms in the expansion of $(1+2x+x^2)^{20}$

Type II EXPANDING A GIVEN EXPRESSION USING THE BINOMIAL THEOREM

EXAMPLE 2 *Expand $(x^2+2a)^5$ by binomial theorem.*

SOLUTION Using binomial theorem,

$$(x^2+2a)^5={}^5C_0\,(x^2)^5\,(2a)^0+{}^5C_1\,(x^2)^4\,(2a)^1+{}^5C_2\,(x^2)^3\,(2a)^2+{}^5C_3\,(x^2)^2\,(2a)^3+{}^5C_4\,(x^2)\,(2a)^4+{}^5C_5\,(x^2)^0\,(2a)^5$$

$$=x^{10}+5\,(x^8)\,(2a)+10\,(x^6)\,(4a^2)+10\,(x^4)\,(8a^3)+5\,(x^2)\,(16\,a^4)+32\,a^5$$

$$=x^{10}+10\,x^8\,a+40\,x^6\,a^2+80\,x^4\,a^3+80\,x^2\,a^4+32\,a^5$$

EXAMPLE 3 *Expand $(2x-3y)^4$ by binomial theorem.*

SOLUTION Using binomial theorem, we obtain

$$(2x-3y)^4=\{2x+(-3y)\}^4$$

$$={}^4C_0\,(2x)^4\,(-3y)^0+{}^4C_1\,(2x)^3\,(-3y)+{}^4C_2\,(2x)^2\,(-3y)^2+{}^4C_3\,(2x)^1\,(-3y)^3+{}^4C_4\,(-3y)^4$$

$$=16\,x^4+4\,(8x^3)\,(-3y)+6\,(4x^2)\,(9y^2)+4\,(2x)\,(-27\,y^3)+81\,y^4$$

$$=16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$$

EXAMPLE 4 *By using binomial theorem, expand:*

(i) $(1+x+x^2)^3$ (ii) $(1-x+x^2)^4$

SOLUTION (i) Let $y=x+x^2$. Then, [NCERT EXEMPLAR]

$$(1+x+x^2)^3=(1+y)^3={}^3C_0+{}^3C_1\,y+{}^3C_2\,y^2+{}^3C_3\,y^3=1+3y+3y^2+y^3$$

$$=1+3\,(x+x^2)+3\,(x+x^2)^2+(x+x^2)^3$$

$$=1+3\,(x+x^2)+3\,(x^2+2x^3+x^4)+\left\{{}^3C_0\,x^3\,(x^2)^0+{}^3C_1\,x^{3-1}\,(x^2)^1+{}^3C_2\,x^{3-2}\,(x^2)^2+{}^3C_3\,x^0\,(x^2)^3\right\}$$

$$= 1 + 3(x + x^2) + 3(x^2 + 2x^3 + x^4) + (x^3 + 3x^4 + 3x^5 + x^6)$$

$$= x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$$

(ii) Let $y = -x + x^2$. Then,

$$(1 - x + x^2)^4 = (1 + y)^4 = {}^4C_0 + {}^4C_1\, y + {}^4C_2\, y^2 + {}^4C_3\, y^3 + {}^4C_4\, y^4$$

$$= 1 + 4y + 6y^2 + 4y^3 + y^4 = 1 + 4(-x + x^2) + 6(-x + x^2)^2 + 4(-x + x^2)^3 + (-x + x^2)^4$$

$$= 1 - 4x(1 - x) + 6x^2(1 - x)^2 - 4x^3(1 - x)^3 + x^4(1 - x)^4$$

$$= 1 - 4x(1 - x) + 6x^2(1 - 2x + x^2) - 4x^3({}^3C_0 - {}^3C_1 x + {}^3C_2 x^2 - {}^3C_3 x^3)$$
$$+ x^4\left({}^4C_0 - {}^4C_1 x + {}^4C_2 x^2 - {}^4C_3 x^3 + {}^4C_4 x^4\right)$$

$$= 1 - 4x + 4x^2 + 6x^2(1 - 2x + x^2) - 4x^3(1 - 3x + 3x^2 - x^3) + x^4(1 - 4x + 6x^2 - 4x^3 + x^4)$$

$$= 1 - 4x + 4x^2 + 6x^2 - 12x^3 + 6x^4 - 4x^3 + 12x^4 - 12x^5 + 4x^6 + x^4 - 4x^5 + 6x^6 - 4x^7 + x^8$$

$$= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$$

EXAMPLE 5 *Using binomial theorem, expand* $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4$, $x \neq 0$. [NCERT]

SOLUTION We have,

$$\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4 = \left\{1 + \left(\frac{x}{2} - \frac{2}{x}\right)\right\}^4$$

$$= {}^4C_0 + {}^4C_1\left(\frac{x}{2} - \frac{2}{x}\right) + {}^4C_2\left(\frac{x}{2} - \frac{2}{x}\right)^2 + {}^4C_3\left(\frac{x}{2} - \frac{2}{x}\right)^3 + {}^4C_4\left(\frac{x}{2} - \frac{2}{x}\right)^4$$

$$= 1 + 4\left(\frac{x}{2} - \frac{2}{x}\right) + 6\left(\frac{x^2}{4} - 2 + \frac{4}{x^2}\right) + 4\left\{\frac{x^3}{8} - \frac{8}{x^3} - 3\left(\frac{x}{2} - \frac{2}{x}\right)\right\}$$
$$+ \left\{{}^4C_0\left(\frac{x}{2}\right)^4\left(-\frac{2}{x}\right)^0 + {}^4C_1\left(\frac{x}{2}\right)^3\left(-\frac{2}{x}\right) + {}^4C_2\left(\frac{x}{2}\right)^2\left(-\frac{2}{x}\right)^2\right.$$
$$\left. + {}^4C_3\left(\frac{x}{2}\right)\left(-\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^0\left(-\frac{2}{x}\right)^4\right\}$$

$$= 1 + \left(2x - \frac{8}{x}\right) + 6\left(\frac{x^2}{4} - 2 + \frac{4}{x^2}\right) + 4\left(\frac{x^3}{8} - \frac{8}{x^3} - \frac{3x}{2} - \frac{6}{x}\right)$$
$$+ \left(\frac{x^4}{16} + 4 \times \frac{x^3}{8} \times -\frac{2}{x} + 6 \times \frac{x^2}{4} \times \frac{4}{x^2} + 4 \times \frac{x}{2} \times -\frac{8}{x^3} + \frac{16}{x^4}\right)$$

$$= 1 + \left(2x - \frac{8}{x}\right) + \left(\frac{3}{2}x^2 - 12 + \frac{24}{x^2}\right) + \left(\frac{x^3}{2} - \frac{32}{x^3} - 6x + \frac{24}{x}\right) + \left(\frac{x^4}{16} - x^2 + 6 - \frac{16}{x^2} + \frac{16}{x^4}\right)$$

$$= (1-12+6)+(2x-6x)+\left(\frac{3}{2}x^2-x^2\right)+\frac{x^3}{2}+\frac{x^4}{16}+\left(\frac{-8}{x}+\frac{24}{x}\right)+\left(\frac{24}{x^2}-\frac{16}{x^2}\right)-\frac{32}{x^3}+\frac{16}{x^4}$$

$$= -5-4x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}+\frac{16}{x}+\frac{8}{x^2}-\frac{32}{x^3}+\frac{16}{x^4}$$

EXAMPLE 6 *Find the expansion of* $(3x^2 - 2ax + 3a^2)^3$ *using binomial theorem.* [NCERT]

SOLUTION We have,

$$(3x^2 - 2ax + 3a^2)^3$$

$$= \left\{(3x^2 - 2ax) + 3a^2\right\}^3$$

$$= {}^3C_0\,(3x^2-2ax)^3\,(3a^2)^0 + {}^3C_1\,(3x^2-2ax)^2\,(3a^2) + {}^3C_2\,(3x^2-2ax)^1\,(3a^2)^2 + {}^3C_3\,(3x^2-2ax)^0\,(3a^2)^3$$

$$= (3x^2-2ax)^3 + 9a^2\,(3x^2-2ax)^2 + 27a^4\,(3x^2-2ax) + 27a^6$$

$$= \left\{{}^3C_0\,(3x^2)^3(-2ax)^0 + {}^3C_1(3x^2)^2\,(-2ax)^1 + {}^3C_2\,(3x^2)\,(-2ax)^2 + {}^3C_3\,(3x^2)^0\,(-2ax)^3\right\} + 9a^2\,(9x^4-12ax^3+4a^2x^2) + 27a^4\,(3x^2-2ax) + 27a^6$$

$$= (27x^6 - 54x^5a + 36x^4a^2 - 8x^3a^3) + (81x^4a^2 - 108x^3a^3 + 36x^2a^4) + (81x^2a^4 - 54xa^5) + 27a^6$$

$$= 27x^6 - 54x^5a + 117x^5a^2 - 116x^3a^3 + 117x^2a^4 - 54xa^5 + 27a^6$$

EXAMPLE 7 *Using binomial theorem, expand* $\left(x+\frac{1}{y}\right)^{11}$.

SOLUTION We have,

$$\left(x+\frac{1}{y}\right)^{11} = {}^{11}C_0\,x^{11}\left(\frac{1}{y}\right)^0 + {}^{11}C_1\,x^{10}\left(\frac{1}{y}\right) + {}^{11}C_2\,x^9\left(\frac{1}{y}\right)^2 + {}^{11}C_3\,x^8\left(\frac{1}{y}\right)^3$$
$$+ {}^{11}C_4\,x^7\left(\frac{1}{y}\right)^4 + {}^{11}C_5\,x^6\left(\frac{1}{y}\right)^5 + {}^{11}C_6\,x^5\left(\frac{1}{y}\right)^6 + {}^{11}C_7\,x^4\left(\frac{1}{y}\right)^7 + {}^{11}C_8\,x^3\left(\frac{1}{y}\right)^8$$
$$+ {}^{11}C_9\,x^2\left(\frac{1}{y}\right)^9 + {}^{11}C_{10}\,x\left(\frac{1}{y}\right)^{10} + {}^{11}C_{11}\left(\frac{1}{y}\right)^{11}$$

$$= x^{11} + 11\frac{x^{10}}{y} + 55\frac{x^9}{y^2} + 165\frac{x^8}{y^3} + 330\frac{x^7}{y^4} + 462\frac{x^6}{y^5} + 462\frac{x^5}{y^6} + \frac{330x^4}{y^7} + \frac{165x^3}{y^8} + \frac{55x^2}{y^9} + \frac{11x}{y^{10}} + \frac{1}{y^{11}}$$

EXAMPLE 8 *Prove that* $\sum_{r=0}^{n} {}^nC_r\, 3^r = 4^n$ [NCERT]

SOLUTION We have,

$$(1+x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + \ldots + {}^nC_n\,x^n$$

or, $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

Putting $x = 3$ on both sides, we get

$$(1+3)^n = \sum_{r=0}^{n} {}^nC_r \; 3^r \quad \text{or,} \quad 4^n = \sum_{r=0}^{n} {}^nC_r \; 3^r$$

Type III ON APPLICATIONS OF BINOMIAL THEOREM

EXAMPLE 9 *Find an approximation of* $(0.99)^5$ *using the first three terms of its expansion.* [NCERT]

SOLUTION We have,

$$(0.99)^5 = (1-0.01)^5 = \left(1-\frac{1}{100}\right)^5$$

$$= {}^5C_0 - {}^5C_1 \times \frac{1}{100} + {}^5C_2 \times \left(\frac{1}{100}\right)^2 - {}^5C_3\left(\frac{1}{100}\right)^3 + {}^5C_4\left(\frac{1}{100}\right)^4 - {}^5C_5\left(\frac{1}{100}\right)^5$$

$$= 1 - \frac{5}{100} + \frac{10}{10000} - \frac{10}{1000000} + \frac{5}{(100)^4} - \frac{1}{(100)^5}$$

$$= 1 - 0.05 + 0.001 = 0.951 \qquad \text{[Neglecting fourth and other terms]}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 10 *Using binomial theorem, compute the following:*

(i) $(99)^5$ (ii) $(102)^6$ (iii) $(10.1)^5$

SOLUTION (i) We have,

$$(99)^5 = (100-1)^5$$

$$= {}^5C_0 \times (100)^5 - {}^5C_1 \times (100)^4 + {}^5C_2 \times (100)^3 - {}^5C_3 \times (100)^2 + {}^5C_4 \times (100)^1 - {}^5C_5 \times (100)^0$$

$$= (100)^5 - 5 \times (100)^4 + 10 \times (100)^3 - 10 \times (100)^2 + 5 \times 100 - 1$$

$$= 10^{10} - 5 \times 10^8 + 10^7 - 10^5 + 5 \times 10^2 - 1$$

$$= (10^{10} + 10^7 + 5 \times 10^2) - (5 \times 10^8 + 10^5 + 1) = 10010000500 - 500100001 = 9509900499.$$

(ii) We have,

$$(102)^6 = (100+2)^6$$

$$= {}^6C_0 \times (100)^6 + {}^6C_1 \times (100)^5 \times 2 + {}^6C_2 \times (100)^4 \times 2^2$$
$$+ {}^6C_3 \times (100)^3 \times 2^3 + {}^6C_4 \times (100)^2 \times 2^4 + {}^6C_5 \times (100)^1 \times 2^5 + {}^6C_6 \times (100)^0 \times 2^6$$

$$= (100)^6 + 6 \times (100)^5 \times 2 + 15 \times (100)^4 \times 2^2 + 20 \times (100)^3 \times 2^3 + 15 \times (100)^2 \times 2^4$$
$$+ 6 \times (100)^1 \times 2^5 + 2^6$$

$$= 10^{12} + 12 \times 10^{10} + 6 \times 10^9 + 16 \times 10^7 + 24 \times 10^5 + 192 \times 10^2 + 64$$

$$= 1126162419264.$$

(iii) We have,

$$(10.1)^5 = (10+0.1)^5$$

$$= {}^5C_0 \times (10)^5 \times (0.1)^0 + {}^5C_1 \times (10)^4 \times (0.1) + {}^5C_2 \times (10)^3 \times (0.1)^2 + {}^5C_3 \times (10)^2 \times (0.1)^3$$
$$+ {}^5C_4 \times (10)^1 \times (0.1)^4 + {}^5C_5 \times (10)^0 \times (0.1)^5$$

$$= (10)^5 + 5 \times 10^4 \times 0.1 + 10 \times 10^3 \times (0.1)^2 + 10 \times (10)^2 \times (0.1)^3 + 5 \times 10 \times (0.1)^4 + (0.1)^5$$

$$= 10^5 + 5 \times 10^3 + 10^2 + 1 + 5 \times 0.001 + 0.00001$$

$$= 100000 + 5000 + 100 + 1 + 0.005 + 0.00001 = 105101.00501.$$

EXAMPLE 11 *Write down the binomial expansion of* $(1+x)^{n+1}$, *when* $x=8$. *Deduce that* $9^{n+1}-8n-9$ *is divisible by 64, where n is a positive integer.* [NCERT]

SOLUTION We have,

$$(1+x)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1\,x + {}^{n+1}C_2\,x^2 + {}^{n+1}C_3\,x^3 + \ldots + {}^{n+1}C_{n+1}\,x^{n+1}$$

Putting $x=8$, we get

$$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1\,(8)^1 + {}^{n+1}C_2(8)^2 + {}^{n+1}C_3(8)^3 + \ldots + {}^{n+1}C_{n+1}\,(8)^{n+1} \quad \ldots(\text{i})$$

$$\Rightarrow \quad 9^{n+1} = 1 + (n+1)\times 8 + {}^{n+1}C_2\,(8)^2 + {}^{n+1}C_3\,(8)^3 + \ldots + {}^{n+1}C_{n+1}\,(8)^{n+1}$$

$$\Rightarrow \quad 9^{n+1} - 8n - 9 = (8)^2\left\{{}^{n+1}C_2 + {}^{n+1}C_3\,(8) + {}^{n+1}C_4\,(8)^2 + \ldots + {}^{n+1}C_{n+1}\,(8)^{n-1}\right\}$$

$$\Rightarrow \quad 9^{n+1} - 8n - 9 = 64 \times \text{an integer}$$

$$\Rightarrow \quad 9^{n+1} - 8n - 9 \text{ is divisible by } 64.$$

EXAMPLE 12 *Using binomial theorem, prove that* $6^n - 5n$ *always leaves the remainder 1 when divided by 25.* [NCERT]

SOLUTION We have,

$$6^n - 5n = (1+5)^n - 5n$$

$$\Rightarrow \quad 6^n - 5n = \left\{{}^nC_0 + {}^nC_1\times(5) + {}^nC_2\times(5)^2 + {}^nC_3\times(5)^3 + \ldots + {}^nC_n\times(5)^n\right\} - 5n$$

$$\Rightarrow \quad 6^n - 5n = 1 + 5n + {}^nC_2\times 5^2 + {}^nC_3\times 5^3 + \ldots + {}^nC_n\times 5^n - 5n$$

$$\Rightarrow \quad 6^n - 5n - 1 = {}^nC_2\times 5^2 + {}^nC_3\times 5^3 + \ldots + {}^nC_n\times 5^n$$

$$\Rightarrow \quad 6^n - 5n - 1 = 5^2\left\{{}^nC_2 + {}^nC_3\times 5 + {}^nC_4\times 5^2 + \ldots + {}^nC_n\times 5^{n-2}\right\}$$

$$\Rightarrow \quad 6^n - 5n - 1 = 25\times \text{an integer}$$

$$\Rightarrow \quad 6^n - 5n = 25\times \text{an integer} + 1$$

$$\Rightarrow \quad 6^n - 5n \text{ leaves the remainder 1 when divided by 25.}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

Type IV ON EXPANSION OF A BINOMIAL BY USING BINOMIAL THEOREM

EXAMPLE 13 *Using binomial theorem, expand* $\left\{(x+y)^5 + (x-y)^5\right\}$ *and hence find the value of* $\left\{(\sqrt{2}+1)^5 + (\sqrt{2}-1)^5\right\}$.

SOLUTION We have,

$$(x+y)^5 + (x-y)^5 = 2\left\{{}^5C_0\,x^5 + {}^5C_2\,x^3\,y^2 + {}^5C_4\,x^1\,y^4\right\} = 2\left(x^5 + 10x^3y^2 + 5xy^4\right)$$

Putting $x=\sqrt{2}$ and $y=1$, we get

$$(\sqrt{2}+1)^5 + (\sqrt{2}-1)^5 = 2\left\{(\sqrt{2})^5 + 10(\sqrt{2})^3 + 5\sqrt{2}\right\} = 2\left(4\sqrt{2} + 20\sqrt{2} + 5\sqrt{2}\right) = 58\sqrt{2}$$

EXAMPLE 14 *If O be the sum of odd terms and E that of even terms in the expansion of* $(x+a)^n$, *prove that:*

(i) $O^2 - E^2 = (x^2 - a^2)^n$ (ii) $4\,OE = (x+a)^{2n} - (x-a)^{2n}$

(iii) $2(O^2 + E^2) = (x+a)^{2n} + (x-a)^{2n}$. [NCERT EXEMPLAR]

SOLUTION We have,

$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots + {}^nC_{n-1}\, x\, a^{n-1} + {}^nC_n\, a^n$$

$$\Rightarrow \quad (x+a)^n = \left\{{}^nC_0\, x^n a^0 + {}^nC_2\, x^{n-2} a^2 + \ldots\right\} + \left\{{}^nC_1\, x^{n-1} a^1 + {}^nC_3\, x^{n-3} a^3 + \ldots\right\}$$

$$\Rightarrow \quad (x+a)^n = O + E \qquad \ldots(i)$$

and, $$(x-a)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 - {}^nC_3\, x^{n-3} a^3 + \ldots + {}^nC_{n-1}\, x(-1)^{n-1} a^{n-1} + {}^nC_n (-1)^n a^n$$

$$\Rightarrow \quad (x-a)^n = \left\{{}^nC_0\, x^n + {}^nC_2\, x^{n-2} a^2 + \ldots\right\} - \left\{{}^nC_1\, x^{n-1} a^1 + {}^nC_3\, x^{n-3} a^3 + \ldots\right\}$$

$$\Rightarrow \quad (x-a)^n = O - E \qquad \ldots(ii)$$

(i) Multiplying (i) and (ii), we get

$$(x+a)^n (x-a)^n = (O+E)(O-E)$$

$$\Rightarrow \quad (x^2 - a^2)^n = O^2 - E^2$$

(ii) We have,

$$4\,OE = (O+E)^2 - (O-E)^2$$

$$\Rightarrow \quad 4\,OE = \left\{(x+a)^n\right\}^2 - \left\{(x-a)^n\right\}^2 \qquad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad 4\,OE = (x+a)^{2n} - (x-a)^{2n}$$

(iii) Squaring (i) and (ii) and then adding, we get

$$(x+a)^{2n} + (x-a)^{2n} = (O+E)^2 + (O-E)^2 = 2(O^2 + E^2).$$

Type V ON APPLICATIONS OF BINOMIAL THEOREM

EXAMPLE 15 *Which is larger* $(1.01)^{1000000}$ *or, 10,000 ?* [NCERT]

SOLUTION We have,

$$(1.01)^{1000000} - 10000$$

$$= (1 + 0.01)^{1000000} - 10000$$

$$= {}^{1000000}C_0 + {}^{1000000}C_1 (0.01) + {}^{1000000}C_2 (0.01)^2 + \ldots + {}^{1000000}C_{1000000} \times (0.01)^{1000000} - 10000$$

$$= (1 + 1000000 \times 0.01 + \text{other positive terms}) - 10000$$

$$= (1 + 10000 + \text{other positive terms}) - 10000$$

$$= 1 + \text{other positive terms} > 0$$

$$\therefore \quad (1.01)^{1000000} > 10000$$

EXAMPLE 16 *If a and b are distinct integers, prove that* $a^n - b^n$ *is divisible by* $(a-b)$*, whenever* $n \in N$. [NCERT]

SOLUTION We have,

$$a^n = \{(a-b) + b\}^n$$

$$\Rightarrow a^n = {}^nC_0 (a-b)^n + {}^nC_1 (a-b)^{n-1} b^1 + {}^nC_2 (a-b)^{n-2} b^2 + \ldots + {}^nC_{n-1} (a-b)\, b^{n-1} + {}^nC_n\, b^n$$

$$\Rightarrow a^n - b^n = (a-b)^n + {}^nC_1 (a-b)^{n-1} b^1 + {}^nC_2 (a-b)^{n-2} b^2 + \ldots + {}^nC_{n-1} (a-b)\, b^{n-1}$$

$\Rightarrow a^n - b^n = (a-b)\left\{(a-b)^{n-1} + {}^nC_1 (a-b)^{n-2} b + {}^nC_2 (a-b)^{n-3} b^2 + \dots + {}^nC_{n-1} b^{n-1}\right\}$

Clearly, RHS is divisible by $(a-b)$. Hence, $a^n - b^n$ is divisible by $(a-b)$.

EXAMPLE 17 *Using binomial theorem, prove that* $(101)^{50} > 100^{50} + 99^{50}$. [NCERT EXEMPLAR]

SOLUTION Let $x = 101^{50}$ and $y = 100^{50} + 99^{50}$. Then,

$$x - y = 101^{50} - 100^{50} - 99^{50}$$

$$\Rightarrow \quad x - y = 101^{50} - 99^{50} - 100^{50}$$

$$\Rightarrow \quad x - y = (100+1)^{50} - (100-1)^{50} - 100^{50}$$

$$\Rightarrow \quad x - y = 2\left\{{}^{50}C_1 \times 100^{49} + {}^{50}C_3 \times 100^{47} + \dots + {}^{50}C_{49} \times 100\right\} - 100^{50}$$

$$\Rightarrow \quad x - y = 100^{50} + 2 \times {}^{50}C_3 \times 100^{47} + \dots + 2 \times {}^{50}C_{49} \times 100 - 100^{50}$$

$$\Rightarrow \quad x - y = 2 \times {}^{50}C_3 \times 100^{47} + \dots + 2 \times {}^{50}C_{49} \times 100$$

$$\Rightarrow \quad x - y = \text{a positive integer}$$

$$\Rightarrow \quad x - y > 0 \Rightarrow x > y \Rightarrow 101^{50} > 100^{50} + 99^{50}$$

EXERCISE 17.1

BASIC

1. Using binomial theorem, write down the expansions of the following:

(i) $(2x+3y)^5$ (ii) $(2x-3y)^4$ (iii) $\left(x - \frac{1}{x}\right)^6$ (iv) $(1-3x)^7$

(v) $\left(ax - \frac{b}{x}\right)^6$ (vi) $\left(\sqrt{\frac{x}{a}} - \sqrt{\frac{a}{x}}\right)^6$ (vii) $(\sqrt[3]{x} - \sqrt[3]{a})^6$ (viii) $(1+2x-3x^2)^5$

(ix) $\left(x + 1 - \frac{1}{x}\right)^3$ (x) $(1-2x+3x^2)^3$

2. Evaluate the following:

(i) $(1+2\sqrt{x})^5 + (1-2\sqrt{x})^5$ (ii) $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$

(iii) $(3+\sqrt{2})^5 - (3-\sqrt{2})^5$ (iv) $(2+\sqrt{3})^7 + (2-\sqrt{3})^7$

(v) $(\sqrt{3}+1)^5 - (\sqrt{3}-1)^5$ (vi) $(0.99)^5 + (1.01)^5$

(vii) $(\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6$ [NCERT]

(viii) $\left(\sqrt{x+1} + \sqrt{x-1}\right)^6 + \left(\sqrt{x+1} - \sqrt{x-1}\right)^6$ (ix) $\left(x + \sqrt{x^2-1}\right)^6 + \left(x - \sqrt{x^2-1}\right)^6$

(x) $\left\{a^2 + \sqrt{a^2-1}\right\}^4 + \left\{a^2 - \sqrt{a^2-1}\right\}^4$ [NCERT, NCERT EXEMPLAR]

3. Find $(a+b)^4 - (a-b)^4$. Hence, evaluate $\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$. [NCERT]

4. Find $(x+1)^6 + (x-1)^6$. Hence, or otherwise evaluate $(\sqrt{2}+1)^6 + \sqrt{2}-1)^6$. [NCERT]

BASED ON LOTS

5. Using binomial theorem evaluate each of the following:
   (i) $(96)^3$ [NCERT] (ii) $(102)^5$ [NCERT] (iii) $(101)^4$ [NCERT] (iv) $(98)^5$ [NCERT]
6. Using binomial theorem, prove that $2^{3n} - 7n - 1$ is divisible by 49, where $n \in N$.
7. Using binomial theorem, prove that $3^{2n+2} - 8n - 9$ is divisible by 64, $n \in N$.
8. If n is a positive integer, prove that $3^{3n} - 26n - 1$ is divisible by 676.

BASED ON HOTS

9. Using binomial theorem, indicate which is larger $(1.1)^{10000}$ or 1000? [NCERT]
10. Using binomial theorem determine which number is smaller $(1.2)^{4000}$ or 800?
11. Find the value of $(1.01)^{10} + (1 - 0.01)^{10}$ correct to 7 places of decimal.
12. Show that $2^{4n+4} - 15n - 16$, where $n \in N$ is divisible by 225. [NCERT EXEMPLAR]

ANSWERS

1. (i) $32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$
   (ii) $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$
   (iii) $x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$
   (iv) $1 - 21x + 189x^2 - 945x^3 + 2835x^4 - 5103x^5 + 5103x^6 - 2187x^7$
   (v) $a^6x^6 - 6a^5x^4b + 15a^4x^2b^2 - 20a^3b^3 + 15\frac{a^2b^4}{x^2} - \frac{6ab^5}{x^4} + \frac{b^6}{x^6}$
   (vi) $\frac{x^3}{a^3} - 6\frac{x^2}{a^2} + 15\frac{x}{a} - 20 + 15\frac{a}{x} - 6\frac{a^2}{x^2} + \frac{a^3}{x^3}$
   (vii) $x^2 - 6x^{5/3}a^{1/3} + 15x^{4/3}a^{2/3} - 20ax + 15x^{2/3}a^{4/3} - 6x^{1/3}a^{5/3} + a^2$
   (viii) $1 + 10x + 25x^2 - 40x^3 - 190x^4 + 92x^5 + 570x^6 - 360x^7 - 675x^8 + 810x^9 - 243x^{10}$
   (ix) $x^3 + 3x^2 - 5 + \frac{3}{x^2} - \frac{1}{x^3}$
   (x) $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$
2. (i) $2(1 + 40x + 80x^2)$ (ii) 198 (iii) $1178\sqrt{2}$ (iv) 10084 (v) 152 (vi) 2.0020001
   (vii) $396\sqrt{6}$ (viii) $16x(4x^2 - 3)$ (ix) $64x^6 - 96x^4 + 36x^2 - 2$
   (x) $2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$
3. $8(a^3b + ab^3)$, $40\sqrt{6}$ 4. $2(x^6 + 15x^4 + 15x^2 + 1)$, 198
5. (i) 884736 (ii) 11040808032 (iii) 104060401 (iv) 9039207968
9. $(1.1)^{10000} > 1000$ 10. 800 11. 2.0090042

HINTS TO SELECTED PROBLEMS

2. (vii) We know that $(x+a)^n - (x-a)^n = 2\left\{{}^nC_1\, x^{n-1}a^1 + {}^nC_3\, x^{n-3}\, a^3 + ...\right\}$

$$\therefore\ (\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6 = 2\left\{{}^6C_1 (\sqrt{3})^5 (\sqrt{2})^1 + {}^6C_3 (\sqrt{3})^3 (\sqrt{2})^3 + {}^6C_5 (\sqrt{3})^1 (\sqrt{2})^5\right\}$$
$$= 2(6 \times 9 \times \sqrt{6} + 20 \times 6 \times \sqrt{6} + 6 \times 4 \times \sqrt{6})$$
$$= 2\left(54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}\right) = 2 \times 198\sqrt{6} = 396\sqrt{6}$$

(x) Using $(x+a)^n + (x-a)^n = 2\left\{{}^nC_0 x^n a^0 + {}^nC_2 x^{n-2} a^2 + \ldots\right\}$, we get

$$\left\{a^2 + \sqrt{a^2-1}\right\}^4 + \left\{a^2 - \sqrt{a^2-1}\right\}^4$$
$$= 2\left\{{}^4C_0 (a^2)^0 \left(\sqrt{a^2-1}\right)^4 + {}^4C_2 (a^2)^2 \left(\sqrt{a^2-1}\right)^2 + {}^4C_4 (a^2)^4 \left(\sqrt{a^2-1}\right)^0\right\}$$
$$= 2\left\{(a^2-1)^2 + 6a^4 (a^2-1) + a^8\right\}$$
$$= 2(a^8 + 6a^6 - 5a^4 - 2a^2 + 1) = 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$$

3. Using $(x+a)^n - (x-a)^n = 2\left\{{}^nC_1 x^{n-1} a^1 + {}^nC_3 x^{x-3} a^3 + \ldots\right\}$, we get

$$(a+b)^4 - (a-b)^4 = 2\left\{{}^4C_1\, a^3 b^1 + {}^4C_3\, a^1 b^2\right\} = 2(4a^3 b + 4ab^3) = 8ab(a^2+b^2)$$

$$\therefore\quad (\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4 = 8\sqrt{3} \times \sqrt{2}\left\{(\sqrt{3})^2 + (\sqrt{2})^2\right\} = 40\sqrt{6}$$

4. Using $(x+a)^n + (x-a)^n = 2\left({}^nC_0 x^n\, a^0 + {}^nC_2\, x^{x-2}\, a^2 + \ldots\right)$ we get

$$(x+1)^6 + (x-1)^6 = 2\left({}^6C_0 x^6 + {}^6C_2 x^4 + {}^6C_4 x^2 + {}^6C_6 x^0\right) = 2(x^6 + 15x^4 + 15x^2 + 1)$$

Putting $x = \sqrt{2}$, we get

$$(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 2\left\{(\sqrt{2})^6 + 15(\sqrt{2})^4 + 15(\sqrt{2})^2 + 1\right\} = 2(8 + 60 + 30 + 1) = 198$$

5. (i) $96^3 = (100-4)^3$

$$= {}^3C_0\, (100)^3\, (4)^0 - {}^3C_1\, (100)^2\, (4)^1 + {}^3C_2\, (100)^1\, (4)^2 - {}^3C_3\, (100)^0\, (4)^3$$
$$= 10^6 - 12 \times 10^4 + 4800 - 64 = 1000000 - 120000 + 4800 - 64 = 884736$$

(ii) $(102)^5 = (100+2)^5$

$$= {}^5C_0\, (100)^5\, 2^0 + {}^5C_1 (100)^4 \times 2 + {}^5C_2 \times (100)^3 \times 2^2 + {}^5C_3 \times (100)^2 \times 2^3$$
$$+ {}^5C_4 \times (100)^1 \times 2^4 + {}^5C_5 \times (100)^0 \times 2^5$$
$$= 10^{10} + 10^9 + 40 \times 10^6 + 80 \times 10^4 + 80 \times 10^2 + 32 = 11040808032$$

(iii) $(101)^4 = (10^2+1)^4$

$$= {}^4C_0\,(10^2)^0 + {}^4C_1(10^2)^1 + {}^4C_2\,(10^2)^2 + {}^4C_3\,(10^2)^3 + {}^4C_4\,(10^2)^4$$

$$= 1 + 400 + 6\times 10^4 + 4\times 10^6 + 10^8 = 104060401$$

(ix) $(98)^5 = (100-2)^5$

$$= {}^5C_0\,(100)^5 - {}^5C_1(100)^4\times 2 + {}^5C_2\times(100)^3\times 2 - {}^5C_3\times(100)^2\times 2^3 + {}^5C_4\times(100)^1\times 2^4 - {}^5C_5\times(100)^0\times 2^5$$

$$= 10^{10} - 10^9 + 40\times 10^6 + 8000 - 32 = 1039207968$$

9. Using $(x+a)^n = {}^nC_0x^na^0 + {}^nC_1x^{n-1}a^1 + {}^nC_3x^{n-3}a^3 + \ldots + {}^nC_n\,x^0\,a^n$, we get

$$(1.1)^{10000} = \left(1+\frac{1}{10}\right)^{10000}$$

$$= {}^{10000}C_0 + {}^{10000}C_1\times\frac{1}{10} + {}^{10000}C_2\times\left(\frac{1}{10}\right)^2 + \ldots + {}^{10000}C_{10000}\left(\frac{1}{10}\right)^{10000}$$

$$= 1 + 1000 + {}^{10000}C_2\times\left(\frac{1}{10}\right)^2 + \ldots + {}^{10000}C_{10000}\left(\frac{1}{10}\right)^{10000}$$

$$\therefore\quad (1.1)^{10000} - 1000 = 1 + {}^{10000}C_2\times\left(\frac{1}{10}\right)^2 + \ldots + {}^{10000}C_{10000}\left(\frac{1}{10}\right)^{10000}$$

$$\Rightarrow\quad (1.1)^{10000} - 1000 > 0 \Rightarrow (1.1)^{10000} > 1000$$

12. $2^{4n+4} - 15n - 16 = 2^{4(n+1)} - 15n - 15 - 1$

$$= (2^4)^{n+1} - 15\,(n+1) - 1$$

$$= 16^{n+1} - 15\,(n+1) - 1$$

$$= (1+15)^{n+1} - 15\,(n+1) - 1$$

$$= \left\{{}^{n+1}C_0 + {}^{n+1}C_1(15) + {}^{n+1}C_2\,(15)^2 + {}^{n+1}C_3(15)^3 + \ldots\ldots + {}^{n+1}C_{n+1}\,(15)^{n+1}\right\} - 15\,(n+1) - 1$$

$$= \left\{1 + 15\,(n+1) + {}^{n+1}C_2\,(15)^2 + {}^{n+1}C_3\,(15)^3 + \ldots + {}^{n+1}C_{n+1}\,(15)^{n+1}\right\} - 15\,(n+1) - 1$$

$$= 225\left\{{}^{n+1}C_2 + {}^{n+1}C_3\,(15) + \ldots + {}^{n+1}C_{n+1}\,(15)^{n-1}\right\}$$

$$= 225\times \text{A natural number.}$$

Hence, $2^{4n+4} - 15n - 16$ is divisible by 225.

17.4 GENERAL TERM AND MIDDLE TERMS IN A BINOMIAL EXPANSION

We have,

$$(x+a)^n = {}^nC_0\,x^n\,a^0 + {}^nC_1\,x^{n-1}\,a^1 + {}^nC_2\,x^{n-2}\,a^2 + \ldots + {}^nC_r\,x^{n-r}\,a^r + \ldots + {}^nC_n\,x^0\,a^n$$

We find that: The first term $= {}^nC_0\, x^n\, a^0$

The second term $= {}^nC_1\, x^{n-1}\, a^1$

The third term $= {}^nC_2\, x^{n-2}\, a^2$

The fourth term $= {}^nC_3\, x^{n-3}\, a^3$, and so on.

We thus observe that the suffix of C in any term is one less than the number of terms, the index of x is n minus the suffix of C and the index of a is the same as the suffix of C.

Hence, the $(r+1)$th term is given by ${}^nC_r\, x^{n-r}\, a^r$. Thus, if T_{r+1} denotes the $(r+1)^{th}$ term, then

$$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$$

This is called the *general term*, because by giving different values to r we can determine all terms of the expansion.

Since, $(x-a)^n = \{x+(-a)\}^n$. So, the general term in the binomial expansion of $(x-a)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r}(-a)^r = (-1)^r\, {}^nC_r\, x^{n-r} a^r$$

In the binomial expansion of $(1+x)^n$, the general term is given by

$$T_{r+1} = {}^nC_r\, x^r$$

In the binomial expansion of $(1-x)^n$, the general term is given by

$$T_{r+1} = (-1)^r\, {}^nC_r\, x^r$$

NOTE: *In the binomial expansion of $(x+a)^n$, the r^{th} term from the end is $((n+1)-r+1)$ $=(n-r+2)$th term from the beginning.*

17.4.1 MIDDLE TERMS IN A BINOMIAL EXPANSION

The binomial expansion of $(x+a)^n$ contains $(n+1)$ terms. Therefore,

(i) If n is even, then $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term.

(ii) If n is odd, then $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms are the two middle terms.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I ON FINDING THE GENERAL TERM OR AN INDICATED TERM IN THE BINOMIAL EXPANSION OF SOME GIVEN EXPRESSION

EXAMPLE 1 *Write the general term in the expansion of $(x^2-y)^6$.* [NCERT]

SOLUTION We have, $(x^2-y)^6 = \left\{x^2+(-y)\right\}^6$

The general term in the expansion of the above binomial is given by

$$T_{r+1} = {}^6C_r\, (x^2)^{6-r}\, (-y)^r = (-1)^r\, {}^6C_r\, x^{12-2r}\, y^r \quad [\because T_{r+1} = {}^nC_r\, x^{n-r}\, a^r]$$

EXAMPLE 2 *Find the 10th term in the binomial expansion of $\left(2x^2+\frac{1}{x}\right)^{12}$.*

SOLUTION We know that the $(r+1)$th term in the expansion of $(x+a)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$$

Therefore, in the expansion of $\left(2x^2 + \frac{1}{x}\right)^{12}$, the tenth term T_{10} is given by

$$T_{10} = T_{9+1} = {}^{12}C_9\, (2x^2)^{12-9} \left(\frac{1}{x}\right)^9 \qquad \left[\text{Here: } n = 12, r = 9, x = 2x^2 \text{ and } a = \frac{1}{x}\right]$$

$$\Rightarrow \quad T_{10} = {}^{12}C_9\, (2x^2)^3 \times \frac{1}{x^9} = {}^{12}C_9 \times 2^3 \left(\frac{1}{x^3}\right)$$

$$\Rightarrow \quad T_{10} = {}^{12}C_3\, \frac{8}{x^3} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} \times \frac{8}{x^3} = \frac{1760}{x^3} \qquad \left[\because\ {}^{12}C_9 = {}^{12}C_3\right]$$

EXAMPLE 3 *Find the 9th term in the expansion of* $\left(\frac{x}{a} - \frac{3a}{x^2}\right)^{12}$.

SOLUTION We know that the $(r+1)$th term in the expansion of $(x+a)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$$

Therefore, in the expansion of $\left(\frac{x}{a} - \frac{3a}{x^2}\right)^{12}$, the 9th term T_9 is given by

$$\Rightarrow \quad T_9 = T_{8+1} = {}^{12}C_8 \left(\frac{x}{a}\right)^{12-8} \left(-\frac{3a}{x^2}\right)^8 = {}^{12}C_8 \left(\frac{x}{a}\right)^4 \left(-\frac{3a}{x^2}\right)^8 = {}^{12}C_4 \times 3^8 \times \frac{a^4}{x^{12}}$$

$$\Rightarrow \quad T_9 = ({}^{12}C_4\, x^{-12}\, a^4)\, 3^8$$

EXAMPLE 4 *Find the 6th term in the expansion of* $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$.

SOLUTION Clearly, $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9 = \left\{\frac{4x}{5} + \left(-\frac{5}{2x}\right)\right\}^9$

$$\therefore \quad T_6 = T_{5+1} = {}^9C_5 \left(\frac{4x}{5}\right)^{9-5} \left(-\frac{5}{2x}\right)^5 \qquad [\because\ T_{r+1} = {}^nC_r\, x^{n-r}\, a^r]$$

$$\Rightarrow \quad T_6 = {}^9C_5 \left(\frac{4x}{5}\right)^4 (-1)^5 \left(\frac{5}{2x}\right)^5 = -\,{}^9C_4 \left(\frac{4x}{5}\right)^4 \left(\frac{5}{2x}\right)^5 \qquad [\because\ {}^9C_5 = {}^9C_4]$$

$$\Rightarrow \quad T_6 = -\frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \left(\frac{2^8 x^4}{5^4}\right) \left(\frac{5^5}{2^5 x^5}\right) = -\frac{5040}{x}$$

EXAMPLE 5 *Find 13th term in the expansion of* $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$, $x \neq 0$. [NCERT]

SOLUTION Clearly,

$$\left(9x - \frac{1}{3\sqrt{x}}\right)^{18} = \left\{9x + \left(\frac{-1}{3\sqrt{x}}\right)\right\}^{18}$$

$$\therefore \quad T_{13} = T_{12+1} = {}^{18}C_{12}\,(9x)^{18-12}\left(-\frac{1}{3\sqrt{x}}\right)^{12} = {}^{18}C_{12}\,(9x)^{6}\left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$= {}^{18}C_6 \times 9^6 \times x^6 \times \frac{1}{3^{12}\,x^6} = {}^{18}C_6 = \frac{18!}{12!\,6!} = 18564$$

EXAMPLE 6 *Find the 4th term from the end in the expansion of* $\left(\frac{3}{x^2} - \frac{x^3}{6}\right)^7$.

SOLUTION Clearly, the given expansion contains 8 terms.

So, 4th term from the end $= (8-4+1)$th $=$ 5th term from the beginning

$$\therefore \quad \text{Required term} = T_5 = T_{4+1} = {}^7C_4\left(\frac{3}{x^2}\right)^{7-4}\left(-\frac{x^3}{6}\right)^4$$

$$= {}^7C_3\left(\frac{3}{x^2}\right)^3\left(\frac{x^3}{6}\right)^4 = \frac{7\times6\times5}{3\times2\times1}\left(\frac{3^3}{x^6}\right)\left(\frac{x^{12}}{6^4}\right) = \frac{35}{48}x^6 \quad \left[\because\ {}^7C_4 = {}^7C_3\right]$$

EXAMPLE 7 *Find the 11th term from the end in the expansion of* $\left(2x - \frac{1}{x^2}\right)^{25}$.

SOLUTION Clearly, the given expansion contains 26 terms.

So, 11th term from the end $= (26-11+1)$th term from the beginning i.e. 16th term from the beginning

$$\therefore \quad \text{Required term} = T_{16} = T_{15+1} = {}^{25}C_{15}\,(2x)^{25-15}\left(-\frac{1}{x^2}\right)^{15}$$

$$= {}^{25}C_{15} \times 2^{10} \times x^{10} \times \frac{(-1)^{15}}{x^{30}} = -\,{}^{25}C_{15} \times \frac{2^{10}}{x^{20}}$$

Type II ON FINDING THE MIDDLE TERM(S)

EXAMPLE 8 *Find the middle term in the expansion of* $\left(\frac{2}{3}x^2 - \frac{3}{2x}\right)^{20}$.

SOLUTION Here $n = 20$, which is an even number. So, $\left(\frac{20}{2}+1\right)^{\text{th}}$ term i.e. 11th term is the middle term.

Hence, the middle term $= T_{11} = T_{10+1} = {}^{20}C_{10}\left(\frac{2}{3}x^2\right)^{20-10}\left(-\frac{3}{2x}\right)^{10} = {}^{20}C_{10}\,x^{10}$

EXAMPLE 9 *Find the middle terms in the expansion of* $\left(3x - \frac{x^3}{6}\right)^7$.

SOLUTION The given expression is $\left(3x - \frac{x^3}{6}\right)^7$. Here $n = 7$, which is an odd number.

So, $\left(\frac{7+1}{2}\right)^{th}$ and $\left(\frac{7+1}{2}+1\right)^{th}$ i.e. 4th and 5th terms are two middle terms.

Now, $T_4 = T_{3+1} = {}^7C_3\,(3x)^{7-3}\left(-\frac{x^3}{6}\right)^3 = (-1)^3\ {}^7C_3\,(3x)^4\left(\frac{x^3}{6}\right)^3 = -\frac{105x^{13}}{8}$

and, $T_5 = T_{4+1} = {}^7C_4\,(3x)^{7-4}\left(-\frac{x^3}{6}\right)^4 = {}^7C_4\,(3x)^3\left(-\frac{x^3}{6}\right)^4 = \frac{35\,x^{15}}{48}$

Hence, the middle terms are $-\frac{105\,x^{13}}{8}$ and $\frac{35\,x^{15}}{48}$

Type III ON FINDING THE COEFFICIENT FOR A GIVEN INDEX (EXPONENT) OF THE VARIABLE

EXAMPLE 10 *Find the coefficient of x^{10} in the binomial expansion of $\left(2x^2-\frac{3}{x}\right)^{11}$, when $x \neq 0$.*

SOLUTION Suppose $(r+1)$th term contains x^{10} in the binomial expansion of $\left(2x^2-\frac{3}{x}\right)^{11}$.

Now, $T_{r+1} = {}^{11}C_r\,(2x^2)^{11-r}\left(-\frac{3}{x}\right)^r = (-1)^r\ {}^{11}C_r\ 2^{(11-r)}\,.\,3^r\,.\,x^{22-3r}$ ...(i)

If T_{r+1} contains x^{10}, then $22-3r=10 \Rightarrow r=4$. So, $(4+1)$th i.e. 5th term contains x^{10}.

Putting $r=4$ in (i), we get

$$T_5 = (-1)^4\ {}^{11}C_4\ 2^{11-4}\times 3^4\times x^{10} = {}^{11}C_4\times 2^7\times 3^4\times x^{10}$$

$\therefore$ Coefficient of $x^{10} = {}^{11}C_4\times 2^7\times 3^4$

EXAMPLE 11 *Find the coefficients of x^{32} and x^{-17} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$.*

SOLUTION Suppose $(r+1)$th term involves x^{32} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$.

Now, $T_{r+1} = {}^{15}C_r\,(x^4)^{15-r}\left(-\frac{1}{x^3}\right)^r = (-1)^r\ {}^{15}C_r\ x^{60-7r}$ ...(i)

For this term to contain x^{32}, we must have: $60-7r=32 \Rightarrow r=4$.

So, $(4+1)$th i.e. 5th term contains x^{32}.

Putting $r=4$ in (i), we get

$$T_5 = (-1)^4\ {}^{15}C_4\ x^{(60-28)} = {}^{15}C_4\,x^{32}.$$

$\therefore$ Coefficient of $x^{32} = {}^{15}C_4 = 1365$.

Suppose $(s+1)$th term in the binomial expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$ contains x^{-17}.

Now, $T_{s+1} = {}^{15}C_s\,(x^4)^{15-s}\left(-\frac{1}{x^3}\right)^s = (-1)^s\ {}^{15}C_s\ x^{60-7s}$ ...(ii)

If this term contains x^{-17}, we must have: $60 - 7s = -17 \Rightarrow s = 11$

So, $(11+1)$th i.e. 12th term contains x^{-17}.

Putting $s = 11$ in (ii), we get

$$T_{12} = (-1)^{11}\ {}^{15}C_{11}\, x^{-17} = -\,{}^{15}C_{11}\, x^{-17} = -\,{}^{15}C_4\, x^{-17} \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

$\therefore$ Coefficient of $x^{-17} = -\,{}^{15}C_4 = -1365.$

EXAMPLE 12 *Find the coefficient of $x^6 y^3$ in the expansion of $(x+2y)^9$.* [NCERT]

SOLUTION Suppose $x^6 y^3$ occurs in $(r+1)^{th}$ term of the expansion of $(x+2y)^9$.

Now,

$$T_{r+1} = {}^9C_r \times (x)^{9-r} \times (2y)^r = {}^9C_r \times 2^r \times x^{9-r} \times y^r$$

This will contain $x^6 y^3$, if $9 - r = 6$ and $r = 3 \Rightarrow r = 3$

$\therefore$ Coefficient of $x^6 y^3 = {}^9C_3 \times 2^3 = \dfrac{9!}{3!\,6!} \times 2^3 = \dfrac{9 \times 8 \times 7 \times 6!}{3! \times 6!} \times 8 = 672$

EXAMPLE 13 *Find the coefficient of x^{40} in the expansion of $(1+2x+x^2)^{27}$.*

SOLUTION We have,

$$(1+2x+x^2)^{27} = \left\{(1+x)^2\right\}^{27} = (1+x)^{54}$$

Suppose x^{40} occurs in $(r+1)^{\text{th}}$ term in the expansion of $(1+x)^{54}$.

Now, $T_{r+1} = {}^{54}C_r\, x^r$

For this term to contain x^{40}, we must have $r = 40$. So, coefficient of $x^{40} = {}^{54}C_{40}$.

ALITER We know that the coefficient of x^r in $(1+x)^n$ is ${}^nC_r$.

$\therefore$ Coefficient of x^{40} in $(1+x)^{54}$ is ${}^{54}C_{40}$.

EXAMPLE 14 *Prove that there is no term involving x^6 in the expansion of $\left(2x^2 - \dfrac{3}{x}\right)^{11}$, where $r \neq 0$.*

SOLUTION Suppose x^6 occurs in $(r+1)^{\text{th}}$ term in the expansion of $\left(2x^2 - \dfrac{3}{x}\right)^{11}$.

$$\text{Now,}\quad T_{r+1} = {}^{11}C_r\,(2x^2)^{11-r}\left(-\frac{3}{x}\right)^r = {}^{11}C_r\,(-1)^r\, 2^{11-r}\, 3^r\, x^{22-3r} \qquad \text{...(i)}$$

For this term to contain x^6, we must have: $22 - 3r = 6 \Rightarrow r = \dfrac{16}{3}$, which is a fraction. But, r is a natural number. Hence, there is no term containing x^6.

Type IV ON FINDING THE TERM INDEPENDENT OF THE VARIABLE

EXAMPLE 15 *Find the term independent of x in the expansion of $\left(3x^2 - \dfrac{1}{2x^3}\right)^{10}$.*

SOLUTION Let $(r+1)$th term be independent of x in the given expression.

Now, $T_{r+1} = {}^{10}C_r \, (3x^2)^{10-r} \left(-\frac{1}{2x^3}\right)^r = {}^{10}C_r \, 3^{10-r} \left(-\frac{1}{2}\right)^r x^{20-5r}$ ...(i)

This term will be independent of x, if $20 - 5r = 0 \Rightarrow r = 4$.

So, $(4+1)$th i.e. 5th term is independent of x. Putting $r = 4$ in (i), we get

$$T_5 = {}^{10}C_4 \, 3^6 \left(-\frac{1}{2}\right)^4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \times \frac{729}{16} = \frac{76545}{8}$$

Hence, required term $= \dfrac{76545}{8}$

EXAMPLE 16 *Find the term independent of x in the expansion of*

(i) $\left(x - \dfrac{1}{x}\right)^{12}$ (ii) $\left(2x - \dfrac{1}{x}\right)^{10}$

SOLUTION (i) Let $(r+1)$th term be independent of x in the given expression.

Now, $T_{r+1} = {}^{12}C_r \, x^{12-r} \left(-\frac{1}{x}\right)^r = {}^{12}C_r \, (-1)^r \, x^{12-2r}$ ...(i)

For this term to be independent of x, we must have $12 - 2r = 0 \Rightarrow r = 6$.

So, $(6+1)$th i.e. 7th term is independent of x. Putting $r = 6$ in (i), we get

$$T_7 = {}^{12}C_6 \, (-1)^6 = {}^{12}C_6$$

Hence, required term $= {}^{12}C_6$

(ii) Let $(r+1)$th term be independent of x in the given expression.

Now, $T_{r+1} = {}^{10}C_r \, (2x)^{10-r} \left(-\frac{1}{x}\right)^r = (-1)^r \, {}^{10}C_r \, 2^{10-r} \, x^{10-2r}$ ...(i)

For this term to be independent of x, we must have $10 - 2r = 0 \Rightarrow r = 5$.

So, $(5+1)$th i.e. 6th term is independent of x. Putting $r = 5$ in (i), we get

$$T_6 = (-1)^5 \, {}^{10}C_5 \,.\, 2^{10-5} = -\,{}^{10}C_5 \times 2^5 = -\frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} \times 32 = -8064$$

Hence, required term $= -8064$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type I ON FINDING THE UNKNOWN WHEN A RELATION BETWEEN TWO OR MORE TERMS IS GIVEN.

EXAMPLE 17 *Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2} + \dfrac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$.* [NCERT]

SOLUTION We find that

Fifth term from the end $= (n + 1 - 5 + 1)^{\text{th}}$ term from the beginning

$= (n-3)^{\text{th}}$ term from the beginning

Now, $T_5 = T_{4+1} = {}^nC_4 \left\{\sqrt[4]{2}\right\}^{n-4} \left(\dfrac{1}{\sqrt[4]{3}}\right)^4 = {}^nC_4 \times 2^{\frac{n-4}{4}} \times \dfrac{1}{3}$

and, $T_{n-3} = T_{(n-4)+1} = {}^nC_{n-4} \left\{\sqrt[4]{2}\right\}^{n-(n-4)} \left(\dfrac{1}{\sqrt[4]{3}}\right)^{n-4} = {}^nC_{n-4} \times 2 \times \dfrac{1}{3^{\frac{n-4}{4}}}$

It is given that $\frac{T_5}{T_{n-3}} = \frac{\sqrt{6}}{1}$

$$\Rightarrow \quad \frac{{}^nC_4 \times 2^{\frac{n-4}{4}} \times \frac{1}{3}}{{}^nC_{n-4} \times 2 \times 3^{-\frac{n-4}{4}}} = \frac{\sqrt{6}}{1} \Rightarrow 2^{\frac{n-4}{4}-1} \times 3^{\frac{n-4}{4}-1} = 6^{1/2} \qquad \left[\because {}^nC_4 = {}^nC_{n-4}\right]$$

$$\Rightarrow \quad 2^{\frac{n-8}{4}} \times 3^{\frac{n-8}{4}} = 6^{1/2}$$

$$\Rightarrow \quad (2 \times 3)^{\frac{n-8}{4}} = 6^{1/2} \Rightarrow 6^{\frac{n-8}{4}} = 6^{1/2} \Rightarrow \frac{n-8}{4} = \frac{1}{2} \Rightarrow n-8 = 2 \Rightarrow n = 10$$

EXAMPLE 18 *Find a, if 17th and 18th terms in the expansion of $(2+a)^{50}$ are equal.* [NCERT]

SOLUTION We have,

$$T_{17} = T_{16+1} = {}^{50}C_{16}\,(2)^{50-16}\,a^{16} = {}^{50}C_{16} \times 2^{34} \times a^{16}$$

and, $$T_{18} = T_{17+1} = {}^{50}C_{17}\,(2)^{50-17}\,a^{17} = {}^{50}C_{17} \times 2^{33} \times a^{17}$$

It is given that 17^{th} and 18^{th} terms are equal.

i.e. $T_{17} = T_{18}$

$$\Rightarrow \quad {}^{50}C_{16} \times 2^{34} \times a^{16} = {}^{50}C_{17} \times 2^{33} \times a^{17}$$

$$\Rightarrow \quad \frac{{}^{50}C_{16}}{{}^{50}C_{17}} \times 2 = \frac{a^{17}}{a^{16}} \Rightarrow a = \frac{50!}{34!\,16!} \times \frac{33!\,17!}{50!} \times 2 = \frac{17}{34} \times 2 = 1$$

Type II ON MIDDLE TERM(S) IN A GIVEN EXPANSION

EXAMPLE 19 *Show that the middle term in the expansion of $(1+x)^{2n}$ is*

$$\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{n!}\, 2^n \cdot x^n \qquad \text{[NCERT]}$$

SOLUTION The exponent of $(1+x)$ in $(1+x)^{2n}$ is an even number $2n$.

So, $\left(\frac{2n}{2}+1\right)^{th}$ i.e. $(n+1)^{th}$ term is the middle term in the binomial expansion of $(1+x)^{2n}$.

Now, $$T_{n+1} = {}^{2n}C_n\,(1)^{2n-n}\,x^n = {}^{2n}C_n\,x^n = \frac{(2n)!}{(2n-n)!\,n!}\,x^n$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \ldots (2n-3)(2n-2)(2n-1)(2n)}{n!\,n!}\,x^n$$

$$= \frac{\{1 \cdot 3 \cdot 5 \ldots (2n-3)\ (2n-1)\}\{2 \cdot 4 \cdot 6 \ldots\ (2n-2)(2n)\}}{n!\,n!}\,x^n$$

$$= \frac{\{1 \cdot 3 \cdot 5 \ldots (2n-3)\ (2n-1)\}\{1 \cdot 2 \cdot 3 \ldots (n-1)\ (n)\}2^n}{n!\,n!}\,x^n$$

$$= \frac{\{1 \cdot 3 \cdot 5 \ldots (2n-3)\ (2n-1)\}\,n! \cdot 2^n \cdot x^n}{n!\,n!} = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{n!}\,2^n\,x^n$$

EXAMPLE 20 *Show that the middle term in the expansion of* $\left(x-\frac{1}{x}\right)^{2n}$ *is*

$$\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}(-2)^n$$ [NCERT EXEMPLAR]

SOLUTION The exponent in $\left(x-\frac{1}{x}\right)^{2n}$ is an even natural number. So, $\left(\frac{2n}{2}+1\right)^{\text{th}}$ i.e. $(n+1)^{\text{th}}$ term is the middle term and is given by

$$T_{n+1} = {}^{2n}C_n\,(x)^{2n-n}\left(-\frac{1}{x}\right)^n = \frac{(2n)!}{n!\,n!}x^n \times \frac{(-1)^n}{x^n} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\ldots(2n-1)(2n)}{n!n!}\times(-1)^n$$

$$\Rightarrow \quad T_{n+1} = \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\}\{2\cdot 4\cdot 6\ldots(2n-2)(2n)\}}{n!n!}\times(-1)^n$$

$$\Rightarrow \quad T_{n+1} = \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\}\{1\cdot 2\cdot 3\ldots(n-1)\,n\}}{n!n!}\times(-1)^n$$

$$\Rightarrow \quad T_{n+1} = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\times 2^n\times(-1)^n = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\times(-2)^n$$

EXAMPLE 21 *Prove that the coefficient of the middle term in the expansion of* $(1+x)^{2n}$ *is equal to the sum of the coefficients of middle terms in the expansion of* $(1+x)^{2n-1}$. [NCERT]

SOLUTION As discussed in the previous example, the middle term in the expansion of $(1+x)^{2n}$ is given by $T_{n+1} = {}^{2n}C_n\,x^n$.

So, the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is ${}^{2n}C_n$.

Now, consider the expansion of $(1+x)^{2n-1}$. Here, the index $(2n-1)$ is odd.

So, $\left(\frac{(2n-1)+1}{2}\right)^{\text{th}}$ and $\left(\frac{(2n-1)+1}{2}+1\right)^{\text{th}}$ i.e. n^{th} and $(n+1)^{\text{th}}$ terms are middle terms.

Now, $T_n = T_{(n-1)+1} = {}^{2n-1}C_{n-1}\,(1)^{(2n-1)-(n-1)}\,x^{n-1} = {}^{2n-1}C_{n-1}\,x^{n-1}$

and, $T_{n+1} = {}^{2n-1}C_n\,(1)^{(2n-1)-n}\,x^n = {}^{2n-1}C_n\,x^n$

So, the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$ are ${}^{2n-1}C_{n-1}$ and ${}^{2n-1}C_n$.

$\therefore$ Sum of these coefficients $= {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$

$= {}^{(2n-1)+1}C_n = {}^{2n}C_n$ $\qquad [\because\ {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]$

$=$ Coefficient of middle term in the expansion of $(1+x)^{2n}$

Type III ON FINDING THE COEFFICIENT OF A GIVEN EXPONENT OF THE VARIABLE

EXAMPLE 22 *Find the coefficient of* x^5 *in the expansion of the product* $(1+2x)^6\,(1-x)^7$. [NCERT]

SOLUTION We have,

$$(1+2x)^6(1-x)^7 = \left\{1 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 + {}^6C_5(2x)^5 + {}^6C_6(2x)^6\right\}$$
$$\times\left\{1 - {}^7C_1 x + {}^7C_2 x^2 - {}^7C_3 x^3 + {}^7C_4 x^4 - {}^7C_5 x^5 + \dots\right\}$$
$$= (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + \dots)$$
$$\times (1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + \dots)$$

$\therefore$ Coefficient of x^5 in the product $= 1\times(-21) + 12\times 35 + 60\times(-35) + 160\times 21 + 240\times -7 + 192\times 1$

$$= -21 + 420 - 2100 + 3360 - 1680 + 192 = 171$$

EXAMPLE 23 *Find the value of a so that the term independent of x in* $\left(\sqrt{x} + \dfrac{a}{x^2}\right)^{10}$ *is 405.*

SOLUTION Let $(r+1)^{\text{th}}$ term in the expansion of $\left(\sqrt{x} + \dfrac{a}{x^2}\right)^{10}$ be independent of x.

Now,

$$T_{r+1} = {}^{10}C_r\,(\sqrt{x})^{10-r}\left(\frac{a}{x^2}\right)^r = {}^{10}C_r\; x^{5-\frac{r}{2}-2r}\, a^r \qquad \text{...(i)}$$

This will be independent of x, if

$$5 - \frac{r}{2} - 2r = 0 \Rightarrow 5 - \frac{5r}{2} = 0 \Rightarrow 5 = \frac{5r}{2} \Rightarrow r = 2$$

Putting $r = 2$ in (i), we get: $T_3 = {}^{10}C_2\, a^2$

It is given that the term independent of x is equal to 405.

$\therefore \quad {}^{10}C_2\, a^2 = 405 \Rightarrow 45a^2 = 405 \Rightarrow a^2 = 9 \Rightarrow a = \pm 3$

Type IV PROBLEMS RELATING TO COEFFICIENTS IN A BINOMIAL EXPANSION

In solving the problems relating the coefficients in the binomial expansion we generally use the following results:

(i) Coefficient of $(r+1)$th term in the binomial expansion of $(1+x)^n$ is ${}^nC_r$.

(ii) Coefficient of x^r in the binomial expansion of $(1+x)^n$ is ${}^nC_r$.

(iii) Coefficient of x^r in the expansion of $(1-x)^n$ is $(-1)^r\ {}^nC_r$.

(iv) Coefficient of $(r+1)$th term in the expansion of $(1-x)^n$ is $(-1)^r\ {}^nC_r$.

EXAMPLE 24 *In the binomial expansion of* $(1+a)^{m+n}$*, prove that the coefficients of* a^m *and* a^n *are equal.* **[NCERT]**

SOLUTION Let A and B be the coefficients of a^m and a^n respectively in the expansion of $(1+a)^{m+n}$. Then,

$$A = \text{Coefficient of } a^m \text{ in the binomial expansion of } (1+a)^{m+n} = {}^{m+n}C_m = \frac{(m+n)!}{m!\,n!} \qquad \text{...(i)}$$

$$B = \text{Coefficient of } a^n \text{ in the binomial expansion of } (1+a)^{m+n} = {}^{m+n}C_n = \frac{(m+n)!}{m!\,n!} \qquad \text{...(ii)}$$

Clearly, $A = B$ i.e. the coefficients of a^m and a^n in the binomial expansion of $(1+a)^{m+n}$ are equal.

EXAMPLE 25 *Prove that the coefficients of* x^n *in* $(1+x)^{2n}$ *is twice the coefficient of* x^n *in* $(1+x)^{2n-1}$. **[NCERT]**

SOLUTION Let A and B be the coefficients of x^n in the binomial expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively. Then,

$$A = \text{Coefficient of } x^n \text{ in } (1+x)^{2n} = {}^{2n}C_n = \frac{(2n)!}{n!\,n!} = \frac{(2n)(2n-1)!}{n(n-1)!\,n!} = 2\,\frac{(2n-1)!}{(n-1)!\,n!} \quad \text{...(i)}$$

and,

$$B = \text{Coefficient of } x^n \text{ in } (1+x)^{2n-1} = {}^{2n-1}C_n = \frac{(2n-1)!}{(n-1)!\,n!} \quad \text{...(ii)}$$

From (i) and (ii), we find that:

$$A = 2B \quad \text{i.e. Coefficient of } x^n \text{ in } (1+x)^{2n} = 2 \times \text{Coefficient of } x^n \text{ in } (1+x)^{2n-1}.$$

EXAMPLE 26 *In the binomial expansion of $(a+b)^n$, the coefficients of the fourth and thirteenth terms are equal to each other. Find n.*

SOLUTION The coefficients of the fourth and thirteenth terms in the binomial expansion of $(a+b)^n$ are ${}^nC_3$ and ${}^nC_{12}$ respectively. It is given that:

Coefficient of 4th term in $(a+b)^n$ = Coefficient of 13th term in $(a+b)^n$

$$\Rightarrow \quad {}^nC_3 = {}^nC_{12} \Rightarrow n = 15 \qquad [\because\ {}^nC_x = {}^nC_y \Rightarrow x = y, \text{ or } x + y = n]$$

EXAMPLE 27 *Find a positive value of m for which the coefficient of x^2 in the expansion of $(1+x)^m$ is 6.* **[NCERT]**

SOLUTION We know that the coefficient of x^r in $(1+x)^n$ is ${}^nC_r$. Therefore, coefficient of x^2 in $(1+x)^m$ is ${}^mC_2$. It is given that the coefficient of x^2 in $(1+x)^m$ is 6.

$$\therefore \quad {}^mC_2 = 6 \Rightarrow \frac{m(m-1)}{2!} = 6$$

$$\Rightarrow m^2 - m = 12 \Rightarrow m^2 - m - 12 = 0 \Rightarrow (m-4)(m+3) = 0 \Rightarrow m - 4 = 0 \Rightarrow m = 4. \qquad [\because\ m + 3 \neq 0]$$

EXAMPLE 28 *If the coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms in the expansion of $(1+x)^{34}$ are equal, find r.* **[NCERT]**

SOLUTION We know that the coefficient of r^{th} term in the expansion of $(1+x)^n$ is ${}^nC_{r-1}$. Therefore, Coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms in the expansion of $(1+x)^{34}$ are ${}^{34}C_{r-6}$ and ${}^{34}C_{2r-2}$ respectively. It is given that these coefficients are equal

$$\therefore \quad {}^{34}C_{r-6} = {}^{34}C_{2r-2}$$

$$\Rightarrow \quad r - 6 = 2r - 2 \text{ or, } r - 6 + 2r - 2 = 34 \qquad \left[\because\ {}^nC_r = {}^nC_s \Rightarrow r = s \text{ or, } r + s = n\right]$$

$$\Rightarrow \quad 3r - 8 = 34 \qquad [\because\ r - 6 = 2r - 2 \Rightarrow r = -4, \text{ which is not possible}]$$

$$\Rightarrow \quad 3r = 42 \Rightarrow r = 14$$

Type V PROBLEMS BASED ON CONSECUTIVE TERMS OR CONSECUTIVE COEFFICIENTS

If consecutive terms or coefficients of consecutive terms in the expansion of $(x+a)^n$ are given, we assume that the consecutive terms are r^{th}, $(r+1)^{\text{th}}$ and $(r+2)^{\text{th}}$ i.e. T_r, T_{r+1} and T_{r+2}.

In case of consecutive terms, we find $\frac{T_{r+1}}{T_r}$ and $\frac{T_r}{T_{r-1}}$. It should be noted that $\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \frac{a}{x}$.

In case of consecutive coefficients, we find the ratios $\frac{r^{\text{th}} \text{ coefficient}}{(r+1)^{\text{th}} \text{ coefficient}}$ and $\frac{(r+1)^{th} \text{ coefficient}}{(r+2)^{\text{th}} \text{ coefficient}}$ etc. to get equations and solve them. In computing these ratios, we may use the following results:

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} \text{ and } \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1}$$

EXAMPLE 29 *The coefficients of three consecutive terms in the expansion of $(1+x)^n$ are in the ratio 1 : 7 : 42. Find n.* [NCERT]

SOLUTION Let the three consecutive terms be rth, $(r+1)$th and $(r+2)$th terms. Their coefficients in the expansion of $(1+x)^n$ are ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ respectively. It is given that,

$${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 1 : 7 : 42 \text{ i.e. } \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{1}{7} \text{ and } \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{7}{42}$$

Now, $\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{1}{7} \Rightarrow \frac{r}{n-r+1} = \frac{1}{7} \Rightarrow n - 8r + 1 = 0$ ...(i) $\left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right]$

and, $\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{7}{42} \Rightarrow \frac{r+1}{n-r} = \frac{1}{6} \Rightarrow n - 7r - 6 = 0$ ...(ii) $\left[\because \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1}\right]$

Solving (i) and (ii), we get $r = 7$ and $n = 55$.

EXAMPLE 30 *In the binomial expansion of $(1+x)^n$, the coefficients of the fifth, sixth and seventh terms are in A.P. Find all values of n for which this can happen.*

SOLUTION The coefficients of fifth, sixth and seventh terms in the binomial expansion of $(1+x)^n$ are ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ respectively. We are given that ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P.

$\therefore \quad 2\,{}^nC_5 = {}^nC_4 + {}^nC_6$

$\Rightarrow \quad 2 = \frac{{}^nC_4}{{}^nC_5} + \frac{{}^nC_6}{{}^nC_5}$ [Dividing both sides by ${}^nC_5$]

$\Rightarrow \quad 2 = \frac{5}{n-4} + \frac{n-5}{6}$ $\left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right]$

$\Rightarrow \quad 2 = \frac{30 + (n-4)(n-5)}{6(n-4)}$

$\Rightarrow \quad 12n - 48 = 30 + n^2 - 9n + 20 \Rightarrow n^2 - 21n + 98 = 0 \Rightarrow (n-14)(n-7) = 0 \Rightarrow n = 7, 14.$

EXAMPLE 31 *If the coefficients of a^{r-1}, a^r, a^{r+1} in the binomial expansion of $(1+a)^n$ are in A.P., prove that $n^2 - n(4r+1) + 4r^2 - 2 = 0$.* [NCERT]

SOLUTION The coefficients of a^{r-1}, a^r and a^{r+1} in the binomial expansion of $(1+a)^n$ are ${}^nC_{r-1}, {}^nC_r$ and ${}^nC_{r+1}$ respectively. It is given that ${}^nC_{r-1}, {}^nC_r$ and ${}^nC_{r+1}$ are in A.P.

$$\therefore \quad 2\,{}^nC_r = {}^nC_{r-1} + {}^nC_{r+1}$$

$$\Rightarrow \quad 2 = \frac{{}^nC_{r-1}}{{}^nC_r} + \frac{{}^nC_{r+1}}{{}^nC_r}$$

$$\Rightarrow \quad 2 = \frac{r}{n-r+1} + \frac{n-r}{r+1} \qquad \left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

$$\Rightarrow \quad 2 = \frac{r(r+1) + (n-r)(n-r+1)}{(r+1)(n-r+1)}$$

$$\Rightarrow \quad 2\left\{(n-r+1)\ (r+1)\right\} = r(r+1) + (n-r)(n-r+1)$$

$$\Rightarrow \quad 2nr - 2r^2 + 2n + 2 = r^2 + r + n^2 - 2nr + r^2 + n - r$$

$$\Rightarrow \quad n^2 - 4nr - n + 4r^2 - 2 = 0 \Rightarrow n^2 - n(4r+1) + 4r^2 - 2 = 0$$

EXAMPLE 32 *The coefficients of $(r-1)^{th}$, rth and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ are in the ratio 1:3:5. Find n and r.* [NCERT]

SOLUTION We know that the coefficient of rth term in the expansion of $(x+1)^n$ is ${}^nC_{r-1}$. Therefore, coefficients of $(r-1)^{\text{th}}$, r^{th} and $(r+1)^{\text{th}}$ terms are ${}^nC_{r-2}$, ${}^nC_{r-1}$ and ${}^nC_r$ respectively. It is given that

$${}^nC_{r-2} : {}^nC_{r-1} : {}^nC_r = 1:3:5$$

$$\Rightarrow \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{5}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_{r-2}} = \frac{3}{1} \Rightarrow \frac{n-r+1}{r} = \frac{5}{3} \text{ and } \frac{n-r+2}{r-1} = \frac{3}{1} \qquad \left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

$$\Rightarrow 3n - 8r + 3 = 0 \text{ and } n - 4r + 5 = 0 \Rightarrow n = 7 \text{ and } r = 3$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

Type I ON FINDING THE UNKNOWN WHEN THE VALUE OF A TERM IS GIVEN

EXAMPLE 33 *If the third term in the expansion of $\left(\frac{1}{x} + x^{\log_{10} x}\right)^5$ is 1000, then find x.*

SOLUTION We have, $T_3 = 1000$

$$\Rightarrow \quad T_{2+1} = 1000$$

$$\Rightarrow \quad {}^5C_2 \left(\frac{1}{x}\right)^{5-2} (x^{\log x})^2 = 1000 \Rightarrow 10\,(x^{\log_{10} x})^2 \times x^{-3} = 1000$$

$$\Rightarrow \quad x^{2\log_{10} x} \times x^{-3} = 100 \qquad \Rightarrow x^{2\log_{10} x - 3} = 10^2 \Rightarrow 2\log_{10} x - 3 = \log_x 10^2$$

$$\Rightarrow \quad 2\log_{10} x - 3 = \frac{2}{\log_{10} x} \qquad \Rightarrow 2y - 3 = \frac{2}{y}, \quad \text{where } y = \log_{10} x$$

$$\Rightarrow \quad 2y^2 - 3y - 2 = 0 \Rightarrow (2y+1)(y-2) = 0 \Rightarrow y = 2 \text{ or } y = -\frac{1}{2}$$

$\Rightarrow \quad \log_{10} x = 2$ or, $\log_{10} x = -\frac{1}{2} \Rightarrow x = 10^2 = 100$ or, $x = 10^{-1/2} = \frac{1}{\sqrt{10}}$.

EXAMPLE 34 *If the fourth term in the expansion of* $\left\{\sqrt{x^{\frac{1}{\log x+1}}} + x^{\frac{1}{12}}\right\}^6$ *is equal to 200 and* $x > 1$, *then find* x.

SOLUTION It is given that $T_4 = 200$

$\Rightarrow T_{3+1} = 200$

$$\Rightarrow {}^6C_3 \left\{\sqrt{x^{\frac{1}{\log x+1}}}\right\}^{6-3} (x^{1/12})^3 = 200 \Rightarrow 20\left(x^{\frac{1}{\log x+1}}\right)^{3/2} x^{1/4} = 200$$

$$\Rightarrow x^{\frac{3}{2}\left(\frac{1}{\log x+1}\right)+\frac{1}{4}} = 10 \Rightarrow \frac{3}{2}\left(\frac{1}{\log x+1}\right) + \frac{1}{4} = \log_x 10$$

$$\Rightarrow \frac{3}{2}\left(\frac{1}{\log_{10} x+1}\right) + \frac{1}{4} = \frac{1}{\log_{10} x} \Rightarrow \frac{3}{2(y+1)} + \frac{1}{4} = \frac{1}{y}, \text{ where } y = \log_{10} x$$

$$\Rightarrow \frac{6+y+1}{4(y+1)} = \frac{1}{y} \Rightarrow y^2 + 3y - 4 = 0 \Rightarrow (y+4)(y-1) = 0 \Rightarrow y = 1, -4$$

$\Rightarrow \log_{10} x = 1, -4 \Rightarrow x = 10$ or, $x = 10^{-4} \Rightarrow x = 10$ $\quad [\because x > 1]$

EXAMPLE 35 *For what value of x is the ninth term in the expansion of*

$$\left\{3^{\log_3 \sqrt{25^{x-1}+7}} + 3^{(-1/8)\log_3 (5^{x-1}+1)}\right\}^{10}$$ *is equal to 180?*

SOLUTION We know that $a^{\log_a N} = N$.

$$\therefore \quad \left\{3^{\log_3 \sqrt{25^{x-1}+7}} + 3^{(-1/8)\log_3 (5^{x-1}+1)}\right\}^{10} = \left\{\sqrt{25^{x-1}+7} + (5^{x-1}+1)^{-1/8}\right\}^{10}$$

Let T_9 be the 9th term in the above expansion. Then,

$$T_9 = 180$$

$$\Rightarrow \quad {}^{10}C_8 \left\{\sqrt{25^{x-1}+7}\right\}^{10-8} \left\{(5^{x-1}+1)^{-1/8}\right\}^8 = 180$$

$$\Rightarrow \quad {}^{10}C_8 \, (25^{x-1}+7)\,(5^{x-1}+1)^{-1} = 180$$

$$\Rightarrow \quad \frac{45(25^{x-1}+7)}{5^{x-1}+1} = 180 \Rightarrow \frac{25^{x-1}+7}{5^{x-1}+1} = 4 \Rightarrow \frac{y^2+7}{y+1} = 4, \text{ where } y = 5^{x-1}$$

$$\Rightarrow \quad y^2 - 4y + 3 = 0 \Rightarrow (y-3)(y-1) = 0 \Rightarrow y = 3, -1$$

$$\Rightarrow \quad 5^{x-1} = 3 \text{ or, } 5^{x-1} = 1 \Rightarrow 5^x = 15 \text{ or, } 5^x = 5 \Rightarrow x = \log_5 15 \text{ or, } x = 1.$$

EXAMPLE 36 *If the fourth term in the expansion of* $\left(ax+\frac{1}{x}\right)^n$ *is* $\frac{5}{2}$, *then find the values of a and n.*

SOLUTION It is given that

$$T_4 = \frac{5}{2} \Rightarrow T_{3+1} = \frac{5}{2} \Rightarrow {}^nC_3\,(ax)^{n-3}\left(\frac{1}{x}\right)^3 = \frac{5}{2} \Rightarrow {}^nC_3\, a^{n-3}\, x^{n-6} = \frac{5}{2} \quad ...(i)$$

Clearly, RHS of the above equality is independent of x. Therefore, $n-6 = 0 \Rightarrow n = 6$.
Putting $n = 6$ in (i), we get

$$^6C_3\, a^3 = \frac{5}{2} \Rightarrow \frac{6\times5\times4}{3\times2\times1}a^3 = \frac{5}{2} \Rightarrow a^3 = \frac{1}{8} \Rightarrow a = \frac{1}{2}$$

Hence, $a=\frac{1}{2}$ and $n=6$.

Type II ON MIDDLE TERM (S) IN A BINOMIAL EXPENSION

EXAMPLE 37 *Find the value of* α *for which the coefficients of the middle terms in the expansions of* $(1+\alpha x)^4$ *and* $(1-\alpha x)^6$ *are equal, find* α.

SOLUTION In the expansion of $(1+\alpha x)^4$. Middle term $= {}^4C_2\ (\alpha x)^2 = 6\alpha^2 x^2$

In the expansion of $(1-\alpha x)^6$. Middle term $= {}^6C_3\,(-\alpha x)^3 = -20\,\alpha^3 x^3$

It is given that:

Coefficient of the middle term in $(1+\alpha x)^4$ = Coefficient of the middle term in $(1-\alpha x)^6$

$$\Rightarrow \quad 6\,\alpha^2 = -20\alpha^3 \Rightarrow \alpha = 0,\ \alpha = -\frac{3}{10}$$

EXAMPLE 38 *If the middle term in the binomial expansion of* $\left(\frac{1}{x}+x\sin x\right)^{10}$ *is equal to* $\frac{63}{8}$, *find the value of x.* [NCERT EXEMPLAR]

SOLUTION In the binomial expansion of $\left(\frac{1}{x}+x\sin x\right)^{10}$, $\left(\frac{10}{2}+1\right)^{\text{th}}$ i.e. 6th term is the middle term. It is given that

$$T_6 = \frac{63}{8}$$

$$\Rightarrow \quad {}^{10}C_5\left(\frac{1}{x}\right)^{10-5}(x\sin x)^5 = \frac{63}{8}$$

$$\Rightarrow \quad \frac{10!}{5!5!}(\sin x)^5 = \frac{63}{8} \Rightarrow (\sin x)^5 = \left(\frac{1}{2}\right)^5 \Rightarrow \sin x = \frac{1}{2} = \sin\frac{\pi}{6} \Rightarrow x = n\pi + (-1)^n\frac{\pi}{6}, n\in Z$$

Type III ON COEFFICIENTS OF TERMS IN A BINOMIAL EXPANSION

EXAMPLE 39 *The sum of the coefficients of first three terms in the expansion of* $\left(x-\frac{3}{x^2}\right)^m$, $x \neq 0$, *m being a natural number, is 559. Find the term of the expansion containing* x^3. [NCERT]

SOLUTION We have,

$$\left(x-\frac{3}{x^2}\right)^m = {}^mC_0\,x^m + {}^mC_1\,x^{m-1}\left(-\frac{3}{x^2}\right) + {}^mC_2\,x^{m-2}\left(-\frac{3}{x^2}\right)^2 + \dots + {}^mC_m\,x^0\left(-\frac{3}{x^2}\right)^m$$

$$\Rightarrow \left(x - \frac{3}{x^2}\right)^m = {}^mC_0\, x^m + (-3 \times {}^mC_1)\, x^{m-3} + (9 \times {}^mC_2)\, x^{m-6} + \ldots + {}^mC_m\, (-3)^m \times x^{-2m}$$

Clearly, the coefficients of first three terms are: ${}^mC_0$, $-3 \times {}^mC_1$ and $9 \times {}^mC_2$

It is given that the sum of these coefficients is 559.

$\therefore \quad {}^mC_0 - 3 \times {}^mC_1 + 9 \times {}^mC_2 = 559$

$\Rightarrow \quad 1 - 3m + \dfrac{9m(m-1)}{2} = 559 \quad \Rightarrow 2 - 6m + 9m(m-1) = 1118$

$\Rightarrow \quad 9m^2 - 15m - 1116 = 0 \quad \Rightarrow 3m^2 - 5m - 372 = 0$

$\Rightarrow \quad 3m^2 - 36m + 31m - 372 = 0 \Rightarrow 3m(m-12) + 31(m-12) = 0$

$\Rightarrow \quad (m-12)(3m+31) = 0 \Rightarrow m = 12 \qquad [\because m \in N \therefore 3m + 31 \neq 0]$

Suppose $(r+1)^{th}$ term contains x^3.

Now,

$$T_{r+1} = {}^mC_r\, (x)^{m-r} \left(-\frac{3}{x^2}\right)^r = {}^mC_r\, (-3)^r\, x^{m-3r} = {}^{12}C_r\, (-3)^r\, x^{12-3r} \qquad [\because m = 12]$$

This will contain x^3, if $12 - 3r = 3$ i.e. $r = 3$. Putting $r = 3$ in T_{r+1}, we get

Required term $= T_5 = {}^{12}C_3\, (-3)^3\, x^{12-9} = -5940x^3$

EXAMPLE 40 *Find the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ and x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$ and find the relation between a and b so that these coefficients are equal.*

SOLUTION Suppose x^7 occurs in $(r+1)^{th}$ term of the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$.

Now,

$$T_{r+1} = {}^{11}C_r\, (ax^2)^{11-r} \left(\frac{1}{bx}\right)^r = {}^{11}C_r\, a^{11-r}\, b^{-r}\, x^{22-3r} \qquad \ldots(i)$$

This will contain x^7, if

$22 - 3r = 7 \Rightarrow 3r = 15 \Rightarrow r = 5.$

Putting $r = 5$ in (i), we obtain that

Coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is ${}^{11}C_5\, a^6\, b^{-5}$.

Suppose x^{-7} occurs in $(r+1)^{th}$ term of the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$.

Now, $$T_{r+1} = {}^{11}C_r\, (ax)^{11-r} \left(-\frac{1}{bx^2}\right)^r = {}^{11}C_r\, a^{11-r}\, (-1)^r\, b^{-r}\, x^{11-3r} \qquad \ldots(ii)$$

This will contain x^{-7}, if

$11 - 3r = -7 \Rightarrow 3r = 18 \Rightarrow r = 6.$

Putting $r = 6$ in (ii), we obtain that

Coefficient of x^{-7} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6\, a^5\, b^{-6}\, (-1)^6$.

If the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$, then

$${}^{11}C_5\, a^6\, b^{-5} = {}^{11}C_6\, a^5\, b^{-6}\, (-1)^6 \Rightarrow {}^{11}C_5\, ab = {}^{11}C_6 \Rightarrow ab = 1 \qquad \left[\because\ {}^{11}C_5 = {}^{11}C_6\right]$$

EXAMPLE 41 *If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$, prove that its coefficient is*

$$\left\{\frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}\right\}.$$

[NCERT EXEMPLAR]

SOLUTION Suppose x^p occurs in $(r+1)^{\text{th}}$ term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$.

Now, $T_{r+1} = {}^{2n}C_r\, (x^2)^{2n-r} \left(\frac{1}{x}\right)^r = {}^{2n}C_r\, x^{4n-3r}$ ...(i)

For this term to contain x^p, we must have $4n - 3r = p \Rightarrow r = \frac{4n-p}{3}$.

$$\therefore \quad \text{Coefficient of } x^p = {}^{2n}C_r \quad \text{where } r = \frac{4n-p}{3}$$

$$= \frac{(2n)!}{(2n-r)!\,r!}, \text{ where } r = \frac{4n-p}{3}$$

$$= \frac{(2n)!}{\left\{2n - \left(\frac{4n-p}{3}\right)\right\}!\left(\frac{4n-p}{3}\right)!} = \frac{(2n)!}{\left(\frac{2n+p}{3}\right)!\left(\frac{4n-p}{3}\right)!}$$

EXAMPLE 42 *Find the coefficient of x^n in the expansion of $(1+x)(1-x)^n$.*

SOLUTION Coefficient of x^n in $(1+x)(1-x)^n$

$= \text{Coefficient of } x^n \text{ in } (1-x)^n + \text{Coefficient of } x^{n-1} \text{ in } (1-x)^n$

$= (-1)^n\ {}^nC_n + (-1)^{n-1}\ {}^nC_{n-1} = (-1)^n\ (1-n)$

EXAMPLE 43 *Find the coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^{11}$.*

[NCERT EXEMPLAR]

SOLUTION $(1+x+x^2+x^3)^{11} = \{(1+x) + x^2(1+x)\}^{11} = \{(1+x)(1+x^2)^{11}\} = (1+x)^{11}(1+x^2)^{11}$

$$= \left({}^{11}C_0 + {}^{11}C_1 + {}^{11}C_2\, x^2 + {}^{11}C_3\, x^3 + {}^{11}C_4\, x^4 + {}^{11}C_5\, x^5 + \ldots\right) \times$$
$$\left({}^{11}C_0 + {}^{11}C_1 x^2 + {}^{11}C_2 (x^2)^2 + {}^{11}C_3 (x^2)^3 + \ldots\right)$$

$\therefore$ Coefficient of x^4 in $(1+x+x^2+x^3)^4 = {}^{11}C_0 \times {}^{11}C_2 + {}^{11}C_2 \times {}^{11}C_1 + {}^{11}C_4 \times {}^{11}C_0$

$$= 55 + 55 \times 11 + 330 = 990$$

EXAMPLE 44 *If the coefficients of x and x^2 in the expansion of $(1+x)^m (1-x)^n$ are 3 and -6 respectively. Find the values of m and n.*

SOLUTION We have,

$$(1+x)^m (1-x)^n$$

$$= \left\{{}^mC_0 + {}^mC_1\, x + {}^mC_2\, x^2 + \dots + {}^mC_m\, x^m\right\} \times \left\{{}^nC_0 - {}^nC_1\, x + {}^nC_2\, x^2 \dots + (-1)^n\, {}^nC_n\, x^n\right\}$$

$$= {}^mC_0\, {}^nC_0 - \left({}^mC_0\, {}^nC_1 - {}^nC_0\, {}^mC_1\right)x + \left({}^mC_0\, {}^nC_2 + {}^nC_0\, {}^mC_2 - {}^mC_1\, {}^nC_1\right)x^2 + \dots$$

It is given that the coefficients of x and x^2 in the expansion of $(1+x)^m (1-x)^n$ are 3 and -6 respectively.

$\therefore \quad -({}^mC_0\ {}^nC_1 - {}^nC_0\ {}^mC_1) = 3$ and, ${}^mC_0\ {}^nC_2 + {}^nC_0\ {}^mC_2 - {}^mC_1\ {}^nC_1 = -6$

$\Rightarrow \quad m - n = 3$ and $n(n-1) + m(m-1) - 2mn = -12$

$\Rightarrow \quad m - n = 3$ and $(m-n)^2 - (m+n) = -12 \Rightarrow m - n = 3$ and $m + n = 21 \Rightarrow m = 12, n = 9$

Type IV ON FINDING THE TERM INDEPENDENT OF THE VARIABLE

EXAMPLE 45 *Find the coefficient of the term independent of x in the expansion of*

$$\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10}.$$

SOLUTION We have,

$$\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}} = \frac{(x^{1/3})^3 + 1^3}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x^{1/2}\,(x^{1/2} - 1)}$$

$$= \frac{(x^{1/3}+1)(x^{2/3} - x^{1/3} + 1)}{x^{2/3} - x^{1/3} + 1} - \frac{\left(x^{1/2}+1\right)\left(x^{1/2}-1\right)}{x^{1/2}\left(x^{1/2}-1\right)}$$

$$= \left(x^{1/3}+1\right) - \left(\frac{x^{1/2}+1}{x^{1/2}}\right) = x^{1/3} + 1 - 1 - x^{-1/2} = x^{1/3} - x^{-1/2}$$

$$\therefore \quad \left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10} = (x^{1/3} - x^{-1/2})^{10}$$

Let T_{r+1} be the general term in $\left(x^{1/3} - x^{-1/2}\right)^{10}$. Then,

$$T_{r+1} = {}^{10}C_r\ (x^{1/3})^{10-r}\,(-1)^r\,(x^{-1/2})^r = (-1)^r\ {}^{10}C_r\ x^{\frac{10-r}{3} - \frac{r}{2}}$$

For this term to be independent of x, we must have

$$\frac{10-r}{3} - \frac{r}{2} = 0 \Rightarrow 20 - 2r - 3r = 0 \Rightarrow r = 4$$

So, required coefficient $= {}^{10}C_4\,(-1)^4 = 210.$

EXAMPLE 46 *Find the greatest value of the term independent of x in the expansion of* $\left(x \sin\alpha + \dfrac{\cos\alpha}{x}\right)^{10}$*, where* $\alpha \in R$.

SOLUTION Let $(r+1)^{th}$ term be independent of x.

Now, $T_{r+1} = {}^{10}C_r\,(x\sin\alpha)^{10-r}\left(\dfrac{\cos\alpha}{x}\right)^r = {}^{10}C_r\,x^{10-2r}(\sin\alpha)^{10-r}(\cos\alpha)^r$

If it is independent of x, then $r=5$.

$\therefore$ Term independent of $x = T_6 = {}^{10}C_5(\sin\alpha\cos\alpha)^5 = {}^{10}C_5\times 2^{-5}(\sin 2\alpha)^5$

Clearly, it is greatest when $2\alpha=\pi/2$ and its greatest value is ${}^{10}C_5\times 2^{-5} = \dfrac{10!}{2^5\,(5!)^2}$

Type V ON COEFFICIENTS OF TERMS IN A BINOMIAL EXPANSION

EXAMPLE 47 *Find the coefficient of x^5 in the expansion of $(1+x)^{21}+(1+x)^{22}+\ldots+(1+x)^{30}$.*

SOLUTION We have,

$$(1+x)^{21}+(1+x)^{22}+\ldots+(1+x)^{30}$$
$$=(1+x)^{21}\left\{\frac{(1+x)^{10}-1}{(1+x)-1}\right\} = \frac{1}{x}\left\{(1+x)^{31}-(1+x)^{21}\right\}$$

$\therefore$ Coefficient of x^5 in the given expression $=$ Coefficient of x^5 in $\left[\dfrac{1}{x}\left\{(1+x)^{31}-(1+x)^{21}\right\}\right]$

$$= \text{Coefficient of } x^6 \text{ in } \left\{(1+x)^{31}-(1+x)^{21}\right\}$$
$$={}^{31}C_6-{}^{21}C_6$$

EXAMPLE 48 *Find the coefficient of x^{50} after simplifying and collecting the like terms in the expansion of $(1+x)^{1000}+x(1+x)^{999}+x^2(1+x)^{998}+\ldots+x^{1000}$.* [NCERT EXEMPLAR]

SOLUTION Let $S=(1+x)^{1000}+x(1+x)^{999}+x^2(1+x)^{998}+\ldots+x^{1000}$. Clearly, it is a G.P. consisting of 1001 terms with first term $(1+x)^{1000}$ and common ratio $\dfrac{x}{1+x}$.

$$\therefore \quad S=(1+x)^{1000}\left\{\frac{1-\left(\frac{x}{1+x}\right)^{1001}}{1-\left(\frac{x}{1+x}\right)}\right\}=(1+x)^{1000}\left\{\frac{(1+x)^{1001}-x^{1001}}{(1+x)^{1000}}\right\}=(1+x)^{1001}-x^{1001}$$

$\therefore$ Coefficient of x^{50} in $S=$ Coefficient of x^{50} in $\left\{(1+x)^{1001}-x^{1001}\right\}$

$$= \text{Coefficient of } x^{50} \text{ in } (1+x)^{1001} = {}^{1001}C_{50}.$$

EXAMPLE 49 *If n is a positive integer, find the coefficient of x^{-1} in the expansion of $(1+x)^n\left(1+\dfrac{1}{x}\right)^n$.*

SOLUTION Clearly, [NCERT EXEMPLAR]

$$(1+x)^n\left(1+\frac{1}{x}\right)^n=\frac{(1+x)^n(1+x)^n}{x^n}=\frac{(1+x)^{2n}}{x^n}$$

$\therefore$ Coefficient of x^{-1} in $(1+x)^n\left(1+\dfrac{1}{x}\right)^n=$ Coefficient of x^{-1} in $\dfrac{(1+x)^{2n}}{x^n}$

$$= \text{Coefficient of } x^{n-1} \text{ in } (1+x)^{2n} = {}^{2n}C_{n-1}$$

EXAMPLE 50 *If in the expansion of* $(1-x)^{2n-1}$, *the coefficient of* x^r *is denoted by* a_r, *then prove that* $a_{r-1}+a_{2n-r}=0$.

SOLUTION We have,

$$a_{r-1} = \text{Coefficient of } x^{r-1} \text{ in } (1-x)^{2n-1} = (-1)^{r-1}\, {}^{2n-1}C_{r-1}$$

$$a_{2n-r} = \text{Coefficient of } x^{2n-r} \text{ in } (1-x)^{2n-1} = (-1)^{2n-r}\, {}^{2n-1}C_{2n-r}$$

$$\therefore\ a_{r-1}+a_{2n-r} = (-1)^{r-1}\, {}^{2n-1}C_{r-1} + (-1)^{2n-r}\, {}^{2n-1}C_{2n-r}$$

$$= (-1)^{r-1}\, {}^{2n-1}C_{(2n-1)-(r-1)} + (-1)^{2n}(-1)^{-r}\, {}^{2n-1}C_{2n-r} \qquad [\because\ {}^nC_r = {}^nC_{n-r}]$$

$$= (-1)^{r-1}\, {}^{2n-1}C_{2n-r} + (-1)^{-r}\, {}^{2n-1}C_{2n-r} \qquad [\because\ (-1)^{2n}=1]$$

$$= \{(-1)^{r-1}+(-1)^{-r}\}\, {}^{2n-1}C_{2n-r} = \left\{(-1)^{r-1}+\frac{1}{(-1)^r}\right\} {}^{2n-1}C_{2n-r}$$

$$= \left\{\frac{(-1)^{2r-1}+1}{(-1)^r}\right\} {}^{2n-1}C_{2n-r} = \left\{\frac{-1+1}{(-1)^r}\right\} {}^{2n-1}C_{2n-r} = 0 \qquad [\because\ (-1)^{2r-1}=-1]$$

Type VI ON CONSECUTIVE TERMS AND THEIR COEFFICIENTS

EXAMPLE 51 *If* a_1, a_2, a_3, a_4 *be the coefficients of four consecutive terms in the expansion of* $(1+x)^n$, *then prove that:* $\dfrac{a_1}{a_1+a_2}+\dfrac{a_3}{a_3+a_4}=\dfrac{2a_2}{a_2+a_3}$. [NCERT EXEMPLAR]

SOLUTION Let a_1, a_2, a_3, a_4 be the coefficients of 4 consecutive terms viz. the rth, the $(r+1)$th, the $(r+2)$th and the $(r+3)$th terms. Then,

$$a_1 = {}^nC_{r-1},\ a_2 = {}^nC_r,\ a_3 = {}^nC_{r+1} \text{ and } a_4 = {}^nC_{r+2}$$

Now, $a_1+a_2 = {}^nC_{r-1}+{}^nC_r = {}^{n+1}C_r$, $a_2+a_3 = {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

and, $a_3+a_4 = {}^nC_{r+1}+{}^nC_{r+2} = {}^{n+1}C_{r+2}$

$$\therefore\ \frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4} = \frac{{}^nC_{r-1}}{{}^{n+1}C_r}+\frac{{}^nC_{r+1}}{{}^{n+1}C_{r+2}}$$

$$= \frac{{}^nC_{r-1}}{\left(\frac{n+1}{r}\right){}^nC_{r-1}}+\frac{{}^nC_{r+1}}{\left(\frac{n+1}{r+2}\right){}^nC_{r+1}} \qquad \left[\because\ {}^nC_r = \frac{n}{r}\cdot{}^{n-1}C_{r-1}\right]$$

$$= \frac{r}{n+1}+\frac{r+2}{n+1} = 2\left(\frac{r+1}{n+1}\right) \qquad \text{...(i)}$$

and, $$2\,\frac{a_2}{a_2+a_3} = 2\,\frac{{}^nC_r}{{}^{n+1}C_{r+1}} = 2\left(\frac{{}^nC_r}{\frac{n+1}{r+1}\cdot{}^nC_r}\right) = 2\left(\frac{r+1}{n+1}\right) \qquad \text{...(ii)}$$

From (i) and (ii), we obtain

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}.$$

EXAMPLE 52 *The 3^{rd}, 4^{th} and 5^{th} terms in the expansion of $(x+a)^n$ are respectively 84, 280 and 560, find the values of x, a and n.*

SOLUTION It is given that: $T_3 = 84$, $T_4 = 280$ and $T_5 = 560$

We have,

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r\, x^{n-r}\, a^r}{{}^nC_{r-1}\, x^{n-r+1}\, a^{r-1}} = \frac{n-r+1}{r} \cdot \frac{a}{x}$$

$$\therefore \quad \frac{T_4}{T_3} = \frac{n-2}{3} \cdot \frac{a}{x} \text{ and } \frac{T_5}{T_4} = \frac{n-3}{4} \cdot \frac{a}{x}$$

$$\Rightarrow \quad \frac{280}{84} = \frac{n-2}{3} \cdot \frac{a}{x} \text{ and } \frac{560}{280} = \frac{n-3}{4} \cdot \frac{a}{x} \qquad [\because T_3 = 84, T_4 = 280 \text{ and } T_5 = 560]$$

$$\Rightarrow \quad \frac{10}{3} = \frac{n-2}{3} \cdot \frac{a}{x} \text{ and } \frac{2}{1} = \frac{n-3}{4} \cdot \frac{a}{x}$$

$$\Rightarrow \quad \frac{a}{x} = \frac{10}{n-2} \text{ and } \frac{a}{x} = \frac{8}{n-3} \Rightarrow \frac{10}{n-2} = \frac{8}{n-3} \Rightarrow 5n - 15 = 4n - 8 \Rightarrow n = 7$$

Putting $n = 7$ in $\frac{a}{x} = \frac{10}{n-2}$, we get: $\frac{a}{x} = \frac{10}{5} \Rightarrow 2x = a$

Now, $T_3 = 84$

$$\Rightarrow \quad {}^nC_2\, x^{n-2}\, a^2 = 84$$

$$\Rightarrow \quad {}^7C_2\, x^5\, (2x)^2 = 84 \qquad [\because a = 2x \text{ and } n = 7]$$

$$\Rightarrow \quad 21 \times 2^4 \times x^7 = 84 \Rightarrow x^7 = 1 \Rightarrow x = 1$$

$$\therefore \quad a = 2x = 2 \times 1 = 2$$

Hence, $n = 7, a = 2$ and $x = 1$.

Type VII ON APPLICATIONS OF BINOMIAL THEOREM

EXAMPLE 53 *How many terms are free from radical signs in the expansion of $(x^{1/5} + y^{1/10})^{55}$.*

SOLUTION The general term in the expansion of $(x^{1/5} + y^{1/10})^{55}$ is given by

$$T_{r+1} = {}^{55}C_r \left(x^{1/5}\right)^{55-r} \left(y^{1/10}\right)^r \Rightarrow T_{r+1} = {}^{55}C_r\, x^{11-r/5}\, y^{r/10}$$

Clearly, T_{r+1} will be free from radical signs, if $\frac{r}{5}$ and $\frac{r}{10}$ are integers for $0 \le r \le 55$

$$\therefore \quad r = 0, 10, 20, 30, 40, 50.$$

Hence, there are 6 terms in the expansion of $(x^{1/5} + y^{1/10})^{55}$ which are independent of radical signs.

EXAMPLE 54 *Find the number of integral terms in the expansion of $\left(5^{1/2} + 7^{1/8}\right)^{1024}$.*

SOLUTION The general term T_{r+1} in the expansion of $\left(5^{1/2} + 7^{1/8}\right)^{1024}$ is given by

$$T_{r+1} = {}^{1024}C_r \left(5^{1/2}\right)^{1024-r} \left(7^{1/8}\right)^r = {}^{1024}C_r\, 5^{512 - \frac{r}{2}}\, 7^{r/8}$$

$$\Rightarrow \quad T_{r+1} = \left\{ {}^{1024}C_r \ 5^{512-r} \right\} \times 5^{r/2} \times 7^{r/8} = \left\{ {}^{1024}C_r \ 5^{512-r} \right\} \times \left(5^4 \times 7\right)^{r/8}$$

Clearly, T_{r+1} will be an integer, iff

$\frac{r}{8}$ is an integer such that $0 \le r \le 1024$

$\Rightarrow$ r is a multiple of 8 satisfying $0 \le r \le 1024 \Rightarrow r = 0, 8, 16, 24, \dots, 1024$

$\Rightarrow$ r can assume 129 values.

Hence, there are 129 integral terms in the expansion of $\left(5^{1/2} + 7^{1/8}\right)^{1024}$.

EXERCISE 17.2

BASIC

1. Find the 11th term from the beginning and the 11th term from the end in the expansion of $\left(2x - \frac{1}{x^2}\right)^{25}$.
2. Find the 7th term in the expansion of $\left(3x^2 - \frac{1}{x^3}\right)^{10}$.
3. Find the 5th term from the end in the expansion of $\left(3x - \frac{1}{x^2}\right)^{10}$.
4. Find the 8th term in the expansion of $(x^{3/2} y^{1/2} - x^{1/2} y^{3/2})^{10}$.
5. Find the rth term in the expansion of $\left(x + \frac{1}{x}\right)^{2r}$. [NCERT EXEMPLAR]
6. Find the 4th term from the beginning and 4th term from the end in the expansion of $\left(x + \frac{2}{x}\right)^9$.
7. Find the 4th term from the end in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^9$. [NCERT EXEMPLAR]
8. Find the 7th term from the end in the expansion of $\left(2x^2 - \frac{3}{2x}\right)^8$.
9. Find the coefficient of:

   (i) x^{10} in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{20}$ (ii) x^7 in the expansion of $\left(x - \frac{1}{x^2}\right)^{40}$.

   (iii) x^{-15} in the expansion of $\left(3x^2 - \frac{a}{3x^3}\right)^{10}$ (iv) x^{11} in the expansion of $\left(x^3 - \frac{2}{x^2}\right)^{12}$.

   [NCERT EXEMPLAR]

   (v) x^m in the expansion of $\left(x + \frac{1}{x}\right)^n$.

   (vi) x in the expansion of $(1 - 2x^3 + 3x^5)\left(1 + \frac{1}{x}\right)^8$.

(vii) $a^5 b^7$ in the expansion of $(a-2b)^{12}$. [NCERT]

(viii) x in the expansion of $(1-3x+7x^2)(1-x)^{16}$. [NCERT EXEMPLAR]

(ix) x^{-1} in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$. [NCERT EXEMPLAR]

(x) $\frac{1}{x^{17}}$ in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$ [NCERT EXEMPLAR]

(xi) x^{15} in the expansion of $(x-x^2)^{10}$. [NCERT EXEMPLAR]

10. Which term in the expansion of $\left\{\left(\frac{x}{\sqrt{y}}\right)^{1/3}+\left(\frac{y}{x^{1/3}}\right)^{1/2}\right\}^{21}$ contains x and y to one and the same power?

11. (i) Does the expansion of $\left(2x^2-\frac{1}{x}\right)^{20}$ contain any term involving x^9?

(ii) Determine whether the expansion of $\left(x^2-\frac{2}{x}\right)^{18}$ will contain a term containing x^{10}?

[NCERT EXEMPLAR]

12. Show that the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$ does not contain any term involving x^{-1}.

13. Find the middle term in the expansion of:

(i) $\left(\frac{2}{3}x-\frac{3}{2x}\right)^{20}$ (ii) $\left(\frac{a}{x}+bx\right)^{12}$ (iii) $\left(x^2-\frac{2}{x}\right)^{10}$

14. Find the middle terms in the expansion of:

(i) $\left(3x-\frac{x^3}{6}\right)^9$ (ii) $\left(2x^2-\frac{1}{x}\right)^7$ (iii) $\left(3x-\frac{2}{x^2}\right)^{15}$ (iv) $\left(x^4-\frac{1}{x^3}\right)^{11}$

[NCERT EXEMPLAR]

15. Find the middle term(s) in the expansion of:

(i) $\left(x-\frac{1}{x}\right)^{10}$ (ii) $(1-2x+x^2)^n$

(iii) $(1+3x+3x^2+x^3)^{2n}$ (iv) $\left(2x-\frac{x^2}{4}\right)^9$

(v) $\left(x-\frac{1}{x}\right)^{2n+1}$ (vi) $\left(\frac{x}{3}+9y\right)^{10}$ [NCERT]

(vii) $\left(3-\frac{x^3}{6}\right)^7$ (viii) $\left(2ax-\frac{b}{x^2}\right)^{12}$ [NCERT EXEMPLAR]

(ix) $\left(\frac{p}{x}+\frac{x}{p}\right)^9$ [NCERT EXEMPLAR] (x) $\left(\frac{x}{a}-\frac{a}{x}\right)^{10}$ [NCERT EXEMPLAR]

16. Find the term independent of x in the expansion of the following expressions:

(i) $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ (ii) $\left(2x+\frac{1}{3x^2}\right)^9$

(iii) $\left(2x^2-\frac{3}{x^3}\right)^{25}$ (iv) $\left(3x-\frac{2}{x^2}\right)^{15}$ [NCERT EXEMPLAR]

(v) $\left(\sqrt{\frac{x}{3}}+\frac{\sqrt{3}}{2x^2}\right)^{10}$ [NCERT EXEMPLAR] (vi) $\left(x-\frac{1}{x^2}\right)^{3n}$

(vii) $\left(\frac{1}{2}x^{1/3}+x^{-1/5}\right)^8$ (viii) $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ [NCERT EXEMPLAR]

(ix) $\left(\sqrt[3]{x}+\frac{1}{2\sqrt[3]{x}}\right)^{18}$, $x>2$ (x) $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^6$ [NCERT]

BASED ON LOTS

17. If the coefficients of $(2r+4)$th and $(r-2)$th terms in the expansion of $(1+x)^{18}$ are equal, find r. [NCERT EXEMPLAR]
18. If the coefficients of $(2r+1)$th term and $(r+2)$th term in the expansion of $(1+x)^{43}$ are equal, find r.
19. Prove that the coefficient of $(r+1)$th term in the expansion of $(1+x)^{n+1}$ is equal to the sum of the coefficients of rth and $(r+1)$th terms in the expansion of $(1+x)^n$.
20. Prove that the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\cdot 2^n$.
21. The coefficients of 5th, 6th and 7th terms in the expansion of $(1+x)^n$ are in A.P., find n.
22. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^{2n}$ are in A.P., show that $2n^2-9n+7=0$. [NCERT EXEMPLAR]
23. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^n$ are in A.P., then find the value of n.
24. If in the expansion of $(1+x)^n$, the coefficients of pth and qth terms are equal, prove that $p+q=n+2$, where $p\neq q$.
25. Find a, if the coefficients of x^2 and x^3 in the expansion of $(3+ax)^9$ are equal. [NCERT]
26. Find the coefficient of a^4 in the product $(1+2a)^4(2-a)^5$ using binomial theorem. [NCERT]

BASED ON HOTS

27. In the expansion of $(1+x)^n$ the binomial coefficients of three consecutive terms are respectively 220, 495 and 792, find the value of n.

28. If in the expansion of $(1 + x)^n$, the coefficients of three consecutive terms are 56, 70 and 56, then find n and the position of the terms of these coefficients.
29. If 3rd, 4th, 5th and 6th terms in the expansion of $(x + \alpha)^n$ be respectively a, b, c and d, prove that $\frac{b^2 - ac}{c^2 - bd} = \frac{5a}{3c}$.
30. If a, b, c and d in any binomial expansion be the 6th, 7th, 8th and 9th terms respectively, then prove that $\frac{b^2 - ac}{c^2 - bd} = \frac{4a}{3c}$.
31. If the coefficients of three consecutive terms in the expansion of $(1 + x)^n$ be 76, 95 and 76, find n.
32. If the 6th, 7th and 8th terms in the expansion of $(x + a)^n$ are respectively 112, 7 and 1/4, find x, a, n.
33. If the 2nd, 3rd and 4th terms in the expansion of $(x + a)^n$ are 240, 720 and 1080 respectively, find x, a, n. [NCERT]
34. Find a, b and n in the expansion of $(a + b)^n$, if the first three terms in the expansion are 729, 7290 and 30375 respectively. [NCERT]
35. If the term free from x in the expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is 405, find the value of k. [NCERT EXEMPLAR]
36. Find the sixth term in the expansion $\left(y^{1/2} + x^{1/3}\right)^n$, if the binomial coefficient of the third term from the end is 45. [NCERT EXEMPLAR]
37. If p is a real number and if the middle term in the expansion of $\left(\frac{p}{2} + 2\right)^8$ is 1120, find p. [NCERT EXEMPLAR]
38. Find n in the binomial $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$, if the ratio of 7th term from the beginning to the 7th term from the end is $\frac{1}{6}$. [NCERT EXEMPLAR]
39. If the seventh term from the beginning and end in the binomial expansion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$ are equal, find n. [NCERT EXEMPLAR]

ANSWERS

1. ${}^{25}C_{10}\left(\frac{2^{15}}{x^5}\right), -{}^{25}C_{15}\left(\frac{2^{10}}{x^{20}}\right)$
2. $\frac{17010}{x^{10}}$
3. $\frac{17010}{x^8}$
4. $-120\, x^8 y^{12}$
5. $\frac{(2r)!}{(r+1)!\,(r-1)!} x^2$
6. $672\, x^3, \frac{5376}{x^3}$
7. $\frac{672}{x^3}$
8. $4032\, x^{10}$
9. (i) ${}^{20}C_{10} \cdot 2^{10}$ (ii) $-{}^{40}C_{11}$ (iii) $-\frac{40}{27} a^7$

(iv) -25344 (v) $\dfrac{n!}{\left(\dfrac{n-m}{2}\right)!\left(\dfrac{n+m}{2}\right)!}$ (vi) 154 (vii) -101376 (viii) -19

(ix) ${}^{2n}C_{n-1}$. (x) -1365 (xi) -252

10. 10^{th} 11. (i) No (ii) No.

13. (i) ${}^{20}C_{10}$ (ii) $924\,a^6b^6$ (iii) $-8064\,x^5$ (iv) -252

14. (i) $\dfrac{189}{8}x^{17}, -\dfrac{21}{16}x^{19}$ (ii) $-560\,x^5, 280\,x^2$

(iii) $\dfrac{-6435\times 3^8\times 2^7}{x^6}, \dfrac{6437\times 3^7\times 2^8}{x^9}$ (iv) $-462\,x^9,\ 462\,x^2$

15. (i) -252 (ii) $\dfrac{(2n)!}{(n!)^2}(-1)^n x^n$ (iii) $\dfrac{(6n)!}{[(3n)!]^2}\,x^{3n}$

(iv) $\dfrac{63}{4}x^{13}, -\dfrac{63}{32}x^{14}$ (v) $(-1)^n.\ {}^{2n+1}C_n\,x,\ (-1)^{n+1}.\ {}^{2n+1}C_n\,\dfrac{1}{x}$

(vi) $61236\,x^5y^5$ (vii) $-\dfrac{105}{8}x^9, \dfrac{35}{48}x^{12}$ (viii) $\dfrac{59136\,a^6b^6}{x^6}$

(ix) $\dfrac{126x}{p}$ (x) -252

16. (i) $\dfrac{7}{18}$ (ii) $\dfrac{64}{27}\times{}^9C_3$ (iii) ${}^{25}C_{10}\,(2^{15}\times 3^{10})$

(iv) $-3003\times 3^{10}\times 2^5$ (v) $\dfrac{5}{12}$ (vi) $(-1)^n\ {}^{3n}C_n$

(vii) 7 (viii) $\dfrac{17}{54}$ (ix) $\dfrac{{}^{18}C_9}{2^9}$ (x) $\dfrac{5}{12}$

17. 6 18. 14 21. 7 or 14 23. 7

25. $\dfrac{8}{7}$ 26. -438 27. 12 28. $n=8$, 4th, 5th, 6th

31. 8 32. $n=8,\ x=4,\ a=\dfrac{1}{2}$ 33. $n=5, x=2, a=3$

34. $a=3, b=5, n=6$ 35. ± 3 36. $252\,y^{5/2}x^{5/3}$ 37. ± 2

38. 9 39. 12

HINTS TO SELECTED PROBLEMS

9. (vii) Let T_{r+1} be the $(r+1)^{th}$ term in the expansion of $(a-2b)^{12}$. Then,

$$T_{r+1} = {}^{12}C_r\,a^{12-r}(-2b)^r = {}^{12}C_r\,(-1)^r\,2^r\,a^{12-r}b^r$$

If a^5b^7 appears in $(r+1)^{th}$ term, then $12-r=5$ and $r=7 \Rightarrow r=7$

Thus, a^5b^7 appears in 8^{th} term given by $T_8 = {}^{12}C_7\,(-1)^7\,2^7\,a^5b^7 = -101376a^5b^7$

Hence, Coefficient of $a^5b^7 = -101376$

(viii) $(1-3x+7x^2)(1-x)^{16} = (1-3x+7x^2)({}^{16}C_0 - {}^{16}C_1\,x\,{}^{16}C_2\,x^2 - {}^{16}C_3\,x^3 + \ldots)$

$\therefore$ Coefficient of x in $(1-3x+7x^2)(1-x)^{16} = 1\times -{}^{16}C_1 - 3\times{}^{16}C_0 = -16-3 = -19$

(ix) $(1+x)^n\left(1+\frac{1}{x}\right)^n = (1+x)^n\left(\frac{1+x}{x^n}\right)^n = \frac{(1+x)^{2n}}{x^n}$

15. (vi) In the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ there are 11 terms. So, $\left(\frac{10}{2}+1\right)^{th}$ i.e. 6th term is the middle term.

$$\text{Now, } T_6 = T_{5+1} = {}^{10}C_5\left(\frac{x}{3}\right)^{10-5}(9y)^5 = 61236x^5y^5$$

16. (x) Let $(r+1)^{th}$ term in the expansion of $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^6$ be independent of x. Then the exponent of x in $(r+1)^{th}$ term must be zero.

$$\text{Now, } T_{r+1} = {}^6C_r\left(\frac{3}{2}x^2\right)^{6-r}\left(-\frac{1}{3x}\right)^r = {}^6C_r\left(\frac{3}{2}\right)^{6-r}\left(-\frac{1}{3}\right)^r x^{12-3r} \quad ...(i)$$

For T_{r+1} to be independent of x, we must have $12-3r=0 \Rightarrow r=4$

Hence, 5^{th} term is independent of x. Puting $r=4$ in (i), we get

$$T_5 = {}^6C_4\left(\frac{3}{2}\right)^2\left(-\frac{1}{3}\right)^4 = 15\times\frac{1}{4\times 9} = \frac{5}{12}$$

17. ${}^{18}C_{2r+3} = {}^{18}C_{r-3} \Rightarrow (2r+3)+(r-3)=18 \Rightarrow r=6$

22. It is given that ${}^{2n}C_1$, ${}^{2n}C_2$ and ${}^{2n}C_3$ are in A.P.

$\therefore \quad 2\times {}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3$

$\Rightarrow \quad 2\times\frac{(2n)(2n-1)}{2.1} = 2n + \frac{2n(2n-1)(2n-2)}{3.2.1}$

$\Rightarrow \quad 2n(2n-1) = 2n + 2n\frac{(2n-1)(2n-2)}{6} \Rightarrow 6(2n-1) = 6+(2n-1)(2n-2) \Rightarrow 2n^2-9n+7=0$

24. We have,

$$(3+ax)^9 = {}^9C_0\times 3^9 + {}^9C_1\times 3^8\times(ax)^1 + {}^9C_2\times 3^7\times(ax)^2 + {}^9C_3\times 3^6\times(ax)^3 + ...+{}^9C_9(ax)^9$$

$\therefore \quad$ Coefficient of $x^2 = {}^9C_2\times 3^7\times a^2$ and, Coefficient of $x^3 = {}^9C_3\times 3^6\times a^3$

Now, Coefficient of x^2 = Coefficient of x^3

$$\Rightarrow \quad {}^9C_2\times 3^7\times a^2 = {}^9C_3\times 3^6\times a^3 \Rightarrow 36\times 3^7\times a^2 = 84\times 3^6\times a^3 \Rightarrow a = \frac{36\times 3^7}{84\times 3^6} = \frac{9}{7}$$

26. $(1+2a)^4(2-a)^5 = \left\{{}^4C_0 + {}^4C_1(2a) + {}^4C_2(2a)^2 + {}^4C_3(2a)^3 + {}^4C_4(2a)^4\right\}$

$$\times\left\{{}^5C_0\,2^5 - {}^5C_1\,2^4a + {}^5C_2\,2^3a^2 - {}^5C_3\,2^2a^3 + {}^5C_4(2)a^4 - {}^5C_5\,a^5\right\}$$

$\therefore \quad$ Coefficient of $a^4 = {}^4C_0\times\left({}^5C_4\times 2\right)+\left({}^4C_1\times 2\right)\times\left(-{}^5C_3\times 2^2\right)+\left({}^4C_2\times 2^2\right)$

$$\times\left({}^5C_2\times 2^3\right)+\left({}^4C_3\times 2^3\right)\times\left(-{}^5C_1\times 2^4\right)+\left({}^4C_4\times 2^4\right)\times\left({}^5C_0\times 2^5\right)$$

$$=10+8\times(-40)+24\times 80+(4\times 8)(-80)+(16\times 32)$$

$$=10-320+1920-2560+512 \;=\; -438$$

33. It is given that in the expansion of $(x+a)^n$

$T_2=240, T_3=720$ and $T_4=1080$

$$\Rightarrow \frac{T_3}{T_2}=3 \text{ and } \frac{T_4}{T_3}=\frac{3}{2} \Rightarrow \frac{n-2+1}{2}\frac{a}{x}=3 \text{ and } \frac{n-3+1}{3}\frac{a}{x}=\frac{3}{2}\quad\left[\because \frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\frac{a}{x}\right]$$

$$\Rightarrow (n-1)\frac{a}{x}=6 \text{ and } (n-2)\frac{a}{x}=\frac{9}{2}\Rightarrow \frac{n-1}{n-2}=\frac{6}{9}\times 2\Rightarrow \frac{n-1}{n-2}=\frac{4}{3}\Rightarrow 4n-8=3n-3$$

$$\Rightarrow n=5$$

Putting $n=5$ in $(n-1)\frac{a}{x}=6$, we get $2a=3x$.

Now, $T_2=240\Rightarrow {}^nC_1\, x^{n-1}a=240\Rightarrow nx^{n-1}a=240$ $\quad\left[\because n=5, a=\frac{3x}{2}\right]$

$$\Rightarrow 5x^4\times\frac{3x}{2}=240$$

$$\Rightarrow x^5=32\Rightarrow x^5=(2)^5\Rightarrow x=2$$

$\therefore\quad 2a=3x\Rightarrow a=3$. Hence, $x=2, a=3$ and $n=5$.

34. We have,

$${}^nC_0\, a^n b^0=729, {}^nC_1\, a^{n-1}b=7290 \text{ and } {}^nC_2\, a^{n-2}b^2=30375$$

$$\Rightarrow a^n=729, n\,a^{n-1}b=7290 \text{ and } n(n-1)\,a^{n-2}b^2=60750$$

$$\therefore\quad \frac{n\,a^{n-1}b}{a^n}=\frac{7290}{729} \text{ and } \frac{n(n-1)\,a^{n-2}b^2}{n\,a^{n-1}b}=\frac{60750}{7290}$$

$$\Rightarrow \frac{nb}{a}=10 \text{ and } \frac{(n-1)\,b}{a}=\frac{25}{3}\Rightarrow \frac{(n-1)\frac{b}{a}}{n\frac{b}{a}}=\frac{25}{30}\Rightarrow\frac{n-1}{n}=\frac{5}{6}\Rightarrow n=6$$

Now, $a^n=729\Rightarrow a^6=3^6\Rightarrow a=3$

$$\therefore\quad \frac{nb}{a}=10\Rightarrow\frac{6\times b}{3}=10\Rightarrow b=5$$

35. Let $(r+1)^{\text{th}}$ term, in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$, be free from x and be equal to T_{r+1}. Then,

$$T_{r+1}={}^{10}C_r\,(\sqrt{x})^{10-r}\left(\frac{-k}{x^2}\right)^r={}^{10}C_r\, x^{5-\frac{5r}{2}}(-k)^r \qquad \text{...(i)}$$

If T_{r+1} is independent of x, then $5-\frac{5r}{2}=0\Rightarrow r=2$

Putting $r=2$ in (i), we obtain: $T_3={}^{10}C_r\,(-k)^2=45k^2$

But, it is given that the value of term free from x is 405.

$\therefore\quad 45k^2=405\Rightarrow k^2=9\Rightarrow k=\pm 3$

36. In the binomial expansion of $(y^{1/2}+x^{1/3})^n$, there are $(n+1)$ terms. The third term from the end is $((n+1)-3+1)^{\text{th}}$ i.e. $(n-1)^{\text{th}}$ term from the beginning.

$\therefore$ The binomial coefficient of 3rd term from the end

= The binomial coefficient of $(n-1)^{\text{th}}$ term from the beginning $= {}^nC_{n-2} = {}^nC_2$

It is given that the binomial coefficient of the third term from the end is 45.

$$\therefore \quad {}^nC_2 = 45 \Rightarrow \frac{n(n-1)}{2} = 45 \Rightarrow n^2 - n - 90 = 0 \Rightarrow (n-10)(n+9) = 0 \Rightarrow n = 10.$$

Let T_6 be the sixth term in the binomial expansion of $(y^{1/2}+x^{1/3})^n$. Then,

$$T_6 = {}^nC_5\,(y^{1/2})^{n-5}\,(x^{1/3})^5 = {}^{10}C_5\, y^{5/2}\, x^{5/3} = 252\, y^{5/2} x^{5/3} \qquad [\because n = 10]$$

37. In the expansion of $\left(\frac{p}{2}+2\right)^8$, we observe that $\left(\frac{8}{2}+1\right)^{\text{th}}$ i.e. 5th term is the middle term. It is given that the middle term is 1120.

$$\therefore \quad T_5 = 1120 \Rightarrow {}^8C_4\left(\frac{p}{2}\right)^{8-4}(2)^4 = 1120 \Rightarrow p^4 = 16 \Rightarrow p = \pm 2$$

38. In the binomial expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$, $\left((n+1)-7+1\right)^{\text{th}}$ i.e. $(n-5)^{\text{th}}$ term from the beginning is 6 times the 7th term from the end i.e. $T_7 : T_{n-5} = 1:6$.

Now,

$$T_7 = {}^nC_6\,(\sqrt[3]{2})^{n-6}\left(\frac{1}{\sqrt[3]{3}}\right)^6 = {}^nC_6 \times 2^{\frac{n}{3}-2} \times \frac{1}{3^2}$$

$$\text{and, } T_{n-5} = {}^nC_{n-6}\,(\sqrt[3]{2})^6\left(\frac{1}{\sqrt[3]{3}}\right)^{n-6} = {}^nC_6 \times 2^2 \times \frac{1}{3^{n/3-2}}$$

It is given that

$$\frac{T_7}{T_{n-5}} = \frac{1}{6} \Rightarrow \frac{{}^nC_6 \times 2^{(n/3)-2} \times \frac{1}{3^2}}{{}^nC_6 \times 2^2 \times \frac{1}{3^{(n/3)-2}}} = \frac{1}{6}$$

$$\Rightarrow \quad 2^{(n/3)-4} \times 3^{(n/3)-4} = 6^{-1} \Rightarrow 6^{(n/3)-4} = 6^{-1} \Rightarrow \frac{n}{3} - 4 = -1 \Rightarrow n = 9$$

39. Given that $T_7 = T_{n-5}$

$$\Rightarrow \quad {}^nC_6 \times 2^{\frac{n}{3}-2} \times \frac{1}{3^2} = {}^nC_{n-6}\,(\sqrt[3]{2})^6\left(\frac{1}{\sqrt[3]{3}}\right)^{n-6} \Rightarrow {}^nC_6 \times 2^{\frac{n}{3}-2} \times \frac{1}{3^2} = {}^nC_6 \times 2^2 \times \frac{1}{3^{\frac{n-6}{3}}}$$

$$\Rightarrow \quad 2^{\frac{n}{3}-4} \times 3^{\frac{n-6}{3}-2} = 1 \Rightarrow 2^{\frac{n}{3}-4} \times 3^{\frac{n}{3}-4} = 1 \Rightarrow (6)^{\frac{n}{3}-4} = 6^0 \Rightarrow \frac{n}{3} - 4 = 0 \Rightarrow n = 12$$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The largest coefficient in $(1+x)^{30}$ is ....
2. The largest coefficient in $(1+x)^{41}$ is ....
3. The number of terms in the expansion of $(x+y+z)^n$ is ....

4. Middle term in the expansion of $(a^3+ba)^{28}$ is ....

5. The ratio of the coefficients of x^m and x^n in the expansion of $(1+x)^{m+n}$ is ....

6. The coefficient of $a^{-6}\,b^4$ in the expansion of $\left(\frac{1}{a}-\frac{2}{3}b\right)^{10}$ is ....

7. In the expansion of $\left(x^2-\frac{1}{x^2}\right)^{16}$, the value of the constant term is ....

8. The position of the term independent of x in the expansion of $\left(\sqrt{\frac{x}{3}}+\frac{3}{2x^2}\right)^{10}$ is ....

9. If 2^{15} is divided by 13, the remainder is ....

10. The sum of the series $\sum\limits_{r=0}^{10} {}^{20}C_r$ is ....

11. The number of terms in the expansion of $\left\{(2x+y^3)^4\right\}^7$ is ....

12. The middle term in the expansion of $\left(x-\frac{1}{x}\right)^{18}$ is ....

13. The coefficient of the middle term in the expansion of $(1+x)^{10}$ is ....

14. The total number of terms in the expansion of $(1+x)^{2n}-(1-x)^{2n}$ ....

15. If x^4 occurs in the r^{th} terms in the expansion of $\left(x^4+\frac{1}{x^3}\right)^{15}$, then $r=$ ....

16. The coefficient of x in the binomial expansion of $\left(x^2+\frac{a}{x}\right)^5$ is ....

17. If A and B are the coefficient of x^n in the expansion of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, then $\frac{A}{B}=$ ....

18. If the coefficients of x^7 and x^8 in $\left(2+\frac{x}{3}\right)^n$ are equal, then $n=$ ....

19. If 13^{th} term in the expansion of $\left(x^2+\frac{2}{x}\right)^n$ is independent of x, then the value of n is.............................. .

20. The term independent of x in the expansion of $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{10}$ is ....

ANSWERS

1. ${}^{30}C_{15}$ 2. ${}^{41}C_{21}$ or ${}^{41}C_{20}$ 3. ${}^{n+2}C_2$ 4. ${}^{28}C_{14}\,a^{56}\,b^{14}$ 5. 1 6. $\frac{1120}{27}$

7. ${}^{16}C_8$ 8. Third term 9. 12 10. $2^{19}+\frac{1}{2}\,{}^{20}C_{10}$ 11. 29

12. ${}^{18}C_9$ 13. ${}^{10}C_5$ 14. n 15. 9 16. $10a^3$ 17. 2 18. 55

19. 18 20. 252

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the number of terms in the expansion of $(2+\sqrt{3}x)^{10}+(2-\sqrt{3}x)^{10}$.
2. Write the sum of the coefficients in the expansion of $(1-3x+x^2)^{111}$.
3. Write the number of terms in the expansion of $(1-3x+3x^2-x^3)^8$.
4. Write the middle term in the expansion of $\left(\frac{2x^2}{3}+\frac{3}{2x^2}\right)^{10}$.
5. Which term is independent of x, in the expansion of $\left(x-\frac{1}{3x^2}\right)^9$?
6. If a and b denote respectively the coefficients of x^m and x^n in the expansion of $(1+x)^{m+n}$, then write the relation between a and b.
7. If a and b are coefficients of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, then write the relation between a and b.
8. Write the middle term in the expansion of $\left(x+\frac{1}{x}\right)^{10}$.
9. If a and b denote the sum of the coefficients in the expansions of $(1-3x+10x^2)^n$ and $(1+x^2)^n$ respectively, then write the relation between a and b.
10. Write the coefficient of the middle term in the expansion of $(1+x)^{2n}$.
11. Write the number of terms in the expansion of $\{(2x+y^3)^4\}^7$.
12. Find the sum of the coefficients of two middle terms in the binomial expansion of $(1+x)^{2n-1}$.
13. Find the ratio of the coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$.
14. Write last two digits of the number 3^{400}.
15. Find the number of terms in the expansion of $(a+b+c)^n$.
16. If a and b are the coefficients of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, find $\frac{a}{b}$.
17. Write the total number of terms in the expansion of $(x+a)^{100}+(x-a)^{100}$.
18. If $(1-x+x^2)^n=a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n}$, find the value of $a_0+a_2+a_4+\ldots+a_{2n}$.

ANSWERS

1. 6 2. -1 3. 25 4. 252 5. 4^{th} term 6. $a=b$ 7. $a=2b$

8. ${}^{10}C_5$ 9. $a=b^3$ 10. ${}^{2n}C_n$ 11. 29 12. ${}^{2n}C_n$ 13. 1 14. 01

15. $\frac{n(n+1)}{2}$ 16. 2 17. 51 18. $\frac{3^n+1}{2}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If in the expansion of $(1 + x)^{20}$, the coefficients of rth and $(r + 4)$th terms are equal, then r is equal to
(a) 7 (b) 8 (c) 9 (d) 10

2. The term without x in the expansion of $\left(2x - \frac{1}{2x^2}\right)^{12}$ is
(a) 495 (b) -495 (c) -7920 (d) 7920

3. If rth term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{12}$ is without x, then r is equal to
(a) 8 (b) 7 (c) 9 (d) 10

4. If in the expansion of $(a + b)^n$ and $(a + b)^{n+3}$, the ratio of the coefficients of second and third terms, and third and fourth terms respectively are equal, then n is
(a) 3 (b) 4 (c) 5 (d) 6

5. If A and B are the sums of odd and even terms respectively in the expansion of $(x + a)^n$, then $(x+a)^{2n} - (x-a)^{2n}$ is equal to
(a) $4(A + B)$ (b) $4(A - B)$ (c) AB (d) $4\,AB$

6. The number of irrational terms in the expansion of $\left(4^{1/5} + 7^{1/10}\right)^{45}$ is
(a) 40 (b) 5 (c) 41 (d) none of these

7. The coefficient of x^{-17} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ is
(a) 1365 (b) -1365 (c) 3003 (d) -3003

8. In the expansion of $\left(x^2 - \frac{1}{3x}\right)^9$, the term without x is equal to
(a) $\frac{28}{81}$ (b) $\frac{-28}{243}$ (c) $\frac{28}{243}$ (d) none of these

9. If in the expansion of $(1 + x)^{15}$, the coefficients of $(2r + 3)^{\text{th}}$ and $(r - 1)^{\text{th}}$ terms are equal, then the value of r is
(a) 5 (b) 6 (c) 4 (d) 3

10. The middle term in the expansion of $\left(\frac{2x^2}{3} + \frac{3}{2x^2}\right)^{10}$ is
(a) 251 (b) 252 (c) 250 (d) none of these

11. If in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$, x^{-17} occurs in rth term, then
(a) $r = 10$ (b) $r = 11$ (c) $r = 12$ (d) $r = 13$

12. In the expansion of $\left(x - \frac{1}{3x^2}\right)^9$, the term independent of x is
(a) T_3 (b) T_4 (c) T_5 (d) none of these

13. If in the expansion of $(1 + y)^n$, the coefficients of 5th, 6th and 7th terms are in A.P., then n is equal to
(a) 7, 11 (b) 7, 14 (c) 8, 16 (d) none of these

14. In the expansion of $\left(\frac{1}{2}x^{1/3} + x^{-1/5}\right)^8$, the term independent of x is
(a) T_5 (b) T_6 (c) T_7 (d) T_8

15. If the sum of odd numbered terms and the sum of even numbered terms in the expansion of $(x + a)^n$ are A and B respectively, then the value of $(x^2 - a^2)^n$ is
(a) $A^2 - B^2$ (b) $A^2 + B^2$ (c) $4AB$ (d) none of these

16. If the coefficient of x in $\left(x^2 + \frac{\lambda}{x}\right)^5$ is 270, then $\lambda =$
(a) 3 (b) 4 (c) 5 (d) none of these

17. The coefficient of x^4 in $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ is
(a) $\frac{405}{256}$ (b) $\frac{504}{259}$ (c) $\frac{450}{263}$ (d) none of these

18. The total number of terms in the expansion of $(x + a)^{100} + (x - a)^{100}$ after simplification is
(a) 202 (b) 51 (c) 50 (d) none of these

[NCERT EXEMPLAR]

19. If T_2/T_3 in the expansion of $(a + b)^n$ and T_3/T_4 in the expansion of $(a + b)^{n+3}$ are equal, then $n =$
(a) 3 (b) 4 (c) 5 (d) 6

20. The coefficient of $\frac{1}{x}$ in the expansion of $(1 + x)^n \left(1 + \frac{1}{x}\right)^n$ is
(a) $\frac{n!}{\{(n-1)!(n+1)!\}}$ (b) $\frac{(2n)!}{[(n-1)!(n+1)!]}$
(c) $\frac{(2n)!}{(2n-1)!(2n+1)!}$ (d) none of these

21. If the sum of the binomial coefficients of the expansion $\left(2x + \frac{1}{x}\right)^n$ is equal to 256, then the term independent of x is
(a) 1120 (b) 1020 (c) 512 (d) none of these

22. If the fifth term of the expansion $(a^{2/3} + a^{-1})^n$ does not contain 'a'. Then n is equal to
(a) 2 (b) 5 (c) 10 (d) none of these

23. The coefficient of x^{-3} in the expansion of $\left(x-\frac{m}{x}\right)^{11}$ is

(a) $-924\,m^7$ (b) $-792\,m^5$ (c) $-792\,m^6$ (d) $-330\,m^7$

24. The coefficient of the term independent of x in the expansion of $\left(ax+\frac{b}{x}\right)^{14}$ is

(a) $14!\,a^7 b^7$ (b) $\frac{14!}{7!}a^7 b^7$ (c) $\frac{14!}{(7!)^2}a^7 b^7$ (d) $\frac{14!}{(7!)^3}a^7 b^7$

25. The coefficient of x^5 in the expansion of $(1+x)^{21}+(1+x)^{22}+\ldots+(1+x)^{30}$ is

(a) ${}^{51}C_5$ (b) ${}^{9}C_5$ (c) ${}^{31}C_6-{}^{21}C_6$ (d) ${}^{30}C_5+{}^{20}C_5$

26. The coefficient of $x^8\,y^{10}$ in the expansion of $(x+y)^{18}$ is

(a) ${}^{18}C_8$ (b) ${}^{18}P_{10}$ (c) 2^{18} (d) none of these

27. If the coefficients of the $(n+1)^{th}$ term and the $(n+3)^{th}$ term in the expansion of $(1+x)^{20}$ are equal, then the value of n is

(a) 10 (b) 8 (c) 9 (d) none of these

28. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^n$, $n \in N$ are in A.P., then $n =$

(a) 7 (b) 14 (c) 2 (d) none of these

29. The middle term in the expansion of $\left(\frac{2x}{3}-\frac{3}{2x^2}\right)^{2n}$ is

(a) ${}^{2n}C_n$ (b) $(-1)^n\ {}^{2n}C_n\,x^{-n}$ (c) ${}^{2n}C_n\,x^{-n}$ (d) none of these

30. If r^{th} term is the middle term in the expansion of $\left(x^2-\frac{1}{2x}\right)^{20}$, then $(r+3)^{th}$ term is

(a) ${}^{20}C_{14}\left(\frac{x}{2^{14}}\right)$ (b) ${}^{20}C_{12}\,x^2\,2^{-12}$ (c) $-{}^{20}C_7\,x\,.\,2^{-13}$ (d) none of these

31. The number of terms with integral coefficients in the expansion of $\left(17^{1/3}+35^{1/2}x\right)^{600}$ is

(a) 100 (b) 50 (c) 150 (d) 101

32. Constant term in the expansion of $\left(x-\frac{1}{x}\right)^{10}$ is

(a) 152 (b) -152 (c) -252 (d) 252

33. If the coefficients of x^2 and x^3 in the expansion of $(3+ax)^9$ are the same, then the value of a is

(a) $-\frac{7}{9}$ (b) $-\frac{9}{7}$ (c) $\frac{7}{9}$ (d) $\frac{9}{7}$

34. Given the integers $r > 1, n > 2$, and coefficient of $(3r)^{th}$ and $(r+2)^{nd}$ terms in the binomial expansion of $(1+x)^{2n}$ are equal, then

(a) $n = 2r$ (b) $n = 3r$ (c) $n = 2r+1$ (d) none of these

35. The two successive terms in the expansion of $(1+x)^{24}$ whose coefficients are in the ratio $1:4$ are

(a) 3^{rd} and 4^{th} (b) 4^{th} and 5^{th} (c) 5^{th} and 6^{th} (d) 6^{th} and 7^{th}

[NCERT EXEMPLAR]

36. If the coefficients of 2^{nd}, 3^{rd} and the 4^{th} terms in the expansion of $(1+x)^n$ are in A.P., then the value of n is

(a) 2 (b) 7 (c) 11 (d) 14

[NCERT EXEMPLAR]

37. If the middle term of $\left(\frac{1}{x} + x\sin x\right)^{10}$ is equal to $7\frac{7}{8}$, then the value of x is

(a) $2n\pi + \frac{\pi}{6}$ (b) $n\pi + \frac{\pi}{6}$ (c) $n\pi + (-1)^n \frac{\pi}{6}$ (d) $n\pi + (-1)^n \frac{\pi}{3}$

[NCERT EXEMPLAR]

38. The total number of terms in the expansion of $(x+a)^{51} - (x-a)^{51}$ after simplification is

(a) 102 (b) 25 (c) 26 (d) none of these

39. If the coefficients of x^7 and x^8 in $\left(2 + \frac{x}{3}\right)^n$ are equal, then n is

(a) 56 (b) 55 (c) 45 (d) 15

40. The ratio of the coefficient of x^{15} to the term independent of x in $\left(x^2 + \frac{2}{x}\right)^{15}$, is

(a) $12:32$ (b) $1:32$ (c) $32:12$ (d) $32:1$

[NCERT EXEMPLAR]

41. If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then

(a) $\text{Re}(z) = 0$ (b) $\text{Im}(z) = 0$

(c) $\text{Re}(z) > 0, \text{Im}(z) > 0$ (d) $\text{Re}(z) > 0, \text{Im}(z) < 0$ [NCERT EXEMPLAR]

42. If $(1 - x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n}$, then $a_0 + a_2 + a_4 + \ldots + a_{2n}$ equals

(a) $\frac{3^n + 1}{2}$ (b) $\frac{3^n - 1}{2}$ (c) $\frac{1 - 3^n}{2}$ (d) $3^n + \frac{1}{2}$

[NCERT EXEMPLAR]

ANSWERS

1. (c)	2. (d)	3. (c)	4. (c)	5. (d)	6. (c)	7. (b)	8. (c)
9. (a)	10. (b)	11. (c)	12. (b)	13. (b)	14. (b)	15. (a)	16. (a)
17. (a)	18. (b)	19. (c)	20. (b)	21. (a)	22. (c)	23. (d)	24. (c)
25. (b)	26. (a)	27. (c)	28. (a)	29. (b)	30. (c)	31. (d)	32. (c)
33. (d)	34. (a)	35. (c)	36. (b)	37. (c)	38. (c)	39. (b)	40. (b)
41. (b)	42. (a)						

ACTIVITY

OBJECTIVE To construct the Pascal's triangle and to write binomial expansion for a given positive integral exponent.

MATERIALS REQUIRED Cardboard, chart paper, thumbpins, match sticks and adhesive.

STEPS OF CONSTRUCTION

Step I Take a cardboard of appropriate size and fix a chart paper on it using thumb pins.

Step II Take some match sticks and fix them on the chart paper with the help of adhesive as shown in Fig. 17.1.

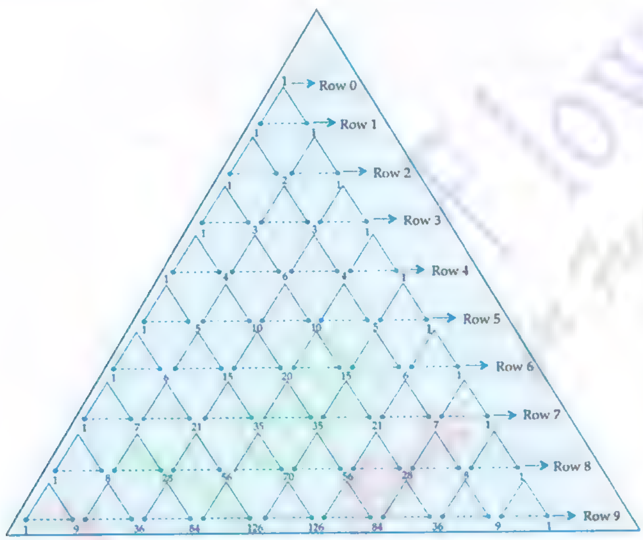

Fig. 17.1

STEPS OF DEMONSTRATION

Step I The figure looks like a triangle known as the Pascal's triangle. At the apex of the Pascal's triangle is 1.

Step II Each of the rows, which follows, begins and ends with 1 and all other numbers in a row is the sum of the two numbers in the preceding row, one on the immediate left and other on the immediate right.

Step III For the expansions of $(a+b)^1, (a+b)^2, (a+b)^3, \ldots\ldots$, we use the numbers obtained in first, second, third, ……rows of the Pascal's triangle.

First row is used to write the binomial expansion of $(a+b)^1$.

The numbers in the first row are 1, 1.

$\therefore \ (a+b)^1 = 1\cdot a + 1\cdot b = a+b$

Second row is used to write the binomial expansion of $(a+b)^2$.

The numbers in the second are 1, 2, 1.

$\therefore \ (a+b)^2 = 1\cdot a^2 + 2\cdot ab + 1\cdot b^2 = a^2 + 2ab + b^2$

To write the binomial expansion of $(a+b)^3$, we use the numbers in third row.

The numbers in the third row are 1, 3, 3, 1.

$\therefore\ (a+b)^3 = 1\cdot a^3 + 3\cdot a^2 b + 3\cdot ab^2 + 1\cdot b^3 = a^3 + 3a^2 b + 3ab^2 + b^3$

We use the numbers in fourth row to write the binomial of $(a+4)^4$. The numbers in the fourth row are 1, 4, 6, 4, 1.

$\therefore\ (a+b)^4 = 1\cdot a^4 + 4\cdot a^3 b + 6\cdot a^2 b^2 + 4\cdot ab^3 + 1\cdot b^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$

and so on.

OBSERVATIONS In the binomial expansion of $(a+b)^n$.

(i) The sum of the indices (exponents) of a and b in each term is n.

(ii) The exponents of a and b in various terms are as follows:

Term	*Exponent of 'a'*	*Exponent of 'b'*	*Sum of the exponents of a and b*
First term	n	*zero*	n
Second term	$(n-1)$	1	n
Third term	$(n-2)$	2	n
Fourth term	$(n-3)$	3	n
Fifth term	$(n-4)$	4	n

and so on.

(iii) The number of terms is $(n+1)$.

SUMMARY

1. (*Binomial theorem*) If x and a are real numbers, then for all $n \in N$, we have

$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \dots\dots\dots + {}^nC_r\, x^{n-r} a^r + \dots + {}^nC_{n-1}\, x^1 a^{n-1} + {}^nC_n\, x^0 a^n$$

i.e., $(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$

This expansion has the following properties:

(i) It has $(n+1)$ terms.

(ii) The sum of the indices of x and a in each term is n.

(iii) The coefficients of terms equidistant from the beginning and the end are equal.

(vi) General term is given by $T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

(v) $(x+a)^n = \sum_{r=0}^{n} {}^nC_r\ x^{n-r}\ a^r$ can also be expressed as $(x+a)^n = \sum_{r+s=n} \frac{n!}{r!\,s!}\ x^r\ a^s$

(vi) Replacing a by $-a$ in the expansion of $(x+a)^n$, we get

$$(x-a)^n = {}^nC_0\ x^n a^0 - {}^nC_1\ x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 - {}^nC_3\, x^{n-3} a^3 + \dots + (-1)^r\ {}^nC_r\ x^{n-r}\ a^r + \dots + (-1)^n\ {}^nC_n\ x^0\ a^n$$

The general term in the expansion of $(x-a)^n$ is $T_{r+1} = (-1)^r \ {}^nC_r \ x^{n-r} \ a^r$

(vii) Putting $x = 1$ and replacing a by x in the expansion of $(x+a)^n$, we get

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n = \sum_{r=0}^{n} {}^nC_r \ x^r$$

This is expansion of $(1+x)^n$ is ascending powers of x. In this case, $T_{r+1} = {}^nC_r \ x^r$

(viii) Putting $a = 1$ in the expansion of $(x+a)^n$, we get

$$(1+x)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \ldots + {}^nC_n x^0 = \sum_{r=0}^{n} {}^nC_r \ x^{n-r}$$

This is the expansion of $(1+x)^n$ in descending powers of x. In this case, $T_{r+1} = {}^nC_r \ x^{n-r}$

(ix) $$(x+a)^n + (x-a)^n = 2\left\{{}^nC_0 x^n a^0 + {}^nC_2 x^{n-2} a^2 + \ldots\right\}$$

$$= 2\,\{\text{Sum of the odd terms in the expansion of } (x+a)^n\}$$

$$(x+a)^n - (x-a)^n = 2\left\{{}^nC_1 x^{n-1} a^1 + {}^nC_3 x^{n-3} a^3 + \ldots\right\}$$

$$= 2\,\{\text{Sum of the even terms in the expansion of } (x+a)^n\}$$

If n is odd, then $\left\{(x+a)^n + (x-a)^n\right\}$ and $\left\{(x+a)^n - (x-a)^n\right\}$ both have $\left(\frac{n+1}{2}\right)$ terms.

If n is even, then $\left\{(x+a)^n + (x-a)^n\right\}$ has $\left(\frac{n}{2}+1\right)$ terms whereas $\left\{(x+a)^n - (x-a)^n\right\}$ has $\left(\frac{n}{2}\right)$ terms.

(x) If O and E denote respectively the sums of odd terms and even terms in the expansion of $(x+a)^n$, then

(a) $(x+a)^n = O + E$ and $(x-a)^n = O - E$

(b) $(x^2 - a^2)^n = O^2 - E^2$

(c) $4\,OE = (x-a)^{2n} - (n-a)^{2n}$

(d) $(x+a)^{2n} + (x-a)^{2n} = 2\,(O^2 + E^2)$

(xi) If n is even, then $\left(\frac{n}{2}+1\right)^{\text{th}}$ term is the middle term.

If n is odd, then $\left(\frac{n+1}{2}\right)$ and $\left(\frac{n+3}{2}\right)$ are middle terms.

CHAPTER 18

ARITHMETIC PROGRESSIONS

18.1 SEQUENCE

A sequence is a function whose domain is the set N of natural numbers.

It is customary to denote a sequence by a letter 'a' and the image $a(n)$ of $n \in N$ under a by a_n. Since the domain for every sequence is the set N of natural numbers, therefore a sequence is represented by its range. The images of 1, 2, 3, ..., n, ... under a sequence 'a' are generally denoted by $a_1, a_2, a_3, \dots a_n, \dots$ respectively. $a_1, a_2, a_3, \dots a_n, \dots$ are known as first term, second term ..., nth term,... respectively of the sequence. If a_n is the nth term of a sequence, 'a' then we write $a = < a_n >$.

REAL SEQUENCE *A sequence whose range is a subset of R is called a real sequence.*

In other words, a real sequence is a function with domain N and the range a subset of the set R of real numbers.

REPRESENTATION OF A SEQUENCE *There are several ways of representing a real sequence.*

One way to represent a real sequence is to list its first few terms till the rule for writing down other terms becomes clear. For example, 1, 3, 5, ... is a sequence whose nth term is $(2n-1)$.

Another way to represent a real sequence is to give a rule of writing the nth term of the sequence. For example, the sequence 1, 3, 5, 7, ... can be written as $a_n = 2n - 1$.

Sometimes we represent a real sequence by using a recursive relation. For example, the Fibonacci sequence is given by

$$a_1 = 1, a_2 = 1 \text{ and } a_{n+1} = a_n + a_{n-1}, n \geq 2$$

The terms of this sequence are 1, 1, 2, 3, 5, 8, ...

ILLUSTRATION 1 *Give first 3 terms of the sequence defined by* $a_n = \dfrac{n}{n^2+1}$.

SOLUTION Putting $n = 1, 2, 3$ in $a_n = \dfrac{n}{n^2+1}$, we get

$$a_1 = \frac{1}{1^2+1} = \frac{1}{2}, \quad a_2 = \frac{2}{2^2+1} = \frac{2}{5} \text{ and } a_3 = \frac{3}{3^2+1} = \frac{3}{10}.$$

ILLUSTRATION 2 *Find the first four terms of the sequence whose first term is 1 and whose $(n+1)$th term is obtained by subtracting n from its nth term.*

SOLUTION We are given that $a_1 = 1$ and $a_{n+1} = a_n - n$.

Putting $n = 1$, we obtain

$$a_2 = a_1 - 1 \Rightarrow a_2 = 1 - 1 = 0 \qquad [\because a_1 = 1]$$

Putting $n = 2$, we obtain

$$a_3 = a_2 - 2 \Rightarrow a_3 = 0 - 2 = -2$$

Similarly, by putting $n = 3$, we obtain

$$a_4 = a_3 - 3 = -2 - 3 = -5$$

SERIES *If $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ is a sequence, then the expression $a_1 + a_2 + a_3 + a_4 + \ldots + a_n + \ldots$ is a series.*

A series is finite or infinite according as the number of terms in the corresponding sequence is finite or infinite.

PROGRESSIONS *It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the nth term. Those sequences whose terms follow certain patterns are called progressions.*

In this chapter, we shall study arithmetical progressions as defined below.

18.2 ARITHMETIC PROGRESSION (A.P.)

A sequence is called an arithmetic progression if the difference of a term and the previous term is always same.

i.e. $a_{n+1} - a_n = constant\ (= d)\ for\ all\ n \in N$

The constant difference, generally denoted by d is called the common difference.

ILLUSTRATION 1 *1, 4, 7, 10, ... is an A.P. whose first term is 1 and the common difference is equal to* $4 - 1 = 3$.

ILLUSTRATION 2 *11, 7, 3, –1, ... is an A.P. whose first term is 11 and the common difference is equal to* $7 - 11 = -4$.

In order to determine whether a sequence is an A.P. or not when its nth term is given, we may use the following algorithm.

ALGORITHM

Step I *Obtain a_n.*

Step II *Replace n by $n+1$ in a_n to get a_{n+1}.*

Step III *Calculate $a_{n+1} - a_n$.*

Step IV *If $a_{n+1} - a_n$ is independent of n, the given sequence is an A.P. Otherwise it is not an A.P.*

Following examples illustrate the procedure:

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Show that the sequence defined by $a_n = 4n + 5$ is an A.P. Also, find its common difference.*

SOLUTION We have, $a_n = 4n + 5$.

Replacing n by $(n+1)$, we obtain: $a_{n+1} = 4(n+1) + 5 = 4n + 9$

$\therefore \quad a_{n+1} - a_n = (4n+9) - (4n+5) = 4$

We find that $a_{n+1} - a_n$ is independent of n and is equal to 4. So, the given sequence is an A.P. with common difference 4.

EXAMPLE 2 *Show that the sequence defined by $a_n = 2n^2 + 1$ is not an A.P.*

SOLUTION We have, $a_n = 2n^2 + 1$

Replacing n by $(n+1)$ in a_n, we obtain : $a_{n+1} = 2(n+1)^2 + 1 = 2n^2 + 4n + 3$

$\therefore \quad a_{n+1} - a_n = (2n^2 + 4n + 3) - (2n^2 + 1) = 4n + 2$

We find that $a_{n+1} - a_n$ is not independent of n and is therefore not constant. So, the given sequence is not an A.P.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Show that the sequence* $\log a, \log\left(\frac{a^2}{b}\right), \log\left(\frac{a^3}{b^2}\right), \log\left(\frac{a^4}{b^3}\right), \ldots\ldots$ *forms an A.P.*

SOLUTION We find that

$$\log\left(\frac{a^2}{b}\right) - \log a = \log\left(\frac{a^2}{b} \times \frac{1}{a}\right) = \log\left(\frac{a}{b}\right)$$

$$\log\left(\frac{a^3}{b^2}\right) - \log\left(\frac{a^2}{b}\right) = \log\left(\frac{a^3}{b^2} \times \frac{b}{a^2}\right) = \log\left(\frac{a}{b}\right)$$

$$\log\left(\frac{a^4}{b^3}\right) - \log\left(\frac{a^3}{b^2}\right) = \log\left(\frac{a^4}{b^3} \times \frac{b^2}{a^3}\right) = \log\left(\frac{a}{b}\right) \text{ and so on.}$$

This shows that the difference of a term and the preceding term is always same. Hence, the given sequence forms an A.P.

ALITER From the symmetry, we obtain

$$a_n = \log\left(\frac{a^n}{b^{n-1}}\right) \Rightarrow a_{n+1} = \log\left(\frac{a^{n+1}}{b^n}\right)$$

$$\therefore \quad a_{n+1} - a_n = \log\left(\frac{a^{n+1}}{b^n}\right) - \log\left(\frac{a^n}{b^{n-1}}\right) = \log\left(\frac{a^{n+1}}{b^n} \times \frac{b^{n-1}}{a^n}\right) = \log\left(\frac{a}{b}\right)$$

Clearly, $a_{n+1} - a_n$ is constant for all values of n. So, the given sequence is an A.P. with common difference $\log\left(\frac{a}{b}\right)$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Show that a sequence is an A.P. if its nth term is a linear expression in n and in such a case the common difference is equal to the coefficient of n.*

SOLUTION Let a_n be the n^{th} term of a sequence. Let a_n be a linear expression in n.

i.e. $\quad a_n = An + B,$ where A, B are constants.

$\Rightarrow \quad a_{n+1} = A(n+1) + B$

$\therefore \quad a_{n+1} - a_n = \{A(n+1) + B\} - \{An + B\} = A$

Clearly, $a_{n+1} - a_n$ is independent of n and is therefore a constant. Hence, the sequence is an A.P. with common difference A.

NOTE *Students are advised to use the statement of the above example as a standard result.*

EXAMPLE 5 *The* n^{th} *term of a sequence is* $3n - 2$. *Is the sequence an A.P. ? If so, find its 10th term.*

SOLUTION Here, $a_n = 3n - 2$. Clearly, a_n is a linear expression in n. So, the given sequence is an A.P. with common difference 3. Putting $n = 10$, we get: $a_{10} = 3 \times 10 - 2 = 28$

REMARK *It is evident from the above examples that a sequence is not an A.P. if its nth term is not a linear expression in n.*

EXERCISE 18.1

BASIC

1. If the n^{th} term a_n of a sequence is given by $a_n = n^2 - n + 1$, write down its first five terms.
2. A sequence is defined by $a_n = n^3 - 6n^2 + 11n - 6, n \in N$. Show that the first three terms of the sequence are zero and all other terms are positive.
3. Find the first four terms of the sequence defined by $a_1 = 3$ and, $a_n = 3a_{n-1} + 2$, for all $n > 1$. [NCERT]
4. Write the first five terms in each of the following sequences:
   (i) $a_1 = 1, a_n = a_{n-1} + 2, n > 1$ (ii) $a_1 = 1 = a_2, a_n = a_{n-1} + a_{n-2}, n > 2$
   (iii) $a_1 = a_2 = 2, a_n = a_{n-1} - 1, n > 2$ [NCERT]
5. The Fibonacci sequence is defined by $a_1 = 1 = a_2, a_n = a_{n-1} + a_{n-2}$ for $n > 2$.
   Find $\frac{a_{n+1}}{a_n}$ for $n = 1, 2, 3, 4, 5$. [NCERT]
6. Show that each of the following sequences is an A.P. Also, find the common difference and write 3 more terms in each case.
   (i) $3, -1, -5, -9 \ldots$ (ii) $-1, 1/4, 3/2, 11/4, \ldots$
   (iii) $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, \ldots$ (iv) $9, 7, 5, 3, \ldots$

BASED ON LOTS

7. The n^{th} term of a sequence is given by $a_n = 2n + 7$. Show that it is an A.P. Also, find its 7th term.
8. The n^{th} term of a sequence is given by $a_n = 2n^2 + n + 1$. Show that it is not an A.P.

ANSWERS

1. $a_1 = 1, a_2 = 3, a_3 = 7, a_4 = 13, a_5 = 21$ 3. $a_1 = 3, a_2 = 11, a_3 = 35, a_4 = 107$
4. (i) $a_1 = 1, a_2 = 3, a_3 = 5, a_4 = 7, a_5 = 9$ (ii) $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5$
   (iii) $a_1 = 2, a_2 = 2, a_3 = 1, a_4 = 0, a_5 = -1$
6. (i) -4 (ii) $\frac{5}{4}$ (iii) $2\sqrt{2}$ (iv) -2 7. 21

18.3 GENERAL TERM OF AN A.P.

THEOREM *Let a be the first term and d be the common difference of an A.P. Then, its nth* term is $a + (n-1)\,d$ *i.e.* $a_n = a + (n-1)\,d$.

PROOF *Let $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ be the given A.P. Then,*

$$a_1 = \text{First term} = a \Rightarrow a_1 = a + (1-1)\,d.$$

Using the definition, we obtain:

$$a_2 - a_1 = d \Rightarrow a_2 = a_1 + d \Rightarrow a_2 = a + d \Rightarrow a_2 = a + (2-1)\,d$$
$$a_3 - a_2 = d \Rightarrow a_3 = a_2 + d \Rightarrow a_3 = (a + d) + d \Rightarrow a_3 = a + 2d \Rightarrow a_3 = a + (3-1)\,d$$
$$a_4 - a_3 = d \Rightarrow a_4 = a_3 + d \Rightarrow a_4 = (a + 2d) + d \Rightarrow a_4 = a + 3d \Rightarrow a_4 = a + (4-1)\,d$$

Similarly, $a_5 = a + (5-1)\,d, a_6 = a + (6-1)\,d, \ldots, a_n = a + (n-1)\,d$.

Hence, nth term of an A.P. with first term a and common difference d is $a_n = a + (n-1)\,d$.

Q.E.D.

18.3.1 *n*th TERM OF AN A.P. FROM THE END

Let a be the first term and d be the common difference of an A.P. having m terms. Then, n^{th} term from the end is $(m-n+1)^{th}$ term from the beginning.

$\therefore$ $\quad n^{th}$ term from the end $= a_{m-n+1} = a + (m-n+1-1)\, d \;=\; a + (m-n)\, d$

For finding the n^{th} term from the end, we may take a_m as the first term and $-d$ as the common difference.

Taking a_m as the first term and common difference equal to '$-d$', we find that

n^{th} term from the end $= a_m + (n-1)(-d)$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **ON FINDING THE INDICATED TERM OF AN A.P.**

EXAMPLE 1 *Show that the sequence 9, 12, 15, 18, ... is an A.P. Find its 16th term and the general term.*

SOLUTION Clearly, $(12-9) = (15-12) = (18-15) = 3$, so the given sequence is an A.P. with common difference $d = 3$ and first term $a = 9$.

$\therefore$ $\quad$ 16th term $= a_{16} = a + (16-1)\, d = a + 15d = 9 + 15 \times 3 = 54 \quad [\because\; a_n = a + (n-1)\, d]$

and, $\quad$ General term $=$ nth term $= a_n = a + (n-1)\, d = 9 + (n-1) \times 3 = 3n + 6$

EXAMPLE 2 *Show that the sequence* $\log a, \log (ab), \log (ab^2), \log (ab^3), \ldots$ *is an A.P. Find its nth term.*

SOLUTION We have,

$$\log (ab) - \log a = \log \left(\frac{ab}{a}\right) = \log b,\; \log (ab^2) - \log (ab) = \log \left(\frac{ab^2}{ab}\right) = \log b$$

$$\log (ab^3) - \log (ab^2) = \log \left(\frac{ab^3}{ab^2}\right) = \log b \text{ and so on.}$$

It follows from the above results that the difference of a term and the preceding term is always same. So, the given sequence is an A.P. with common difference $\log b$.

$\therefore$ $\quad a_n = a + (n-1)\, d = \log a + (n-1) \log b = \log a + \log b^{n-1} = \log (ab^{n-1})$

EXAMPLE 3 *Which term of the sequence 72, 70, 68, 66, ... is 40?*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 72$ and common difference $d = -2$. Let its nth term be 40.

i.e. $\quad a_n = 40$

$\Rightarrow \quad a + (n-1)\, d = 40$

$\Rightarrow \quad 72 + (n-1)(-2) = 40 \qquad [\because\; a_n = a + (n-1)\, d]$

$\Rightarrow \quad 72 - 2n + 2 = 40 \Rightarrow 2n = 34 \Rightarrow n = 17$

Hence, 17th term of the given sequence is 40.

EXAMPLE 4 *Which term of the sequence 4, 9, 14, 19, ... is 124?*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 4$ and common difference $d = 5$. Let 124 be the nth term of the given sequence. Then,

$a_n = 124 \Rightarrow a + (n-1)\, d = 124 \Rightarrow 4 + (n-1) \times 5 = 124 \Rightarrow n = 25$

Hence, 25th term of the given sequence is 124.

EXAMPLE 5 *How many terms are there in the sequence 3, 6, 9, 12, ..., 111?*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 3$ and common difference $d = 3$. Let there be n terms in the given sequence. Then,

$$n^{\text{th}} \text{ term} = 111 \Rightarrow a + (n-1)\, d = 111 \Rightarrow 3 + (n-1) \times 3 = 111 \Rightarrow n = 37$$

Thus, the given sequence contains 37 terms.

EXAMPLE 6 *Is 184 a term of the sequence 3, 7, 11, ... ?*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 3$ and common difference $d = 4$. Let the nth term of the given sequence be 184. Then,

$$a_n = 184 \Rightarrow a + (n-1)\, d = 184 \Rightarrow 3 + (n-1) \times 4 = 184 \Rightarrow 4n = 185 \Rightarrow n = 46\frac{1}{4}.$$

Since n is not a natural number. So, 184 is not a term of the given sequence.

EXAMPLE 7 *Which term of the sequence* $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$ *is the first negative term?*

SOLUTION The given sequence is an A.P. in which first term $a = 20$ and common difference $d = -\frac{3}{4}$. Let the nth term of the given A.P. be the first negative term. Then,

$$a_n < 0$$

$$\Rightarrow \quad a + (n-1)\, d < 0$$

$$\Rightarrow \quad 20 + (n-1) \times (-3/4) < 0 \Rightarrow \frac{83}{4} - \frac{3n}{4} < 0 \Rightarrow 83 - 3n < 0 \Rightarrow 3n > 83 \Rightarrow n > 27\frac{2}{3}$$

Since 28 is the natural number just greater than $27\frac{2}{3}$. So, $n = 28$. Thus, 28th term of the given sequence is the first negative term.

EXAMPLE 8 *Which term of the sequence* $8 - 6i, 7 - 4i, 6 - 2i, \ldots$ *is (i) purely real (ii) purely imaginary?*

SOLUTION The given sequence is clearly an A.P. with first term $a = 8 - 6i$ and common difference $d = -1 + 2i$. The nth term of the given A.P. is given by

$$a_n = a + (n-1)\, d = 8 - 6i + (n-1)(-1 + 2i) = (9 - n) + i\,(2n - 8)$$

(i) Let the nth term of the given sequence be purely real. Then, a_n is purely real.

$\therefore \quad (9 - n) + i(2n - 8)$ is purely real $\Rightarrow 2n - 8 = 0 \Rightarrow n = 4$

So, 4th term of the given sequence is purely real.

(ii) Let the nth term of the given sequence be purely imaginary. Then, a_n is purely imaginary

$\therefore \quad (9 - n) + i(2n - 8)$ is purely imaginary $\Rightarrow 9 - n = 0 \Rightarrow n = 9$

Thus, 9th term of the given sequence is purely imaginary.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type II PROBLEMS BASED UPON $a_n = a + (n-1)\, d$

EXAMPLE 9 *If pth, qth and rth terms of an A.P. are a, b, c respectively, then show that:*

(i) $a(q - r) + b(r - p) + c(p - q) = 0$ *(ii)* $(a - b)\, r + (b - c)\, p + (c - a)\, q = 0$ [NCERT]

SOLUTION Let A be the first term and D be the common difference of the given A.P. Then,

$$a = p\text{th term} \Rightarrow a = A + (p-1)\, D \quad \text{...(i)}$$

$$b = q\text{th term} \Rightarrow b = A + (q-1)\, D \quad \text{...(ii)}$$

$$c = r\text{th term} \Rightarrow c = A + (r-1)\, D \quad \text{...(iii)}$$

(i) Substituting these values of a, b, c, in $a\,(q - r) + b\,(r - p) + c\,(p - q)$, we obtain

$$a\,(q - r) + b\,(r - p) + c\,(p - q)$$

$$= \{A+(p-1)D\}(q-r)+\{A+(q-1)D\}(r-p)+\{A+(r-1)D\}(p-q)$$
$$= A\{(q-r)+(r-p)+(p-q)\}+D\{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)\}$$
$$= A.0+D\{p(q-r)+q(r-p)+r(p-q)-(q-r)-(r-p)-(p-q)\}$$
$$= A.0+D.0 = 0$$

(ii) On subtracting (ii) from (i); (iii) from (ii) and (i) from (iii), we get

$$a-b = (p-q)D \quad ...(iv) \quad (b-c) = (q-r)D \quad ...(v) \quad c-a = (r-p)D \quad ...(vi)$$

$\therefore \quad (a-b)r+(b-c)p+(c-a)q = (p-q)Dr+(q-r)Dp+(r-p)Dq$

$$= D[(p-q)r+(q-r)p+(r-p)q] = D\times 0 = 0$$

EXAMPLE 10 *Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ term of an A.P. is equal to twice the m^{th} term.* [NCERT]

SOLUTION Let a be the first term and d be the common different of the AP. Then,

$$a_{m+n} = (m+n)^{th} \text{ term } = a+(m+n-1)d \text{ and, } a_{m-n} = (m-n)^{th} \text{ term } = a+(m-n-1)d$$

$\therefore \quad a_{m+n}+a_{m-n} = \{a+(m+n-1)d\}+\{a+(m-n-1)d\}$

$$= 2a+(m+n-1+m-n-1)d = 2a+2(m-1)d = 2\{a+(m-1)d\} = 2a_m.$$

EXAMPLE 11 *If m times the mth term of an A.P. is equal to n times its nth term, show that the $(m+n)$th term of the A.P. is zero.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

m times m^{th} term $= n$ times nth term

$\Rightarrow \quad m\,a_m = n\,a_n$

$\Rightarrow \quad m\{a+(m-1)d\} = n\{a+(n-1)d\}$

$\Rightarrow \quad m\{a+(m-1)d\} - n\{a+(n-1)d\} = 0$

$\Rightarrow \quad a(m-n)+\{m(m-1)-n(n-1)\}d = 0$

$\Rightarrow \quad a(m-n)+\{(m^2-n^2)-(m-n)\}d = 0$

$\Rightarrow \quad a(m-n)+(m-n)(m+n-1)d = 0$

$\Rightarrow \quad (m-n)\{a+(m+n-1)d\} = 0 \Rightarrow a+(m+n-1)d = 0 \Rightarrow a_{m+n} = 0 \quad [\because m \neq n]$

Hence, the $(m+n)^{th}$ term of the given A.P. is zero.

EXAMPLE 12 *If the pth term of an A.P. is q and the q^{th} term is p, prove that its nth term is $(p+q-n)$.* [NCERT]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$pth \text{ term } = q \Rightarrow a+(p-1)d = q \quad ...(i)$$
$$qth \text{ term } = p \Rightarrow a+(q-1)d = p \quad ...(ii)$$

Subtracting (ii) from (i), we get

$$(p-q)d = (q-p) \Rightarrow d = -1$$

Putting $d=-1$ in (i), we get: $a = (p+q-1)$

$\therefore \quad nth \text{ term } = a+(n-1)d = (p+q-1)+(n-1)\times(-1) = (p+q-n)$

EXAMPLE 13 *If the mth term of an A.P. be 1/n, and nth term be 1/m then show that its (mn)th term is 1.*

SOLUTION Let a and d be the first term and common difference respectively of the given A.P. Then,

$$\frac{1}{n} = mth \text{ term} \Rightarrow \frac{1}{n} = a+(m-1)d \quad ...(i)$$

$$\frac{1}{m} = n\text{th term} \Rightarrow \frac{1}{m} = a + (n-1)\,d \quad \text{...(ii)}$$

On subtracting (ii) from (i), we obtain

$$\frac{1}{n} - \frac{1}{m} = (m-n)\,d \Rightarrow \frac{m-n}{mn} = (m-n)\,d \Rightarrow d = \frac{1}{mn}$$

Putting $d = \frac{1}{mn}$ in (i), we obtain

$$\frac{1}{n} = a + \frac{(m-1)}{mn} \Rightarrow \frac{1}{n} = a + \frac{1}{n} - \frac{1}{mn} \Rightarrow a = \frac{1}{mn}$$

$\therefore$ (mn)th term $= a + (mn-1)\,d = \frac{1}{mn} + (mn-1)\frac{1}{mn} = 1$

EXAMPLE 14 *Determine the number of terms in the A.P. 3, 7, 11, ... 407. Also, find its 20th term from the end.*

SOLUTION Clearly, the given sequence is an A.P. with first term 3 and the common difference 4. Let there be n terms in the given A.P. Then,

$$407 = n\text{th term} \Rightarrow 407 = 3 + (n-1) \times 4 \Rightarrow 4n = 408 \Rightarrow n = 102$$

Now,

20th term from the end $= [102 - 20 + 1]$th term from the beginning

$= 83$rd term from the beginning $= 3 + (83-1) \times 4 = 331$

ALITER To find 20th term from the end, we consider the given sequence as an A.P. with first term $= 407$ and common difference -4.

$\therefore$ 20th term from the end $= 407 + (20-1) \times (-4) = 331$.

EXAMPLE 15 *How many numbers of two digits are divisible by 7?*

SOLUTION First two digit number divisible by 7 is 14 and last two digit number divisible by 7 is 98. So, we have to determine the number of terms in the sequence 14, 21, 28, ..., 98. Let there be n terms in this sequence. Then,

$$98 = n\text{th term} \Rightarrow 98 = 14 + (n-1) \times 7 \Rightarrow 7n = 91 \Rightarrow n = 13$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 16 *Show that there is no A.P. which consists of only distinct prime numbers.*

SOLUTION Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be an A.P. consisting only of prime numbers. Let d be the common difference of the A.P. Since the difference of two consecutive prime numbers is greater than or equal to 1. Therefore, $d > 1$.

Now,

$(a_1 + 1)^{\text{th}}$ term of this A.P. $= a_1 + (a_1 + 1 - 1)\,d = a_1(1+d)$

$\Rightarrow$ $(a_1 + 1)^{\text{th}}$ term is not a prime number

This is a contradiction that the A.P. consists of only prime numbers as its terms.
Hence, there cannot be an A.P. which consists only of distinct prime numbers.

EXAMPLE 17 *Show that in an A.P. the sum of the terms equidistant from the beginning and end is always same and equal to the sum of first and last terms.* [NCERT EXEMPLAR]

SOLUTION Let $a_1, a_2, a_3, \ldots, a_n$ be an A.P. with common difference 'd'. Then,

kth term from the beginning $= a_k = a_1 + (k-1)\,d$

and, kth term from the end $= (n-k+1)$th term from the beginning

$= a_{n-k+1} = a_1 + (n-k+1-1)\,d = a_1 + (n-k)\,d$

$\therefore$ (kth term from the beginning) + (kth term from the end)

$$= a_k + a_{n-k+1}$$
$$= \{a_1 + (k-1)\,d\} + \{a_1 + (n-k)\,d\} = 2a_1 + (n-1)\,d = a_1 + \{a_1 + (n-1)\,d\} = a_1 + a_n$$

Thus, $a_k + a_{n-k+1} = a_1 + a_n$ for all $k = 1, 2, \ldots n$

$\Rightarrow \quad a_2 + a_{n-1} = a_3 + a_{n-2} = a_4 + a_{n-3} = \ldots = a_1 + a_n$

Hence, the sum of the terms equidistant from the beginning and end is always same and equal to the sum of first and last terms.

NOTE *The statement of the above example may be treated as a standard result.*

EXAMPLE 18 *In the arithmetic progressions 2, 5, 8, ... upto 50 terms, and 3, 5, 7, 9, ... upto 60 terms, find how many terms are identical.*

SOLUTION Let the mth term of the first A.P. be equal to the nth term of the second A.P. Then,

$$2 + (m-1) \times 3 = 3 + (n-1) \times 2$$
$$\Rightarrow \quad 3m - 1 = 2n + 1$$
$$\Rightarrow \quad 3m = 2n + 2 \Rightarrow \frac{m}{2} = \frac{n+1}{3} = k \text{ (say)}$$
$$\Rightarrow \quad m = 2k \text{ and } n = 3k - 1$$
$$\Rightarrow \quad 2k \le 50 \text{ and } 3k - 1 \le 60 \qquad [\because m \le 50 \text{ and } n \le 60]$$
$$\Rightarrow \quad k \le 25 \text{ and } k \le 20\frac{1}{3}$$
$$\Rightarrow \quad k \le 20 \qquad [\because k \text{ is a natural number}]$$
$$\Rightarrow \quad k = 1, 2, 3, \ldots 20$$

Corresponding to each value of k, we get a pair of identical terms.

Hence, there are 20 identical terms in the two A.P.'s.

EXAMPLE 19 *Find the number of terms common to the two A.P.'s: 3, 7, 11, ... 407 and 2, 9, 16, ..., 709.*

SOLUTION Let the number of terms in two A.P.'s be m and n respectively. Then,

$$407 = m\text{th term of first A.P. and, } 709 = n\text{th term of second A.P.}$$
$$\Rightarrow \quad 407 = 3 + (m-1) \times 4 \text{ and } 709 = 2 + (n-1) \times 7$$
$$\Rightarrow \quad m = 102 \text{ and } n = 102$$

So, each A.P. consists of 102 terms.

Let pth term of first A.P. be identical to qth term of the second A.P. Then,

$$3 + (p-1) \times 4 = 2 + (q-1) \times 7$$
$$\Rightarrow \quad 4p - 1 = 7q - 5$$
$$\Rightarrow \quad 4p + 4 = 7q$$
$$\Rightarrow \quad 4(p+1) = 7q \Rightarrow \frac{p+1}{7} = \frac{q}{4} = k \text{ (say)} \Rightarrow p = 7k - 1 \text{ and } q = 4k$$

Since each A.P. consists of 102 terms.

$$\therefore \quad p \le 102 \text{ and } q \le 102$$
$$\Rightarrow \quad 7k - 1 \le 102 \text{ and } 4k \le 102 \Rightarrow k \le 14\frac{5}{7} \text{ and } k \le 25\frac{1}{2} \Rightarrow k \le 14 \Rightarrow k = 1, 2, 3, \ldots, 14$$

Corresponding to each value of k, we get a pair of identical terms.

Hence, there are 14 identical terms in two A.P.'s.

EXAMPLE 20 *If $a_1, a_2, a_3, \ldots, a_n$ are in A.P., where $a_i > 0$ for all i, show that*

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}.$$

[NCERT EXEMPLAR]

SOLUTION Let d be the common difference of the given A.P. Then

$$a_2 - a_1 = a_3 - a_2 = \dots = a_n - a_{n-1} = d \quad \text{...(i)}$$

Now, $$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \frac{1}{\sqrt{a_3}+\sqrt{a_4}} + \dots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

$$= \frac{\sqrt{a_2}-\sqrt{a_1}}{(\sqrt{a_2}+\sqrt{a_1})(\sqrt{a_2}-\sqrt{a_1})} + \frac{\sqrt{a_3}-\sqrt{a_2}}{(\sqrt{a_3}+\sqrt{a_2})(\sqrt{a_3}-\sqrt{a_2})} + \dots + \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{(\sqrt{a_n}+\sqrt{a_{n-1}})(\sqrt{a_n}-\sqrt{a_{n-1}})}$$

$$= \frac{\sqrt{a_2}-\sqrt{a_1}}{a_2-a_1} + \frac{\sqrt{a_3}-\sqrt{a_2}}{a_3-a_2} + \dots + \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}$$

$$= \frac{\sqrt{a_2}-\sqrt{a_1}}{d} + \frac{\sqrt{a_3}-\sqrt{a_2}}{d} + \dots + \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{d} \quad \text{[Using (i)]}$$

$$= \frac{1}{d}\left\{(\sqrt{a_2}-\sqrt{a_1}) + (\sqrt{a_3}-\sqrt{a_2}) + \dots + (\sqrt{a_n}-\sqrt{a_{n-1}})\right\}$$

$$= \frac{1}{d}\left\{\sqrt{a_n} - \sqrt{a_1}\right\}$$

$$= \frac{(\sqrt{a_n}-\sqrt{a_1})(\sqrt{a_n}+\sqrt{a_1})}{d(\sqrt{a_n}+\sqrt{a_1})} = \frac{a_n-a_1}{d(\sqrt{a_n}+\sqrt{a_1})} = \frac{a_1+(n-1)d-a_1}{d(\sqrt{a_n}+\sqrt{a_1})} = \frac{n-1}{\sqrt{a_n}+\sqrt{a_1}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

EXAMPLE 21 *If $a_1, a_2, a_3, \dots, a_n$ be an A.P. of non-zero terms, prove that*
$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}.$$

SOLUTION Let 'd' be the common difference of the given A.P. Then,
$$a_2 - a_1 = a_3 - a_2 = \dots = a_n - a_{n-1} = d \text{ (say).} \quad \text{...(i)}$$

Now,
$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4} + \dots + \frac{1}{a_{n-1} a_n}$$

$$= \frac{1}{d}\left\{\frac{d}{a_1 a_2} + \frac{d}{a_2 a_3} + \frac{d}{a_3 a_4} + \dots + \frac{d}{a_{n-1} a_n}\right\}$$

$$= \frac{1}{d}\left\{\frac{(a_2-a_1)}{a_1 a_2} + \frac{(a_3-a_2)}{a_2 a_3} + \frac{(a_4-a_3)}{a_3 a_4} + \dots + \frac{(a_n-a_{n-1})}{a_{n-1} a_n}\right\}$$

$$= \frac{1}{d}\left\{\left(\frac{1}{a_1}-\frac{1}{a_2}\right) + \left(\frac{1}{a_2}-\frac{1}{a_3}\right) + \left(\frac{1}{a_3}-\frac{1}{a_4}\right) + \dots + \left(\frac{1}{a_{n-1}}-\frac{1}{a_n}\right)\right\}$$

$$= \frac{1}{d}\left\{\frac{1}{a_1}-\frac{1}{a_n}\right\} = \frac{1}{d}\left\{\frac{a_n-a_1}{a_1 a_n}\right\} = \frac{1}{d}\left\{\frac{a_1+(n-1)d-a_1}{a_1 a_n}\right\} = \frac{n-1}{a_1 a_n}$$

EXAMPLE 22 *If $a_1, a_2, a_3, \dots, a_n$ are in AP with common difference d (where $d \neq 0$), then the sum of series.*
$\sin d(\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \dots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n)$ *is equal to* $\cot a_1 - \cot a_n$.

[NCERT EXEMPLAR]

SOLUTION We have,

$\sin d(\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \ldots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n)$

$$= \frac{\sin d}{\sin a_1 \sin a_2} + \frac{\sin d}{\sin a_2 \sin a_3} + \frac{\sin d}{\sin a_3 \sin a_4} + \ldots + \frac{\sin d}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\sin (a_2 - a_1)}{\sin a_1 \sin a_2} + \frac{\sin (a_3 - a_2)}{\sin a_2 \sin a_3} + \frac{\sin (a_4 - a_3)}{\sin a_3 \sin a_4} + \ldots + \frac{\sin (a_n - a_{n-1}}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\sin a_2 \cos a_1 - \cos a_1 \sin a_2}{\sin a_1 \sin a_2} + \frac{\sin a_3 \cos a_2 - \cos a_3 \sin a_2}{\sin a_2 \sin a_3} + \ldots + \frac{\sin a_n \cos a_{n-1} - \cos a_n \sin a_{n-1}}{\sin a_{n-1} \sin a_n}$$

$$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3) + \ldots + (\cot a_{n-1} - \cot a_n)$$

$$= \cot a_1 - \cot a_n$$

EXERCISE 18.2

BASIC

1. Find:
   (i) 10th term of the A.P. 1, 4, 7, 10, ... (ii) 18th term of the A.P. $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, \ldots$
   (iii) *nth* term of the A.P. 13, 8, 3, –2, ...
2. In an A.P., show that $a_{m+n} + a_{m-n} = 2a_m$.
3. (i) Which term of the A.P. 3, 8, 13, ... is 248 ? (ii) Which term of the A.P. 84, 80, 76, ... is 0 ?
   (iii) Which term of the A.P. 4, 9, 14, ... is 254 ?
4. (i) Is 68 a term of the A.P. 7, 10, 13, ... ? (ii) Is 302 a term of the A.P. 3, 8, 13, ... ?
5. (i) Which term of the sequence 24, $23\frac{1}{4}, 22\frac{1}{2}, 21\frac{3}{4} \ldots$ is the first negative term ?
   (ii) Which term of the sequence 12 + 8*i*, 11 + 6*i*, 10 + 4*i*, ... is (a) purely real (b) purely imaginary ?
6. (i) How many terms are there in the A.P. 7, 10, 13, ... 43 ?
   (ii) How many terms are there in the A.P. $-1, -\frac{5}{6}, -\frac{2}{3}, -\frac{1}{2}, \ldots, \frac{10}{3}$?
7. The first term of an A.P. is 5, the common difference is 3 and the last term is 80; find the number of terms.
8. The 6th and 17th terms of an A.P. are 19 and 41 respectively, find the 40th term.
9. If 9th term of an A.P. is zero, prove that its 29th term is double the 19th term.
10. If 10 times the 10th term of an A.P. is equal to 15 times the 15th term, show that 25th term of the A.P. is zero.
11. The 10th and 18th terms of an A.P. are 41 and 73 respectively. Find 26th term.
12. In a certain A.P. the 24th term is twice the 10th term. Prove that the 72nd term is twice the 34th term.

BASED ON LOTS

13. If $(m + 1)$th term of an A.P. is twice the $(n + 1)$th term, prove that $(3m + 1)$th term is twice the $(m + n + 1)$th term.
14. If the nth term of the A.P. 9, 7, 5, ... is same as the nth term of the A.P. 15, 12, 9, ... find *n*.
15. Find the 12th term from the end of the following arithmetic progressions:
    (i) 3, 5, 7, 9, ... 201 (ii) 3, 8, 13, ... , 253 (iii) 1, 4, 7, 10, ..., 88
16. The 4th term of an A.P. is three times the first and the 7th term exceeds twice the third term by 1. Find the first term and the common difference.
17. Find the second term and *nth* term of an A.P. whose 6th term is 12 and the 8th term is 22.

18. How many numbers of two digit are divisible by 3 ?
19. An A.P. consists of 60 terms. If the first and the last terms be 7 and 125 respectively, find 32nd term.
20. The sum of 4th and 8th terms of an A.P. is 24 and the sum of the 6th and 10th terms is 34. Find the first term and the common difference of the A.P.
21. How many numbers are there between 1 and 1000 which when divided by 7 leave remainder 4?

BASED ON HOTS

22. The first and the last terms of an A.P. are a and l respectively. Show that the sum of nth term from the beginning and nth term from the end is $a + l$.
23. If an A.P. is such that $\frac{a_4}{a_7} = \frac{2}{3}$, find $\frac{a_6}{a_8}$.
24. If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP, whose common difference is d, show that
$\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n = \frac{\tan\theta_n - \tan\theta_1}{\sin d}$ [NCERT EXEMPALR]

ANSWERS

1. (i) 28 (ii) $35\sqrt{2}$ (iii) $-5n+18$ 3. (i) 50 (ii) 22 (iii) 51
4. (i) No (ii) No 5. (i) 34th (ii) (a) 5 (b) 13
6. (i) 13 (ii) 27 7. 26 8. 87 11. 105 14. 7
15. (i) 179 (ii) 198 (iii) 55 16. First term = 3 , common difference = 2
17. $a_2 = -8$, $a_n = 5n - 18$ 18. 30 19. 69 20. $-\frac{1}{2}, \frac{5}{2}$ 23. $\frac{4}{5}$

18.4 SELECTION OF TERMS IN AN A.P.

Sometimes we require certain number of terms in A.P. The following ways of selecting terms are generally very convenient.

Number of terms	Terms	Common difference
3	$a-d, a, a+d$	d
4	$a-3d, a-d, a+d, a+3d$	$2d$
5	$a-2d, a-d, a, a+d, a+2d$	d
6	$a-5d, a-3d, a-d, a+d, a+3d, a+5d$	$2d$

It should be noted that in case of an odd number of terms, the middle term is a and the common difference is d while in case of an even number of terms the middle terms are $a-d$, $a+d$ and the common differences is $2d$.

The following examples will illustrate the use of such representations.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *The sum of three numbers in A.P. is – 3, and their product is 8. Find the numbers.*

SOLUTION Let the numbers be $(a-d)$, a, $(a+d)$. Then,

$$\text{Sum} = -3 \Rightarrow (a-d)+a+(a+d) = -3 \Rightarrow 3a = -3 \Rightarrow a = -1$$

and, $\quad$ Product $= 8$

$$\Rightarrow \quad (a-d)(a)(a+d) = 8$$

$$\Rightarrow \quad a(a^2-d^2) = 8 \Rightarrow (-1)(1-d^2) = 8 \Rightarrow d^2 = 9 \Rightarrow d = \pm 3 \qquad [\because a=-1]$$

When $a=-1$ and $d=3$, the numbers are $-4, -1, 2$. When $a=-1$ and $d=-3$, the numbers are $2, -1, -4$. So, the numbers are $-4, -1, 2$, or $2, -1, -4$.

EXAMPLE 2 *Find four numbers in A.P. whose sum is 20 and the sum of whose squares is 120.*

SOLUTION Let the numbers be $(a-3d), (a-d), (a+d), (a+3d)$. Then,

$$\text{Sum} = 20 \Rightarrow (a-3d)+(a-d)+(a+d)+(a+3d) = 20 \Rightarrow 4a = 20 \Rightarrow a = 5$$

and, $\quad$ Sum of the squares $= 120$

$$\Rightarrow \quad (a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2 = 120$$

$$\Rightarrow \quad 4a^2+20d^2 = 120 \Rightarrow a^2+5d^2 = 30 \Rightarrow 25+5d^2 = 30 \Rightarrow 5d^2 = 5 \Rightarrow d = \pm 1 \quad [\because a=5]$$

If $d=1$, and $a=5$, then the numbers are $2, 4, 6, 8$. If $d=-1$, and $a=5$, then the numbers are $8, 6, 4, 2$.

Thus, the numbers are $2, 4, 6, 8$ or $8, 6, 4, 2$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Divide 32 into four parts which are in A.P. such that the product of extremes is to the product of means is 7 : 15.*

SOLUTION Let the four parts be $(a-3d), (a-d), (a+d)$ and $(a+3d)$. Then,

$$\text{Sum} = 32 \Rightarrow (a-3d)+(a-d)+(a+d)+(a+3d) = 32 \Rightarrow 4a = 32 \Rightarrow a = 8$$

It is given that the product of extremes is to the product of means is 7 : 15.

$$\therefore \quad \frac{(a-3d)(a+3d)}{(a-d)(a+d)} = \frac{7}{15}$$

$$\Rightarrow \quad \frac{a^2-9d^2}{a^2-d^2} = \frac{7}{15} \Rightarrow \frac{64-9d^2}{64-d^2} = \frac{7}{15} \Rightarrow 128d^2 = 512 \Rightarrow d^2 = 4 \Rightarrow d = \pm 2$$

Thus, the four of 32 parts are 2, 6, 10, 14.

When $a=8$ and $d=2$ four parts are: 2, 6, 10, and 14. When $a=8$ and $d=-2$ four parts are 14, 10, 6 and 2.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *The product of three numbers in A.P. is 224, and the largest number is 7 times the smallest. Find the numbers.* [NCERT EXEMPLAR]

SOLUTION Let the three numbers in A.P. be $a-d, a, a+d$, where $d>0$. Clearly, $a+d$ is the largest number and $a-d$ is the smallest number.

It is given that :

Product of numbers $= 224$ $\quad$ and, $\quad$ The largest number $= 7$ (The smallest numbers)

$$\Rightarrow \quad (a-d)\,a\,(a+d) = 224 \text{ and, } a+d = 7(a-d)$$

$$\Rightarrow \quad a(a^2-d^2) = 224 \quad \text{and,} \quad 6a = 8d$$

$\Rightarrow \quad a(a^2 - d^2) = 224$ and, $d = \dfrac{3a}{4} \Rightarrow a\left(a^2 - \dfrac{9}{16}a^2\right) = 224$ [On eliminating d]

$\Rightarrow \quad \dfrac{7a^3}{16} = 224 \Rightarrow a^3 = 512 = 8^3 \Rightarrow a = 8.$

Putting $a = 8$ in $d = \dfrac{3a}{4}$, we obtain $d = 6$. Hence, three numbers are 2, 8, 14.

EXAMPLE 5 *If the fourth power of the common difference of an A.P. with integer entries is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer.*

SOLUTION Let $a - 3d$, $a - d$, $a + d$, $a + 3d$ be four consecutive terms of an A.P. with integer terms. Clearly, the common difference is $2d$. Since the terms are integers, therefore a and d are also integers.

Now, Given sum $= (a - 3d)(a - d)(a + d)(a + 3d) + (2d)^4$

$$= (a^2 - 9d^2)(a^2 - d^2) + 16d^4$$

$$= a^4 - 10a^2 d^2 + 9d^4 + 16d^4$$

$$= a^4 - 10a^2 d^2 + 25d^4$$

$= (a^2 - 5d^2)^2$, which is square of an integer as a and d are integers.

EXERCISE 18.3

BASIC

1. The sum of three terms of an A.P. is 21 and the product of the first and the third terms exceeds the second term by 6, find three terms.
2. Three numbers are in A.P. If the sum of these numbers be 27 and the product 648, find the numbers.

BASED ON LOTS

3. Find the four numbers in A.P., whose sum is 50 and in which the greatest number is 4 times the least.

BASED ON HOTS

4. The sum of three numbers in A.P. is 12 and the sum of their cubes is 288. Find the numbers.
5. If the sum of three numbers in A.P. is 24 and their product is 440, find the numbers. [NCERT]
6. The angles of a quadrilateral are in A.P. whose common difference is 10°. Find the angles.

ANSWERS

1. 1, 7, 13
2. 6, 9, 12
3. 5, 10, 15, 20
4. 2, 4, 6 or 6, 4, 2
5. 5, 8, 11
6. 75°, 85°, 95°, 105°

HINTS TO SELECTED PROBLEMS

5. Let the three numbers be $a - d, a, a + d$. It is given that the sum and product of these numbers are 24 and 440 respectively. Therefore,

$$a - d + a + a + d = 24 \text{ and } (a - d)\, a\, (a + d) = 440$$

$$\Rightarrow \quad 3a = 24 \text{ and } a(a^2 - d^2) = 440 \Rightarrow a = 8 \text{ and } a(a^2 - d^2) = 440$$

Now,

$$a(a^2 - d^2) = 440 \Rightarrow 8(64 - d^2) = 440 \Rightarrow 64 - d^2 = 55 \Rightarrow d^2 = 9 \Rightarrow d = \pm 3$$

Hence, three numbers are 5, 8, 11 or 11, 8, 5.

18.5 SUM TO n TERMS OF AN A.P.

THEOREM *Show that the sum S_n of n terms of an A.P. with first term 'a' and common difference 'd' is*

$$S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$$

or, $$S_n = \frac{n}{2}(a + l), \text{ where } l = \text{last term} = a + (n-1)\,d$$

PROOF Let $a_1, a_2, a_3, \ldots$ be an A.P. with first term a and common difference d. Then

$$a_1 = a,\ a_2 = a + d,\ a_3 = a + 2d,\ a_4 = a + 3d, \ldots, a_n = a + (n-1)\,d$$

Now,

$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

$$\Rightarrow \quad S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n-2)\,d) + \{a + (n-1)\,d\} \quad \ldots(i)$$

Writing the above series in a reverse order, we get

$$S_n = \{a + (n-1)\,d\} + \{a + (n-2)\,d\} + \ldots + (a + d) + a \quad \ldots(ii)$$

Adding the corresponding terms of (i) and (ii), we get

$$2S_n = \{2a + (n-1)\,d\} + \{2a + (n-1)\,d\} + \ldots + \{2a + (n-1)\,d\}$$

$$\Rightarrow \quad 2S_n = n\{2a + (n-1)\,d\} \quad [\because 2a + (n-1)\,d \text{ repeats } n \text{ times}]$$

$$\Rightarrow \quad S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$$

Now, l = last term $= a + (n-1)\,d$

$$\therefore \quad S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\} = \frac{n}{2}\left[a + \left\{a + (n-1)\,d\right\}\right] = \frac{n}{2}\left(a + l\right)$$

ALITER We have,

$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-2} + a_{n-1} + a_n \quad \ldots(i)$$

or, $$S_n = a_n + a_{n-1} + a_{n-2} + \ldots + a_3 + a_2 + a_1 \quad \ldots(ii)$$

Adding corresponding terms in (i) and (ii), we get

$$2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \ldots + (a_{n-1} + a_2) + (a_n + a_1)$$

$$\Rightarrow \quad 2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n) + (a_1 + a_n)$$

$$\Rightarrow \quad 2S_n = n(a_1 + a_n) \quad [\because a_1 + a_n = a_k + a_{n-k+1} \text{ for } k = 2, 3, \ldots, n]$$

$$\Rightarrow \quad S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}\left\{a_1 + a_1 + (n-1)\,d\right\} \quad [\because a_n = a_1 + (n-1)\,d]$$

$$\Rightarrow \quad S_n = \frac{n}{2}\left\{2a_1 + n - 1)\,d\right\}$$

NOTE 1 *In the formula* $S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$, *there are four quantities viz.* S_n, a, n and d. *If any three of these are known, the fourth can be determined. Sometimes two of these quantities are given, in such cases remaining two quantities are provided by some other relations.*

NOTE 2 *If the sum* S_n *of* n *terms of a sequence is given, then nth term* a_n *of the sequence can be determined by the using formula:* $a_n = S_n - S_{n-1}$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I ON FINDING THE SUM OF GIVEN NUMBER OF TERMS OF AN A.P.

Formula: $S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$ **or,** $S_n = \frac{n}{2}\left\{a + l\right\}$.

EXAMPLE 1 *Find the sum of 20 terms of the A.P. 1, 4, 7, 10, ...*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Clearly, $a = 1$, $d = 3$. We have to find the sum of 20 terms of the given A.P. Putting $a = 1$, $d = 3$, $n = 20$ in $S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$, we get

$$S_{20} = \frac{20}{2}\left\{2 \times 1 + (20-1) \times 3\right\} = 10 \times 59 = 590$$

EXAMPLE 2 *Find the sum of the series : 5 + 13 + 21 + ... + 181.*

SOLUTION The terms of the given series form an A.P. with first term $a = 5$ and common difference $d = 8$. Let there be n terms in the given series. Clearly,

$$a_n = 181 \Rightarrow a + (n-1)\,d = 181 \Rightarrow 5 + (n-1) \times 8 = 181 \Rightarrow 8n = 184 \Rightarrow n = 23$$

$$\therefore \quad \text{Required sum} = \frac{n}{2}(a + l) = \frac{23}{2}(5 + 181) = 2139.$$

EXAMPLE 3 *Find the sum of all three digit natural numbers, which are divisible by 7.*

SOLUTION The smallest and the largest numbers of three digits, which are divisible by 7 are 105 and 994 respectively. So, the sequence of three digit numbers which are divisible by 7 are 105, 112, 119, ..., 994. Clearly, these numbers are in A.P. with first term $a = 105$ and common difference $d = 7$. Let there be n terms in this sequence. Then,

$$a_n = 994 \Rightarrow a + (n-1)\,d = 994 \Rightarrow 105 + (n-1) \times 7 = 994 \Rightarrow n = 128$$

$$\therefore \quad \text{Required sum} = \frac{n}{2}\left\{2a + (n-1)\,d\right\} = \frac{128}{2}\left\{2 \times 105 + (128-1) \times 7\right\} = 70336$$

EXAMPLE 4 *Find the sum of all natural numbers between 250 and 1000 which are exactly divisible by 3.*

SOLUTION Clearly, the numbers between 250 and 1000 which are divisible by 3 are 252, 255, 258, ..., 999. These numbers are in A.P. with first term $a = 252$, common difference = 3 and last term = 999. Let there be n terms in this A.P. Then,

$$a_n = 999 \Rightarrow a + (n-1)\,d = 999 \Rightarrow 252 + (n-1) \times 3 = 999 \Rightarrow n = 250$$

$$\therefore \quad \text{Required sum} = S_n = \frac{n}{2}(a + l) = \frac{250}{2}(252 + 999) = 156375$$

EXAMPLE 5 *Find the sum of all odd integers between 2 and 100 divisible by 3.*

SOLUTION The odd integers between 2 and 100 which are divisible by 3 are 3, 9, 15, 21, ... , 99. Clearly, these numbers are in A.P. with first term $a = 3$ and common difference $d = 6$. Let there be n terms in this sequence. Then,

$$a_n = 99 \Rightarrow a + (n-1)\,d \Rightarrow 3 + (n-1) \times 6 = 99 \Rightarrow n = 17$$

$$\therefore \quad \text{Required sum} = \frac{n}{2}(a + l) = \frac{17}{2}(3 + 99) = 867.$$

EXAMPLE 6 *Find the sum of first 20 terms of an A.P., in which 3rd term is 7 and 7th term is two more than thrice of its 3rd term.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. It is given that

$$a_3 = 7 \quad \text{and} \quad a_7 = 3a_3 + 2$$

$$\Rightarrow \quad a + 2d = 7 \text{ and } a + 6d = 3(a + 2d) + 2 \Rightarrow a + 2d = 7 \text{ and } a = -1 \Rightarrow a = -1, d = 4$$

$$\therefore \quad S_{20} = \frac{20}{2}\left\{2 \times -1 + (20 - 1) \times 4\right\} = \frac{20}{2}(-2 + 76) = 740 \left[\text{Using: } S_n = \frac{n}{2}\left\{2a + (n-1)d\right\}\right]$$

EXAMPLE 7 *The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.* [NCERT]

SOLUTION Let there be n terms in the A.P. with first term $a = 11$ and common difference d. It is given that

$$\text{Sum of first four terms} = 56 \Rightarrow \frac{4}{2}\left\{2 \times 11 + (4-1)d\right\} = 56 \Rightarrow 22 + 3d = 28 \Rightarrow 3d = 6 \Rightarrow d = 2$$

It is also given that

$$\text{Sum of last four terms} = 112$$

$$\Rightarrow \quad a_n + a_{n-1} + a_{n-2} + a_{n-3} = 112$$

$$\Rightarrow \quad \frac{4}{2}(a_n + a_{n-3}) = 112 \qquad \left[\text{Using: } S_n = \frac{n}{2}(a + l)\right]$$

$$\Rightarrow \quad a_n + a_{n-3} = 56$$

$$\Rightarrow \quad \{11 + (n-1) \times 2\} + \{11 + (n-4) \times 2\} = 56 \Rightarrow 22 + 2(2n-5) = 56 \Rightarrow 4n = 44 \Rightarrow n = 11.$$

Hence, there are 11 terms in the A.P.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type I ON FINDING THE VALUE OF UNKNOWN WHEN THE SUM OF TERMS OF AN A.P. IS GIVEN

EXAMPLE 8 *If the sum of n terms of an A.P. is $pn + qn^2$, where p and q are constants, find the common difference.* [NCERT]

SOLUTION Let S_n denote the sum of n terms and a_n denote the nth term of the A.P. Then,

$$S_n = pn + qn^2$$

$$\Rightarrow \quad S_{n-1} = p(n-1) + q(n-1)^2 \qquad [\text{On replacing } n \text{ by } (n-1) \text{ in } S_n]$$

Now, $a_n = S_n - S_{n-1}$

$$\Rightarrow \quad a_n = \{pn + qn^2\} - \{p(n-1) + q(n-1)^2\}$$

$$\Rightarrow \quad a_n = pn - p(n-1) + qn^2 - q(n-1)^2 = p\{n - (n-1)\} + q\{n^2 - (n-1)^2\}$$

$$\Rightarrow \quad a_n = p + q(2n-1)$$

$$\therefore \quad a_{n-1} = p + q\{2(n-1) - 1\} \qquad [\text{Replacing } n \text{ by } (n-1) \text{ in } a_n]$$

Let d be the common difference of the A.P. Then,

$$d = a_n - a_{n-1}$$

$$\Rightarrow \quad d = \{p + q(2n-1)\} - [p + q\{2(n-1) - 1\}] = \{p + q(2n-1)\} - \{p + q(2n-3)\}$$

$$\Rightarrow \quad d = q(2n - 1 - 2n + 3) = 2q$$

Hence, the common difference $= 2q$.

EXAMPLE 9 *If the sum of n terms of an A.P. is $3n^2 + 5n$ and its mth term is 164, find the value of m.* [NCERT]

SOLUTION Let S_n denote the sum of n terms and a_n be the nth term of the given A.P. Then,

$$S_n = 3n^2 + 5n \Rightarrow S_{n-1} = 3(n-1)^2 + 5(n-1) = 3n^2 - n - 2 \quad \text{[On replacing } n \text{ by } (n-1) \text{ in } S_n]$$

Now, $a_n = S_n - S_{n-1} = (3n^2 + 5n) - (3n^2 - n - 2) = 6n + 2$

It is given that $a_m = 164 \Rightarrow 6m + 2 = 164 \Rightarrow 6m = 162 \Rightarrow m = 27$

EXAMPLE 10 *Find the sum to n terms of the sequence given by $a_n = 5 - 6n, n \in N$.*

SOLUTION We have, $a_n = 5 - 6n$. Replacing n by $(n+1)$, we obtain

$$a_{n+1} = 5 - 6(n+1) = -1 - 6n$$

$$\therefore \quad a_{n+1} - a_n = (-1 - 6n) - (5 - 6n) = -6 \text{ for all } n \in N$$

Since $a_{n+1} - a_n$ is constant for all $n \in N$. So, the given sequence is an A.P. with common difference -6. Putting $n = 1$, in $a_n = 5 - 6n$, we get: $a_1 = -1$.

So, the sum S_n to n terms is given by

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(-1 + 5 - 6n) = n(2 - 3n)$$

EXAMPLE 11 *If the m^{th} term of an A.P. is $\frac{1}{n}$ and the n^{th} term is $\frac{1}{m}$, show that the sum of mn terms is $\frac{1}{2}(mn+1)$, where $m \neq n$.* [NCERT]

SOLUTION Let a be the first term and d be the common difference of the given A.P. It is given that

$$a_m = \frac{1}{n} \text{ and } a_n = \frac{1}{m}$$

Now, $a_m = \frac{1}{n} \Rightarrow a + (m-1)\,d = \frac{1}{n}$ ...(i)

and, $a_n = \frac{1}{m} \Rightarrow a + (n-1)\,d = \frac{1}{m}$ ...(ii)

Subtracting (ii) from (i), we get

$$(m-n)\,d = \frac{1}{n} - \frac{1}{m} \Rightarrow (m-n)\,d = \frac{m-n}{mn} \Rightarrow d = \frac{1}{mn}$$

Putting $d = \frac{1}{mn}$ in (i), we get

$$a + (m-1)\frac{1}{mn} = \frac{1}{n} \Rightarrow a + \frac{1}{n} - \frac{1}{mn} = \frac{1}{n} \Rightarrow a = \frac{1}{mn}$$

$$\therefore \quad S_{mn} = \frac{mn}{2}\left\{2a + (mn-1)\,d\right\} = \frac{mn}{2}\left\{\frac{2}{mn} + (mn-1) \times \frac{1}{mn}\right\} = \frac{1}{2}(mn+1)$$

Type II FINDING THE NUMBER OF TERMS IN AN A.P. WHEN THE SUM OF ITS n TERMS IS GIVEN

EXAMPLE 12 *How many terms of the series 54, 51, 48, ... be taken so that their sum is 513 ? Explain the double answer.*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 54$ and common difference $d = -3$. Let the sum of n terms be 513. Then,

$$S_n = 513$$

$$\Rightarrow \frac{n}{2}\left\{2a+(n-1)\,d\right\} = 513$$

$$\Rightarrow \frac{n}{2}\left\{108+(n-1)\times -3\right\} = 513 \Rightarrow n^2 - 37n + 342 = 0 \Rightarrow (n-18)(n-19) = 0 \Rightarrow n = 18 \text{ or, } 19$$

Here, the common difference is negative. So, the terms are in decreasing order and the value of 19^{th} term is $54 + (19-1) \times -3 = 0$. Thus, the sum of 18 terms as well as that of 19 terms is 513.

EXAMPLE 13 *Find the number of terms in the series* $20, 19\frac{1}{3}, 18\frac{2}{3}, \ldots$ *of which the sum is 300, explain the double answer.*

SOLUTION The given sequence is an A.P. with first term $a = 20$ and the common difference $d = -\frac{2}{3}$. Let the sum of n terms be 300. Then,

$$S_n = 300$$

$$\Rightarrow \frac{n}{2}\left\{2a+(n-1)\,d\right\} = 300$$

$$\Rightarrow \frac{n}{2}\left\{2\times 20+(n-1)\left(-\frac{2}{3}\right)\right\} = 300$$

$$\Rightarrow n^2 - 61n + 900 = 0 \Rightarrow (n-25)(n-36) = 0 \Rightarrow n = 25 \text{ or, } 36$$

$\therefore$ Sum of 25 terms = Sum of 36 terms = 300.

Here, the common difference is negative therefore terms go on diminishing and $31st$ term becomes zero. All terms following $31st$ term are negative. These negative terms when added to positive terms from $26th$ term to $30th$ *term*, they cancel out each other and the sum remains same. Hence, the sum of 25 terms as well as that of 36 terms is 300.

EXAMPLE 14 *Solve* $1 + 6 + 11 + 16 + \ldots + x = 148$.

SOLUTION Clearly, terms of the given series form an A.P. with first term $a = 1$ and common difference $d = 5$. Let there be n terms in this series. Then,

$$1 + 6 + 11 + 16 + \ldots + x = 148$$

$$\Rightarrow \text{Sum of } n \text{ terms} = 148$$

$$\Rightarrow \frac{n}{2}\left\{2a+(n-1)\,d\right\} = 148$$

$$\Rightarrow \frac{n}{2}\left\{2+(n-1)\times 5\right\} = 148 \Rightarrow 5n^2 - 3n - 296 = 0 \Rightarrow (n-8)(5n+37) = 0 \Rightarrow n = 8$$

Clearly, $x = n^{th}$ term $= a + (n-1)\,d = 1 + (8-1)\times 5 = 36$ $[\because a = 1, d = 5, n = 8]$

Type III PROVING RESULTS RELATED TO THE SUM OF n TERMS OF AN A.P.

EXAMPLE 15 *The sum of the first p, q, r terms of an A.P. are a, b, c respectively. Show that*

$$\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = 0$$ [NCERT]

SOLUTION Let A be the first term and D be the common difference of the given A.P. Then,

$$a = \text{Sum of } p \text{ terms} \Rightarrow a = \frac{p}{2}\left\{2A+(q-1)\,D\right\} \Rightarrow \frac{2a}{p} = \left\{2A+(p-1)\,D\right\} \quad \ldots(i)$$

$$b = \text{Sum of } q \text{ terms} \Rightarrow b = \frac{q}{2}\left\{2A + (q-1)D\right\} \Rightarrow \frac{2b}{q} = \left\{2A + (q-1)D\right\} \qquad \text{...(ii)}$$

and, $$c = \text{Sum of } r \text{ terms} \Rightarrow c = \frac{r}{2}\left\{2A + (r-1)D\right\} \Rightarrow \frac{2c}{r} = \left\{2A + (r-1)D\right\} \qquad \text{...(iii)}$$

Multiplying (i), (ii) and (iii) by $(q-r)$, $(r-p)$ and $(p-q)$ respectively and adding, we get

$$\frac{2a}{p}(q-r) + \frac{2b}{q}(r-p) + \frac{2c}{r}(p-q)$$
$$= \{2A + (p-1)D\}(q-r) + \{2A + (q-1)D\}(r-p) + \{2A + (r-1)D\}(p-q)$$
$$= 2A(q-r+r-p+p-q) + D\{(p-1)(q-r) + (q-1)(r-p) + (r-1)(p-q)\}$$
$$= 2A \times 0 + D \times 0 = 0$$

EXAMPLE 16 *The sum of n, $2n$, $3n$ terms of an A.P. are S_1, S_2, S_3 respectively. Prove that:* $S_3 = 3(S_2 - S_1)$. [NCERT]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_1 = \text{Sum of } n \text{ terms} \Rightarrow S_1 = \frac{n}{2}\left\{2a + (n-1)d\right\} \qquad \text{...(i)}$$

$$S_2 = \text{Sum of } 2n \text{ terms} \Rightarrow S_2 = \frac{2n}{2}\left\{2a + (2n-1)d\right\} \qquad \text{...(ii)}$$

and, $$S_3 = \text{Sum of } 3n \text{ terms} \Rightarrow S_3 = \frac{3n}{2}\left\{2a + (3n-1)d\right\} \qquad \text{...(iii)}$$

Now, $$S_2 - S_1 = \frac{2n}{2}\left\{2a + (2n-1)d\right\} - \frac{n}{2}\left\{2a + (n-1)d\right\}$$

$$\Rightarrow \quad S_2 - S_1 = \frac{n}{2}\left[2\left\{2a + (2n-1)d\right\} - \left\{2a + (n-1)d\right\}\right] = \frac{n}{2}\left\{2a + (3n-1)d\right\}$$

$$\therefore \quad 3(S_2 - S_1) = \frac{3n}{2}\left\{2a + (3n-1)d\right\} \qquad \text{...(iv)}$$

From (iii) and (iv), we get: $3(S_2 - S_1) = S_3$

EXAMPLE 17 *The sums of n terms of three arithmetical progressions are S_1, S_2 and S_3. The first term of each is unity and the common differences are 1, 2 and 3 respectively. Prove that $S_1 + S_3 = 2S_2$.*

SOLUTION We have,

S_1 = Sum of n terms of an A.P. with first term 1 and common difference 1

$$\Rightarrow \quad S_1 = \frac{n}{2}\left\{2 \times 1 + (n-1) \times 1\right\} = \frac{n}{2}(n+1)$$

S_2 = Sum of n terms of an A.P. with first term 1 and common difference 2

$$\Rightarrow \quad S_2 = \frac{n}{2}\left\{2 \times 1 + (n-1) \times 2\right\} = n^2$$

S_3 = Sum of n terms of and A.P. with first term 1 and common difference 3

$$\Rightarrow \quad S_2 = \frac{n}{2}\left\{2 \times 1 + (n-1) \times 3\right\} = \frac{n}{2}(3n-1)$$

$$\therefore \quad S_1 + S_3 = \frac{n}{2}(n+1) + \frac{n}{2}(3n-1) = 2n^2$$

Hence, $S_1 + S_3 = 2S_2$ $\qquad [\because S_2 = n^2]$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 18 *If in an A.P. the sum of m terms is equal to n and the sum of n terms is equal to m, then prove that the sum of $(m+n)$ terms is $-(m+n)$. Also, find the sum of first $(m-n)$ terms $(m>n)$.* [NCERT EXEMPLAR]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_m = n \Rightarrow \frac{m}{2}\left\{2a+(m-1)\,d\right\} = n \Rightarrow 2am + m(m-1)\,d = 2n \quad \text{...(i)}$$

and, $$S_n = m \Rightarrow \frac{n}{2}\left\{2a+(n-1)\,d\right\} = m \Rightarrow 2an + n(n-1)\,d = 2m \quad \text{...(ii)}$$

Subtracting (ii) from (i), we get

$$2a(m-n) + \{m(m-1) - n(n-1)\}\,d = 2n - 2m$$

$$\Rightarrow \quad 2a(m-n) + \{(m^2-n^2) - (m-n)\}\,d = -2(m-n)$$

$$\Rightarrow \quad 2a + (m+n-1)\,d = -2 \qquad \text{[On dividing both sides by } (m-n)] \quad \text{...(iii)}$$

Now, $$S_{m+n} = \frac{m+n}{2}\left\{2a+(m+n-1)\,d\right\}$$

$$\Rightarrow \quad S_{m+n} = \frac{(m+n)}{2}(-2) \qquad \text{[Using (iii)]}$$

$$\therefore \quad S_{m+n} = -(m+n)$$

From (iii), we obtain

$$2a = -2-(m+n-1)\,d \quad \text{...(iv)}$$

Substituting this value of $2a$ in (i), we obtain

$$-2m - m(m+n-1)\,d + m(m-1)\,d = 2n \Rightarrow d = -2\left(\frac{m+n}{mn}\right) \quad \text{...(v)}$$

Putting $d = -2\left(\frac{m+n}{mn}\right)$ in (iv), we obtain

$$2a = -2 + \frac{2}{mn}(m+n-1)(m+n) \quad \text{...(vi)}$$

Now,

$$S_{m-n} = \frac{m-n}{2}\left\{2a+(m-n-1)\,d\right\}$$

$$\Rightarrow \quad S_{m-n} = \frac{m-n}{2}\left\{-2+\frac{2}{mn}(m+n-1)(m+n) - \frac{2}{mn}(m-n-1)(m+n)\right\} \qquad \text{[Using (v) and (vi)]}$$

$$\Rightarrow \quad S_{m-n} = \frac{m-n}{2}\left\{-2+\frac{4n}{mn}(m+n)\right\} = \frac{1}{m}(m-n)(m+2n)$$

EXAMPLE 19 *If the sum of first m terms of an A.P. is the same as the sum of its first n terms, show that the sum of its $(m+n)$ terms is zero.* [NCERT]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_m = S_n$$

$$\Rightarrow \quad \frac{m}{2}\left\{2a+(m-1)\,d\right\} = \frac{n}{2}\left\{2a+(n-1)\,d\right\}$$

$$\Rightarrow \quad 2a(m-n) + \{m(m-1) - n(n-1)\}\,d = 0$$

$$\Rightarrow \quad 2a(m-n) + \{(m^2-n^2) - (m-n)\}\,d = 0$$

$$\Rightarrow \quad (m-n)\{2a + (m+n-1)\,d\} = 0$$

$\Rightarrow \quad 2a + (m + n - 1)\, d = 0 \qquad [\because m - n \neq 0] \quad \ldots \text{(i)}$

$\therefore \quad S_{m+n} = \dfrac{m+n}{2}\left\{2a + (m + n - 1)\, d\right\} = \dfrac{m+n}{2} \times 0 = 0 \qquad$ [Using (i)]

EXAMPLE 20 *The ratio of the sums of m and n terms of an A.P. is $m^2 : n^2$. Show that the ratio of the mth and nth terms is* $(2m-1):(2n-1)$. **[NCERT]**

SOLUTION Let a be the first term and d the common difference of the given A.P. Then, the sums of m and n terms are given by

$$S_m = \frac{m}{2}\left\{2a + (m-1)\, d\right\} \quad \text{and,} \quad S_n = \frac{n}{2}\left\{2a + (n-1)\, d\right\} \text{ respectively.}$$

It is given that

$$\frac{S_m}{S_n} = \frac{m^2}{n^2}$$

$$\Rightarrow \quad \frac{\frac{m}{2}\left\{2a + (m-1)\, d\right\}}{\frac{n}{2}\left\{2a + (n-1)\, d\right\}} = \frac{m^2}{n^2}$$

$$\Rightarrow \quad \frac{2a + (m-1)\, d}{2a + (n-1)\, d} = \frac{m}{n}$$

$$\Rightarrow \quad \{2a + (m-1)\, d\}\, n = \{2a + (n-1)\, d\}\, m$$

$$\Rightarrow \quad 2a\,(n-m) = d\,\{(n-1)\, m - (m-1)\, n\} \Rightarrow 2a\,(n-m) = d\,(n-m) \Rightarrow d = 2a$$

$$\therefore \quad \frac{T_m}{T_n} = \frac{a + (m-1)\, d}{a + (n-1)\, d} = \frac{a + (m-1)\, 2a}{a + (n-1)\, 2a} = \frac{2m-1}{2n-1}$$

EXAMPLE 21 *The interior angles of a polygon are in A.P. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon.* **[NCERT]**

SOLUTION Let there be n sides of the polygon. Then, the sum of its interior angles is given by

$$S_n = (2n-4) \text{ right angles} = (n-2) \times 180° \qquad \ldots\text{(i)}$$

Thus, the interior angles form an A.P. with first term $a = 120°$ and common difference $d = 5°$.

$$\therefore \quad S_n = \frac{n}{2}\left\{2 \times 120° + (n-1) \times 5°\right\} \qquad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$(n-2) \times 180° = \frac{n}{2}\left\{2 \times 120° + (n-1) \times 5°\right\}$$

$$\Rightarrow \quad (n-2) \times 360 = n\,(5n + 235)$$

$$\Rightarrow \quad n^2 - 25n + 144 = 0 \Rightarrow (n-16)\,(n-9) = 0 \Rightarrow n = 16 \text{ or, } n = 9$$

For $n = 16$, we obtain

Last angle $= a_n = a + (n-1)\, d = 120° + (16-1) \times 5 = 195°$, which is not possible.

Hence, $n = 9$.

EXAMPLE 22 *The first, second and the last terms of an A.P. are a, b, c respectively. Prove that the sum is* $\dfrac{(a+c)\,(b+c-2a)}{2\,(b-a)}$. **[NCERT EXEMPLAR]**

SOLUTION Let d be the common difference of the given A.P. Then, $d = b - a$. Let there be n terms in the given A.P. Then,

$$c = n\text{th term}$$

$\Rightarrow \quad c = a + (n-1)\, d$

$\Rightarrow \quad c = a + (n-1)(b-a) \qquad [\because\ d = b-a]$

$\Rightarrow \quad n - 1 = \dfrac{c-a}{b-a} \Rightarrow n = \dfrac{c-a}{b-a} + 1 \Rightarrow n = \dfrac{b+c-2a}{b-a}$

$\therefore$ Sum of the A.P. = Sum of its n terms

$$= \frac{n}{2}(a+c) \qquad \left[\text{Using} : S_n = \frac{n}{2}(a+l)\right]$$

$$= \frac{(a+c)(b+c-2a)}{2(b-a)}.$$

EXAMPLE 23 *Let S_n denote the sum of the first n terms of an A.P. If $S_{2n} = 3S_n$, then prove that $\dfrac{S_{3n}}{S_n} = 6$.*

SOLUTION Let a be the first term and d the common difference of the given A.P. Then,

$$S_{2n} = 3\, S_n$$

$$\Rightarrow \quad \frac{2n}{2}\left\{2a + (2n-1)\, d\right\} = \frac{3n}{2}\left\{2a + (n-1)\, d\right\}$$

$$\Rightarrow \quad 2\{2a + (2n-1)\, d\} = 3\{2a + (n-1)\, d\}$$

$$\Rightarrow \quad 2a - (3n - 3 - 4n + 2)\, d = 0 \Rightarrow 2a - (n+1)\, d = 0 \Rightarrow 2a = (n+1)\, d \qquad \text{...(i)}$$

Now, $$\frac{S_{3n}}{S_n} = \frac{\frac{3n}{2}\left\{2a + (3n-1)\, d\right\}}{\frac{n}{2}\left\{2a + (n-1)\, d\right\}}$$

$$\Rightarrow \quad \frac{S_{3n}}{S_n} = \frac{3\left\{(n+1)\, d + (3n-1)\, d\right\}}{\left\{(n+1)\, d + (n-1)\, d\right\}} \qquad [\text{Using (i)}]$$

$$\Rightarrow \quad \frac{S_{3n}}{S_n} = \frac{12\, n\, d}{2nd} = 6.$$

Type IV ON SUM OF TERMS OF AN A.P.

EXAMPLE 24 *Prove that a sequence is an A.P. iff the sum of its n terms is of the form $An^2 + Bn$, where A, B are constants.*

SOLUTION Let S_n be the sum of n terms of an A.P. with first term a and common difference d. Then,

$$S_n = \frac{n}{2}\left\{2a + (n-1)\, d\right\} = an + \frac{n^2}{2} d - \frac{n}{2} d = \left(\frac{d}{2}\right) n^2 + \left(a - \frac{d}{2}\right) n$$

$$\Rightarrow \quad S_n = An^2 + Bn, \text{ where } A = \frac{d}{2} \text{ and } B = a - \frac{d}{2}$$

Thus, the sum of n terms of an A.P. is of the form $An^2 + Bn$.

Conversely, let the sum S_n of n terms of a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ be of the form $An^2 + Bn$. Then, we have to show that the sequence is an A.P.

We have, $S_n = An^2 + Bn$

$$\Rightarrow \quad S_{n-1} = A(n-1)^2 + B(n-1) \qquad [\text{On replacing } n \text{ by } n-1]$$

Now, $a_n = S_n - S_{n-1}$

$$\Rightarrow \quad a_n = \{An^2 + Bn\} - \{A(n-1)^2 + B(n-1)\} = 2A\, n + (B - A)$$

$$\Rightarrow \quad a_{n+1} = 2A(n+1) + (B - A) \qquad [\text{On replacing } n \text{ by } n+1]$$

$\therefore \quad a_{n+1} - a_n = \{2A(n+1) + B - A\} - \{2An + (B-A)\} = 2A$

Clearly, $a_{n+1} - a_n = 2A$ for all $n \in N$. So, the sequence is an A.P. with common difference $2A$.

REMARK *It follows from this example that a sequence is an A.P. iff the sum of its n terms is of the form $An^2 + Bn$ i.e. a quadratic expression in n and in such a case the common difference is twice the coefficient of n^2. For example, if $S_n = 3n^2 + 2n$, one can easily say that it is the sum of n terms of an A.P. with common difference 6. Similarly, $S_n = nP + \frac{1}{2}n(n-1)Q = \frac{Q}{2}n^2 + \left(P - \frac{Q}{2}\right)n$ is the sum of n terms of an A.P, with common difference Q.*

EXAMPLE 25 *Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3, \ldots$, if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.* [NCERT EXEMPLAR]

SOLUTION We know that in an A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term i.e. $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$ So, if an A.P. consists of 24 terms, then $a_1 + a_{24} = a_5 + a_{20} = a_{10} + a_{15}$.

Now, $\quad a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$

$$\Rightarrow (a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{15}) = 225 \Rightarrow 3(a_1 + a_{24}) = 225 \Rightarrow a_1 + a_{24} = \frac{225}{3} = 75 \quad \ldots(i)$$

$$\therefore \quad S_{24} = \frac{24}{2}(a_1 + a_{24}) \quad \left[\text{Using } S_n = \frac{n}{2}(a_1 + a_n)\right]$$

$$\Rightarrow S_{24} = 12(75) = 900 \quad [\text{Using (i)}]$$

EXAMPLE 26 *The first term of an A.P. is a and the sum of first p terms is zero, show that the sum of its next q terms is $-\frac{a(p+q)q}{p-1}$.* [NCERT EXEMPLAR]

SOLUTION Let d be the common difference of the A.P. It is gtiven that the sum of first p terms is zero

i.e. $\quad S_p = 0 \Rightarrow \frac{p}{2}\left\{2a + (p-1)d\right\} = 0 \Rightarrow d = -\frac{2a}{p-1}$.

Let S be the required sum. Then,

$$S = a_{p+1} + a_{p+2} + \ldots + a_{p+q}$$

$$\Rightarrow \quad S = (a_1 + a_2 + \ldots + a_p + a_{p+1} + \ldots + a_{p+q}) - (a_1 + a_2 + \ldots + a_p)$$

$$\Rightarrow \quad S = S_{p+q} - S_p$$

$$\Rightarrow \quad S = S_{p+q} - 0 \quad [\because S_p = 0 \text{ (given)}]$$

$$\Rightarrow \quad S = \frac{p+q}{2}\left\{2a + (p+q-1)d\right\} = \frac{p+q}{2}\left\{2a + (p+q-1)\left(-\frac{2a}{p-1}\right)\right\}$$

$$\Rightarrow \quad S = (p+q)a\left\{1 - \left(\frac{p+q-1}{p-1}\right)\right\} = (p+q)a\left(\frac{p-1-p-q+1}{p-1}\right) = -\frac{a(p+q)q}{p-1}$$

EXAMPLE 27 *If the first term of an A.P. is 2 and the sum of first five terms is equal to one-fourth of the sum of the next five terms, find the sum of first 30 terms.*

SOLUTION Let $a_1, a_2, a_3, \ldots$ be given A.P. with common difference d. It is given that $a_1 = 2$ and the sum of first five terms is equal to one fourth of the sum of next five terms.

i.e. $\quad a_1 + a_2 + a_3 + a_4 + a_5 = \frac{1}{4}(a_6 + a_7 + a_8 + a_9 + a_{10})$

$$\Rightarrow \quad 4(a_1 + a_2 + a_3 + a_4 + a_5) = (a_6 + a_7 + a_8 + a_9 + a_{10})$$

$$\Rightarrow \quad 5(a_1 + a_2 + a_3 + a_4 + a_5) = (a_1 + a_2 + \ldots + a_{10})$$

$\Rightarrow \quad 5 S_5 = S_{10}$

$\Rightarrow \quad 5\left[\frac{5}{2}\left\{2\times 2+(5-1)\, d\right\}\right] = \frac{10}{2}\left\{2\times 2+(10-1)\, d\right\} \Rightarrow 50(1+d) = 20+45d \Rightarrow d=-6$

Thus, we have $a = 2$ and $d = -6$.

$\therefore \quad$ Required sum $= S_{30} = \frac{30}{2}\left\{2\times 2+(30-1)\times -6\right\} = -2550.$

EXAMPLE 28 *The p^{th} term of an A.P. is a and qth term is b. Prove that the sum of its (p + q) terms is* $\frac{p+q}{2}\left\{a+b+\frac{a-b}{p-q}\right\}$. [NCERT EXEMPLAR]

SOLUTION Let A and D be the first term and common difference respectively of the given A.P. Then,

$a =$ pth term $\Rightarrow a = A+(p-1)D$ ...(i)

$b =$ qth term $\Rightarrow b = A+(q-1)D$ ...(ii)

Subtracting (ii) from (i), we get: $D = \frac{a-b}{p-q}$

Adding (i) and (ii), we get $a+b = 2A+(p+q-2)D$

$\Rightarrow \quad a+b = 2A+(p+q-1)D-D$

$\Rightarrow \quad (a+b)+D = 2A+(p+q-1)D \Rightarrow (a+b)+\frac{a-b}{p-q} = 2A+(p+q-1)D$ ...(iii)

Now, $\quad S_{p+q} =$ Sum of $(p+q)$ terms

$\Rightarrow \quad S_{p+q} = \frac{p+q}{2}\left\{2A+(p+q-1)D\right\} = \frac{p+q}{2}\left\{a+b+\frac{a-b}{p-q}\right\}$ [Using (iii)]

EXAMPLE 29 *The ratio of the sum of n terms of two A.P.'s is (7n + 1) : (4n + 27). Find the ratio of their mth terms.*

SOLUTION Let a_1, a_2 be the first terms and d_1, d_2 the common differences of the two given A.P.'s. Then, the sums S_n and S_n' of their n terms are given by

$$S_n = \frac{n}{2}\left\{2a_1+(n-1)d_1\right\}, \text{ and } S_n' = \frac{n}{2}\left\{2a_2+(n-1)d_2\right\}$$

$$\therefore \quad \frac{S_n}{S_n'} = \frac{\frac{n}{2}\left\{2a_1+(n-1)d_1\right\}}{\frac{n}{2}\left\{2a_2+(n-1)d_2\right\}} = \frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2}$$

It is given that

$$\frac{S_n}{S_n'} = \frac{7n+1}{4n+27} \Rightarrow \frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2} = \frac{7n+1}{4n+27} \Rightarrow \frac{a_1+\left(\frac{n-1}{2}\right)d_1}{a_2+\left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27} \quad \text{... (i)}$$

We have to find the ratio to m^{th} terms of two A.P.'s i.e., $\frac{a_1+(m-1)d_1}{a_1+(m-1)d_2}$. Clearly, this can be obtained by replacing $\frac{n-1}{2}$ by $(m-1)$ on the LHS of (i). Replacing $\frac{n-1}{2}$ by $m-1$ i.e. n by $(2m-1)$ on both side of (i), we get

$$\frac{a_1+(m-1)\,d_1}{a_2+(m-1)\,d_2}=\frac{7\,(2m-1)+1}{4\,(2m-1)+27}=\frac{14\,m-6}{8m+23}$$

Hence, the ratio of the mth terms of the two A.P.'s is $(14m-6):(8m+23)$.

REMARK *It is evident from the above example that if we are given the ratio of the sums of n terms of two A.P.'s then the ratio of their m^{th} terms is obtained by replacing n by $(2m-1)$.*

EXAMPLE 30 *The sum of n terms of two arithmetic progressions are in the ratio $(3n+8):(7n+15)$. Find the ratio of their 12th terms.* [NCERT]

SOLUTION Let a_1, a_2 be the first terms and d_1, d_2 the common differences of the two given A.P.'s. Then, the sums of their n terms are given by

$$S_n=\frac{n}{2}\ \{2a_1+(n-1)\,d_1\}\ \text{and,}\ \ S_n'=\frac{n}{2}\{2a_2+(n-1)\,d_2\}$$

It is given that

$$\frac{S_n}{S_n'}=\frac{3n+8}{7n+15}$$

$$\Rightarrow\quad \frac{\frac{n}{2}\left\{2a_1+(n-1)\,d_1\right\}}{\frac{n}{2}\left\{2a_2+(n-1)\,d_2\right\}}=\frac{3n+8}{7n+15}\Rightarrow\frac{2a_1+(n-1)\,d_1}{2a_2+(n-1)\,d_2}=\frac{3n+8}{7n+15}\Rightarrow\frac{a_1+\left(\frac{n-1}{2}\right)d_1}{a_2+\left(\frac{n-1}{2}\right)d_2}=\frac{3n+8}{7n+15}$$

Replacing $\frac{n-1}{2}$ by 11 i.e. n by 23 on both sides, we get

$$\frac{a_1+11\,d_1}{a_2+11\,d_2}=\frac{3\times 23+8}{7\times 23+15}=\frac{77}{176}=\frac{7}{16}$$

Hence, the required ratio is $7:16$.

ALITER If the ratio of the sums of n terms of two A.P.s is given, then the ratio of their m^{th} terms is obtained by replacing n by $(2m-1)$ in the given ratio. So, required ratio is obtained by replacing n by $2\times 12-1=23$ in $(3n+8):(7n+15)$.

Hence, required ratio $=(69+8):(161+15)=7:16$.

EXAMPLE 31 *If there are $(2n+1)$ terms in A.P., then prove that the ratio of the sum of odd terms and the sum of even terms is $(n+1):n$.* [NCERT EXEMPLAR]

SOLUTION Let a and d be the first term and common difference respectively of the given A.P. Let a_k denote the k^{th} terms of the given A.P. Then, $a_k=a+(k-1)\,d$.

Now, $\quad S_1=$ Sum of odd terms $=a_1+a_3+a_5+\ldots+a_{2n+1}$

$$\Rightarrow\quad S_1=\frac{n+1}{2}\left(a_1+a_{2n+1}\right)=\frac{n+1}{2}\left\{a+a+(2n+1-1)\,d\right\}=(n+1)\,(a+nd)$$

and, $\quad S_2=$ Sum of even terms $=a_2+a_4+a_6+\ldots+a_{2n}$

$$\Rightarrow\quad S_2=\frac{n}{2}\left(a_2+a_{2n}\right)=\frac{n}{2}\left[(a+d)+\left\{a+(2n-1)\,d\right\}\right]=n\,(a+nd)$$

$$\therefore\quad S_1:S_2=(n+1)\,(a+nd):n\,(a+nd)=(n+1):n$$

EXAMPLE 32 *Let S_k be the sum of first k terms of an A.P. What must this progression be for the ratio $\frac{S_{kx}}{S_x}$ to be independent of x ?*

SOLUTION Let a be the first term and d common difference of the given progression. Then,

$$\frac{S_{kx}}{S_x} = \frac{\frac{kx}{2}\left\{2a+(kx-1)\,d\right\}}{\frac{x}{2}\left\{2a+(x-1)\,d\right\}} = \frac{k\left\{kxd+(2a-d)\right\}}{\left\{xd+(2a-d)\right\}}$$

Clearly, the RHS of the above relation will be independent of x iff $2a-d=0$ i.e. $d=2a$.
Hence, the progression is $a,\ 3a, 5a, 7a, \ldots,$ where a is any non-zero real number.

EXAMPLE 33 *Let S_n be the sum of first n terms of an A.P. with non-zero common difference. Find the ratio of first term and common difference if $\frac{S_{n_1 n_2}}{S_{n_1}}$ is independent of n_1.*

SOLUTION Let the first term and common difference of the A.P. be a and d respectively. Then,

$$S_{n_1 n_2} = \frac{n_1 n_2}{2}\left\{2a+(n_1 n_2-1)\,d\right\} \text{ and, } S_{n_1} = \frac{n_1}{2}\left\{2a+(n_1-1)\,d\right\}$$

$$\therefore \quad \frac{S_{n_1 n_2}}{S_{n_1}} = \frac{n_2\left\{2a+(n_1 n_2-1)\,d\right\}}{\left\{2a+(n_1-1)\,d\right\}} = \frac{n_2\left\{(2a-d)+n_1 n_2\, d\right\}}{\left\{(2a-d)+n_1\, d\right\}}$$

Clearly, RHS will be independent of n_1 iff $2a-d=0$ i.e. $d=2a$. Hence, $\frac{a}{d}=\frac{1}{2}$.

EXAMPLE 34 *If $S_1, S_2, S_3, \ldots S_m$ are the sums of n terms of m A.P.'s whose first terms are 1, 2, 3, ..., m and common differences are 1, 3, 5, ..., (2 m − 1) respectively. Show that*

$$S_1+S_2+\ldots+S_m = \frac{mn}{2}(mn+1)$$

SOLUTION The first terms, common difference and the sums of their n terms are as under:

First terms	*Common differences*	*Sums of n terms*
1	1	$S_1 = \frac{n}{2}\left\{2\times1+(n-1)\times1\right\}$
2	3	$S_2 = \frac{n}{2}\left\{2\times2+(n-1)\times3\right\}$
3	5	$S_3 = \frac{n}{2}\left\{2\times3+(n-1)\times5\right\}$
$\vdots$	$\vdots$	
m	$2m-1$	$S_m = \frac{n}{2}\left\{2m+(n-1)(2m-1)\right\}$

$$\therefore \quad S_1+S_2+\ldots+S_m$$

$$= \frac{n}{2}\left[2\times1+(n-1)\times1\right]+\frac{n}{2}\left[2\times2+(n-1)\times3\right]+\ldots+\frac{n}{2}\left[2m+(n-1)(2m-1)\right]$$

$$= \frac{n}{2}\left[2\times(1+2+3+\ldots+m)+(n-1)(1+3+5+\ldots+(2m-1))\right]$$

$$= \frac{n}{2}\left[2\times\frac{m}{2}(1+m)+(n-1)\frac{m}{2}\{1+(2m-1)\}\right]$$

$$= \frac{n}{2}\left[m(m+1)+m^2(n-1)\right] = \frac{mn}{2}(mn+1)$$

EXAMPLE 35 *If the sum of m terms of an A.P. is equal to the sum of either the next n terms or the next p terms, then prove that*

$$(m+n)\left(\frac{1}{m}-\frac{1}{p}\right)=(m+p)\left(\frac{1}{m}-\frac{1}{n}\right).$$

[NCERT EXEMPLAR]

SOLUTION Let a denote the first term and d the common difference of the A.P. Further, let a_k denote the k^{th} term of the A.P. Then,

Sum of m terms = Sum of next n terms

$$\Rightarrow \quad a_1 + a_2 + a_3 + \dots + a_m = a_{m+1} + a_{m+2} + \dots + a_{m+n}$$

$$\Rightarrow \quad 2(a_1 + a_2 + \dots + a_m) = a_1 + a_2 + \dots + a_m + a_{m+1} + a_{m+2} + \dots + a_{m+n}$$

$$\Rightarrow \quad 2\,S_m = S_{m+n}$$

$$\Rightarrow \quad 2\,\frac{m}{2}\left\{2a + (m-1)\,d\right\} = \frac{m+n}{2}\left\{2a + (m+n-1)\,d\right\}$$

$$\Rightarrow \quad \frac{2m}{m+n} = \frac{2a + (m+n-1)\,d}{2a + (m-1)\,d}$$

$$\Rightarrow \quad \frac{2m}{m+n} - 1 = \frac{2a + (m+n-1)\,d}{2a + (m-1)\,d} - 1 \Rightarrow \frac{m-n}{m+n} = \frac{nd}{2a + (m-1)\,d} \qquad \text{...(i)}$$

Similarly,

$$\text{Sum of } m \text{ terms} = \text{Sum of next } p \text{ terms} \Rightarrow \frac{m-p}{m+p} = \frac{pd}{2a + (m-1)\,d} \qquad \text{...(ii)}$$

Dividing (i) by (ii), we get

$$\frac{m-n}{m+n} \cdot \frac{m+p}{m-p} = \frac{n}{p}$$

$$\Rightarrow \quad \frac{(m-n)(m+p)}{n} = \frac{(m+n)(m-p)}{p}$$

$$\Rightarrow \quad \frac{(m-n)(m+p)}{nm} = \frac{(m+n)(m-p)}{mp}$$

$$\Rightarrow \quad (m+p)\left(\frac{m-n}{mn}\right) = (m+n)\left(\frac{m-p}{mp}\right)$$

$$\Rightarrow \quad (m+n)\left(\frac{1}{p}-\frac{1}{m}\right) = (m+p)\left(\frac{1}{n}-\frac{1}{m}\right) \Rightarrow (m+n)\left(\frac{1}{m}-\frac{1}{p}\right) = (m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$$

EXERCISE 18.4

BASIC

1. Find the sum of the following arithmetic progressions:
   (i) 50, 46, 42, ... to 10 terms
   (ii) 1, 3, 5, 7, ... to 12 terms
   (iii) 3, 9/2, 6, 15/2, ... to 25 terms
   (iv) 41, 36, 31, ... to 12 terms
   (v) $a+b, a-b, a-3b, \dots$ to 22 terms
   (vi) $(x-y)^2, (x^2+y^2), (x+y)^2, \dots$ to n terms
   (vii) $\frac{x-y}{x+y}, \frac{3x-2y}{x+y}, \frac{5x-3y}{x+y}, \dots$ to n terms
2. Find the sum of the following series:
   (i) $2 + 5 + 8 + \dots + 182$
   (ii) $101 + 99 + 97 + \dots + 47$

(iii) $(a-b)^2 + (a^2+b^2) + (a+b)^2 + \ldots + [(a+b)^2 + 6ab]$

3. Find the sum of first n natural numbers.
4. Find the sum of all natural numbers from 1 to 100, which are divisible by 2 or 5. [NCERT]
5. Find the sum of first n odd natural numbers.
6. Find the sum of all odd numbers between 100 and 200.
7. Show that the sum of all odd integers between 1 and 1000 which are divisible by 3 is 83667.
8. Find the sum of all integers between 84 and 719, which are multiples of 5.
9. Find the sum of all integers between 50 and 500 which are divisible by 7.
10. Find the sum of all even integers between 101 and 999.
11. Find the sum of all integers between 100 and 550, which are divisible by 9.
12. Find the sum of the series: $3+5+7+6+9+12+9+13+17+\ldots$ to $3n$ terms.
13. Find the sum of all those integers between 100 and 800 each of which on division by 16 leaves the remainder 7.

BASED ON LOTS

14. Solve: (i) $25+22+19+16+\ldots+x=115$ (ii) $1+4+7+10+\ldots+x=590$.
15. Find the rth term of an A.P., the sum of whose first n terms is $3n^2+2n$. [NCERT EXEMPLAR]
16. How many terms are there in the A.P. whose first and fifth terms are -14 and 2 respectively and the sum of the terms is 40 ?
17. The sum of first 7 terms of an A.P. is 10 and that of next 7 terms is 17. Find the progression.
18. The third term of an A.P. is 7 and the seventh term exceeds three times the third term by 2. Find the first term, the common difference and the sum of first 20 terms.
19. The first term of an A.P. is 2 and the last term is 50. The sum of all these terms is 442. Find the common difference.
20. The number of terms of an A.P. is even; the sum of odd terms is 24, of the even terms is 30, and the last term exceeds the first by 10 1/2, find the number of terms and the series.
21. If $S_n = n^2 p$ and $S_m = m^2 p$, $m \neq n$, in an A.P., prove that $S_p = p^3$.
22. If 12th term of an A.P. is -13 and the sum of the first four terms is 24, what is the sum of first 10 terms ?
23. If the 5*th* and 12*th* terms of an A.P. are 30 and 65 respectively, what is the sum of first 20 terms ?
24. Find the sum of n terms of the A.P. whose kth terms is $5k+1$. [NCERT]
25. Find the sum of all two digit numbers which when divided by 4, yields 1 as remainder. [NCERT]
26. If the sum of a certain number of terms of the AP 25, 22, 19, ... is 116. Find the last term. [NCERT]
27. Find the sum of odd integers from 1 to 2001. [NCERT]
28. How many terms of the A.P. $-6, -\frac{11}{2}, -5, \ldots$ are needed to give the sum -25?
29. In an A.P. the first term is 2 and the sum of the first five terms is one fourth of the next five terms. Show that 20th term is -112. [NCERT]

BASED ON HOTS

30. If S_1 be the sum of $(2n+1)$ terms of an A.P. and S_2 be the sum of its odd terms, then prove that: $S_1 : S_2 = (2n+1):(n+1)$.
31. Find an A.P. in which the sum of any number of terms is always three times the squared number of these terms.
32. If the sum of n terms of an A.P. is $nP + \frac{1}{2}n(n-1)Q$, where P and Q are constants, find the common difference. [NCERT]
33. The sums of n terms of two arithmetic progressions are in the ratio $5n+4 : 9n+6$. Find the ratio of their 18th terms. [NCERT]
34. The sums of first n terms of two A.P.'s are in the ratio $(7n+2):(n+4)$. Find the ratio of their 5*th* terms.

ANSWERS

1. (i) 320 (ii) 144 (iii) 525 (iv) 162 (v) $22a - 440b$ (vi) $n\{(x-y)^2 + (n-1)xy\}$ (vii) $\frac{n}{2(x+y)}\{n(2x-y)-y\}$
2. (i) 5612 (ii) 2072 (iii) $6(a^2+b^2+3ab)$
3. $\frac{n(n+1)}{2}$
4. 3050
5. n^2
6. 7500
8. 50800
9. 17696
10. 246950
11. 16425
12. $3n(2n+3)$
13. 19668
14. (i) -2 (ii) 58
15. $6r-1$
16. 10
17. $a=1,\ d=1/7$
18. $-1, 4, 740$
19. 3
20. 8 terms, $1\frac{1}{2}, 3, 4\frac{1}{2}, \ldots$
22. 0
23. 1150
24. $\frac{n}{2}(5n+7)$
25. 1210
26. 4
27. 1002001
28. 5 or 20
31. 3, 9, 15, 21
32. Q
33. 179 : 321
34. 5 : 1

HINTS TO SELECTED PROBLEMS

3. Required sum $= 1+2+3+\ldots+n = \frac{n}{2}(1+n)$
4. Required sum = Sum of natural numbers between1 and 100 which are divisible by 2
 + Sum of natural numbers between 1 and 100 which are divisible by 5
 − Sum of natural numbers between 1 and 100 which are divisible by 2 and 5 both i.e. by 10

$$= (2+4+\ldots+100) + (5+10+15+\ldots+100) - (10+20+\ldots+100)$$
$$= \frac{50}{2}(2+100) + \frac{20}{2}(5+100) - \frac{10}{2}(10+100)$$
$$= 2550 + 1050 - 550 = 3050$$

5. Required sum $= 1+3+5+\ldots+(2n-1) = \frac{n}{2}\{1+(2n-1)\} = n^2$
6. Required sum $= 101+103+\ldots+199 = \frac{50}{2}(101+199) = 7500$
8. Required sum $= 85+90+\ldots+715 = \frac{127}{2}(85+715) = 50800$
9. Required sum $= 56+63+\ldots+497$
10. Required sum $= 102+104+\ldots+998$
11. Required sum $= 108+117+\ldots+549$

12. Required sum = (3 + 6 + 9 + ... to n terms) + (5 + 9 + 13 + ... to n terms) + (7 + 12 + 17 + ... to n terms)

13. Required sum = 103 + 119 + 135 + ... + 791

15. $a_r = S_r - S_{r-1} = (3r^2 + 2r) - \left\{3(r-1)^2 + 2(r-1)\right\} = 6r - 1$

17. We have, $S_7 = 10$ and $S_{14} = 10 + 17 = 27$

24. We have,

$$a_k = 5k + 1 \Rightarrow a_1 = 6 \text{ and } a_n = 5n + 1$$

$$\therefore \quad S_n = \frac{n}{2}\left(a_1 + a_n\right) \Rightarrow S_n = \frac{n}{2}\left(6 + 5n + 1\right) = \frac{n}{2}\left(5n + 7\right)$$

25. We have to find the sum of all two digit numbers of the form $4k + 1$, $k \in N$. Clearly, such numbers are 13, 17, 21, 25, ..., 97 and are forming an A.P. with common difference 4. Let such numbers be n in number. Then,

$$97 = n^{\text{th}} \text{ term of AP with first term 13 and common difference 4}$$

$$\Rightarrow \quad 97 = 13 + (n-1) \times 4 \Rightarrow n - 1 = 21 \Rightarrow n = 22$$

Let S be the sum of such numbers. Then,

$$S = \frac{n}{2}\left(a_1 + a_n\right) = \frac{22}{2}\left(13 + 97\right) = 1210$$

26. Let the sum of n terms of the A.P. 25, 22, 19, ... be 116. Then,

$$116 = \frac{n}{2}\left\{2 \times 25 + (n-1) \times (-3)\right\}$$

$$\Rightarrow \quad 232 = n(-3n + 53) \Rightarrow 3n^2 - 53n + 232 = 0$$

$$\Rightarrow \quad 3n^2 - 24n - 29n + 232 = 0 \Rightarrow (n-8)(3n-29) = 0 \Rightarrow n = 8 \qquad \left[\because n \neq \frac{29}{3}\right]$$

$$\therefore \quad a_8 = 25 + 7 \times (-3) = 4$$

27. The odd integers from 1 to 2001 are $1, 3, 5, 7, \ldots, 2001$. Let the number of such integers be n. Then,

$$2001 = 1 + (n-1) \times 2 \Rightarrow n = 1001$$

$$\therefore \quad \text{Required sum} = \frac{1001}{2}(1 + 2001) = 1001 \times 1001 = 1002001.$$

ALITER The sum of first n odd integers is n^2. So, the sum of odd integers 1, 3, 5, 7, ..., 2001 is $(1001)^2 = 1002001$.

29. We have, $a_1 = 2$ and $a_1 + a_2 + \ldots + a_5 = \frac{1}{4}(a_6 + a_7 + \ldots + a_{10})$

Now,

$$a_1 + a_2 + \ldots + a_5 = \frac{1}{4}(a_6 + a_7 + \ldots + a_{10})$$

$$\Rightarrow \quad 4(a_1 + a_2 + \ldots + a_5) = a_6 + a_7 + \ldots + a_{10}$$

$$\Rightarrow \quad 4S_5 = S_{10} - S_5 \Rightarrow 5S_5 = S_{10} \Rightarrow 5\left\{\frac{5}{2}\{2 \times 2 + (5-1)d\}\right\} = \frac{10}{2}\left\{2 \times 2 + (10-1)d\right\}$$

$\Rightarrow \quad \frac{25}{2}(4+4d) = \frac{10}{2}(9d+4) \Rightarrow 20(1+d) = 2(9d+4) \Rightarrow 10+10d = 9d+4 \Rightarrow d = -6$

$\therefore \quad a_{20} = a_1 + 19d = 2 + 19 \times (-6) = -112$

31. Use $S_n = 3n^2$ and $a_n = S_n - S_{n-1}$.

32. We have, $S_n = nP + \frac{1}{2}n(n-1)Q \Rightarrow S_{n-1} = (n-1)P + \frac{1}{2}(n-1)(n-2)Q$

Let a_n be the n^{th} term. Then,

$$a_n = S_n - S_{n-1}$$

$$\Rightarrow \quad a_n = \left\{nP + \frac{1}{2}n(n-1)Q\right\} - \left\{(n-1)P + \frac{1}{2}(n-1)(n-2)Q\right\} = P + \frac{1}{2}(n-1)\{n-(n-2)\}Q$$

$$\Rightarrow \quad a_n = P + (n-1)Q \Rightarrow a_{n-1} = P + (n-2)Q$$

Let d be the common difference. Then,

$$d = a_n - a_{n-1} = \{P + (n-1)Q\} - \{P + (n-2)Q\} = Q$$

ALITER We have,

$$S_n = nP + \frac{1}{2}n(n-1)Q \Rightarrow S_n = \frac{1}{2}n^2 Q + \left(P - \frac{Q}{2}\right)n$$

Clearly, S_n is of the form $An^2 + Bn$. Hence, the sequence is an A.P. with common difference $2A = Q$.

33. Let S_n and S'_n be the sums of n terms of two arithmetic progressions. Then,

$$\frac{S_n}{S_n'} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \quad \frac{\frac{n}{2}\{2a_1 + (n-1)d_1\}}{\frac{n}{2}\{2a_2 + (n-1)d_2\}} = \frac{5n+4}{9n+6} \Rightarrow \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{5n+4}{9n+6} \Rightarrow \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{5n+4}{9n+6}$$

Replacing $\frac{n-1}{2}$ by 17 i.e. n by 35, we get $\frac{a_1 + 17d_1}{a_2 + 17d_2} = \frac{179}{321}$

ALITER If the ratio of the sums of n terms is given, then to find the ratio of their n^{th} terms, we replace n by $(2n-1)$. So, to find the ratio of 18th terms, we replace n by $2 \times 18 - 1 = 35$ in the ratio $5n + 4 : 9n + 6$

Hence, required ratio is $(5 \times 35 + 4) : (9 \times 35 + 6)$ i.e. $179 : 321$.

18.6 PROPERTIES OF ARITHMETIC PROGRESSIONS

In this section, we shall discuss some properties of arithmetical progressions which will be frequently used in this chapter and in the subsequent chapters.

PROPERTY 1 *If a constant is added to or subtracted from each term of an A.P., then the resulting sequence is also an A.P. with the same common difference.*

PROOF Let $a_1, a_2, a_3, \dots$ be an A.P. with common difference d, and let k be a fixed constant which is added to each term of this A.P. Then, the resulting sequence is $a_1 + k, a_2 + k, a_3 + k_1 \dots$

Let $b_n = a_n + k, n = 1, 2, \dots$ Then, the new sequence is $b_1, b_2, b_3, \dots$

Now, $\quad b_{n+1} - b_n = (a_{n+1} + k) - (a_n + k) = a_{n+1} - a_n = d$ for all $n \in N$

Thus, the new sequence is also an A.P. with common difference d.

PROPERTY 2 *If each term of a given A.P. is multiplied or divided by a non-zero constant k, then the resulting sequence is also an A.P. with common difference kd or d/k, where d is the common difference of the given A.P.*

PROOF Let $a_1, a_2, a_3, \ldots$ be an A.P. with common difference d and let k be a non-zero constant. Let $b_1, b_2, b_3, \ldots$ be sequence obtained by multiplying each term of the given A.P. by k. Then,

$$b_1 = a_1 k, b_2 = a_2 k, \ldots, b_n = a_n k, \ldots$$

Now, $b_{n+1} - b_n = a_{n+1} k - a_n k = (a_{n+1} - a_n) k = dk$ for all $n \in N$ $[\because a_{n+1} - a_n = d$ for all $n \in N]$

This shows that the new sequence is an A.P. with common difference dk.

Similarly, it can be proved that on dividing each term of a given A.P. by a non-zero constant, we obtain a sequence which is also an A.P.

PROPERTY 3 *In a finite A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.*

i.e. $a_k + a_{n-(k-1)} = a_1 + a_n$ for all $k = 1, 2, 3, \ldots, n-1$.

PROOF Let $a_1, a_2, a_3, \ldots, a_n$ be an A.P. with common difference d. We have to show that

$$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = a_4 + a_{n-3} = \ldots$$

i.e. $a_1 + a_n = a_k + a_{n-(k-1)}$ for all $k = 1, 2, 3, \ldots, n-1$

For any $k = 1, 2, \ldots, n-1$

$$\begin{aligned} a_k + a_{n-(k-1)} &= a_k + a_{n+1-k} \\ &= [a_1 + (k-1) d] + [a_1 + (n+1-k-1) d] \\ &= 2a_1 + (k-1+n+1-k-1) d \\ &= 2a_1 + (n-1) d = a_1 + [a_1 + (n-1) d] = a_1 + a_n. \end{aligned}$$

PROPERTY 4 *Three numbers a, b, c are in A.P. iff $2b = a + c$.*

PROOF First, let a, b, c be in A.P. Then,

$b - a =$ Common difference and, $c - b =$ Common difference

$\Rightarrow \quad b - a = c - b \Rightarrow 2b = a + c$

Conversely, let a, b, c be three numbers such that $2b = a + c$. Then, we have to show that a, b, c are in A.P.

We have, $2b = a + c \Rightarrow b - a = c - b \Rightarrow a, b, c$ are in A.P.

ILLUSTRATION *If $\frac{2}{3}, k, \frac{5}{8}$ are in A.P., find the value of k.*

SOLUTION It is given that,

$$\frac{2}{3}, k, \frac{5}{8} \text{ are in A.P.} \Rightarrow 2k = \frac{2}{3} + \frac{5}{8} \Rightarrow 2k = \frac{31}{24} \Rightarrow k = \frac{31}{48}.$$

PROPERTY 5 *A sequence is an A.P. iff its nth term is a linear expression in n i.e. $a_n = An + B$, where A, B are constants. In such a case the coefficient of n in a_n is the common difference of the A.P.*

PROOF See example 3 on page 18.3.

PROPERTY 6 *A sequence is an A.P. iff the sum of its first n terms is of the form $An^2 + Bn$, where A, B are constants independent of n. In such a case the common difference is 2A i.e. 2 times the coefficient of n^2.*

PROOF See example 6 on page 18.17.

PROPERTY 7 *If the terms of an A.P. are chosen at regular intervals, then they form an A.P.*

PROPERTY 8 *If a_n, a_{n+1} and a_{n+2} are three consecutive terms of an A.P., then $2a_{n+1} = a_n + a_{n+2}$*

.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I TO PROVE THAT THREE NUMBERS ARE IN A.P. WHEN THREE GIVEN NUMBERS ARE IN A.P.

EXAMPLE 1 *If a, b, c are in A.P., prove that the following are also in A.P.*

(i) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ (ii) $b+c, c+a, a+b$

(iii) $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ [NCERT] (iv) $a^2(b+c), b^2(c+a), c^2(a+b)$

(v) $\left\{(b+c)^2-a^2\right\}, \left\{(c+a)^2-b^2\right\}, \left\{(a+b)^2-c^2\right\}$ (vi) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$

SOLUTION (i) a, b, c are in A.P.

$\Rightarrow$ $\frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}$ are in A.P. [On dividing each term by abc and using Property 2]

$\Rightarrow$ $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

Thus, a, b, c are in A.P. $\Rightarrow \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

(ii) It is given that

a, b, c are in A.P.

$\Rightarrow$ $a-(a+b+c), b-(a+b+c), c-(a+b+c)$ are in A.P. [Subtracting $a+b+c$ from each term]

$\Rightarrow$ $-(b+c), -(c+a), -(a+b)$ are in A.P. [Multiplying each term by -1]

$\Rightarrow$ $b+c, c+a, a+b$ are in A.P.

(iii) a, b, c are in A.P.

$\Rightarrow$ $\frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}$ are in A.P. [On dividing each term by abc and using Property 2]

$\Rightarrow$ $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

$\Rightarrow$ $\frac{ab+bc+ca}{bc}, \frac{ab+bc+ca}{ca}, \frac{ab+bc+ca}{ab}$ are in A.P.

[On multiplying each term by $ab+bc+ca$ and using Property 2]

$\Rightarrow$ $\frac{ab+bc+ca}{bc}-1, \frac{ab+bc+ca}{ca}-1, \frac{ab+bc+ca}{ab}-1$ are in A.P.

[On adding -1 to each term and using Property 1]

$\Rightarrow$ $\frac{ab+ac}{bc}, \frac{ab+bc}{ca}, \frac{bc+ca}{ab}$ are in A.P.

$\Rightarrow$ $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

(iv) $a^2(b+c), b^2(c+a), c^2(a+b)$ will be in A.P.

if $b^2(c+a)-a^2(b+c) = c^2(a+b)-b^2(c+a)$

i.e. if $c(b^2-a^2)+ab(b-a) = a(c^2-b^2)+bc(c-b)$

i.e. if $(b-a)(ab+bc+ca) = (c-b)(ab+bc+ca)$

i.e. if $b-a = c-b$
i.e. if $2b = a+c$
i.e. if a, b, c are in A.P.

Thus, a, b, c are in A.P. $\Rightarrow a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

ALITER It is given that

a, b, c are in A.P.

$\Rightarrow$ $\frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}$ are in A.P.

$\Rightarrow$ $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

$\Rightarrow$ $\frac{ab+bc+ca}{bc}, \frac{ab+bc+ca}{ca}, \frac{ab+bc+ca}{ab}$ are in A.P.

$\Rightarrow$ $1+\frac{ab+ca}{bc}, 1+\frac{ab+bc}{ca}, 1+\frac{bc+ca}{ab}$ are in A.P.

$\Rightarrow$ $\frac{a(b+c)}{bc}, \frac{b(a+c)}{ca}, \frac{c(a+b)}{ab}$ are in A.P. [Subtracting 1 from each term]

$\Rightarrow$ $\frac{a^2(b+c)}{abc}, \frac{b^2(a+c)}{abc}, \frac{c^2(a+b)}{abc}$ are in A.P.

$\Rightarrow$ $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P. [Multiplying each term by abc]

(v) It is given that

a, b, c are in A.P.

$\Rightarrow$ $-2a, -2b, -2c$ are in A.P. [Multiplying each term by -2]

$\Rightarrow$ $a+b+c-2a, a+b+c-2b, a+b+c-2c$ are in A.P. [Adding $a+b+c$ to each term]

$\Rightarrow$ $b+c-a, c+a-b, a+b-c$ are in A.P.

$\Rightarrow$ $(a+b+c)(b+c-a), (a+b+c)(c+a-b), (a+b+c)(a+b-c)$ are in A.P.

[Multiplying each terming $a+b+c$]

$\Rightarrow$ $(b+c)^2-a^2, (c+a^2)-b^2, (a+b)^2-c^2$ are in A.P.

(vi) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ will be in A.P.

if $\frac{1}{\sqrt{c}+\sqrt{a}} - \frac{1}{\sqrt{b}+\sqrt{c}} = \frac{1}{\sqrt{a}+\sqrt{b}} - \frac{1}{\sqrt{c}+\sqrt{a}}$

i.e. if $\frac{\sqrt{b}-\sqrt{a}}{(\sqrt{c}+\sqrt{a})(\sqrt{b}+\sqrt{c})} = \frac{(\sqrt{c}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a})}$

i.e. if $\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{c}} = \frac{\sqrt{c}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

i.e. if $b-a = c-b$
i.e. if $2b = a+c$
i.e. if a, b, c are in A.P.

Thus, a, b, c are in A.P. $\Rightarrow \frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P.

ALITER It is given that

a, b, c are in A.P.

$\Rightarrow$ $b-a = c-b$

$$\Rightarrow \quad \left(\sqrt{b}+\sqrt{a}\right)\left(\sqrt{b}-\sqrt{a}\right)=\left(\sqrt{c}-\sqrt{b}\right)\left(\sqrt{c}+\sqrt{b}\right)$$

$$\Rightarrow \quad \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{c}}=\frac{\sqrt{c}-\sqrt{b}}{\sqrt{b}+\sqrt{a}}$$

$$\Rightarrow \quad \frac{\left(\sqrt{b}+\sqrt{c}\right)-\left(\sqrt{a}+\sqrt{c}\right)}{\sqrt{b}+\sqrt{c}}=\frac{\left(\sqrt{c}+\sqrt{a}\right)-\left(\sqrt{b}+\sqrt{a}\right)}{\sqrt{b}+\sqrt{a}}$$

$$\Rightarrow \quad \frac{\left(\sqrt{b}+\sqrt{c}\right)-\left(\sqrt{c}+\sqrt{a}\right)}{\left(\sqrt{b}+\sqrt{c}\right)\left(\sqrt{c}+\sqrt{a}\right)}=\frac{\left(\sqrt{c}+\sqrt{a}\right)-\left(\sqrt{a}+\sqrt{b}\right)}{\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{c}+\sqrt{a}\right)}$$

$$\Rightarrow \quad \frac{1}{\sqrt{c}+\sqrt{a}}-\frac{1}{\sqrt{b}+\sqrt{c}}=\frac{1}{\sqrt{a}+\sqrt{b}}-\frac{1}{\sqrt{c}+\sqrt{a}} \Rightarrow \frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}} \text{ are in A.P.}$$

EXAMPLE 2 *If a^2, b^2, c^2 are in A.P, then prove that the following are also in A.P.*

(i) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ (ii) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$

SOLUTION (i) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ will be in A.P.

if $\quad \frac{1}{c+a}-\frac{1}{b+c} = \frac{1}{a+b}-\frac{1}{c+a}$

i.e. if $\quad \frac{b-a}{(c+a)(b+c)} = \frac{c-b}{(a+b)(c+a)}$

i.e. if $\quad \frac{b-a}{b+c} = \frac{c-b}{a+b}$

i.e. if $\quad b^2-a^2 = c^2-b^2$

i.e. if $\quad 2b^2 = a^2+c^2$

i.e. if $\quad a^2, b^2, c^2$ are in A.P.

Thus, $\quad a^2, b^2, c^2$ are in A.P. $\Rightarrow \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

ALITER (i) It is given that

$a^2,\ b^2,\ c^2$ are in A.P.

$$\Rightarrow \quad b^2-a^2=c^2-b^2$$

$$\Rightarrow \quad (b-a)(b+a)=(c-b)(c+b)$$

$$\Rightarrow \quad \frac{b-a}{b+c}=\frac{c-b}{a+b}$$

$$\Rightarrow \quad \frac{(b+c)-(a+c)}{b+c}=\frac{(c+a)-(b+a)}{a+b}$$

$$\Rightarrow \quad \frac{(b+c)-(a+c)}{(a+c)(b+c)}=\frac{(c+a)-(b+a)}{(a+b)(a+c)}$$ [Multiplying both side by $\frac{1}{a+b}$]

$\Rightarrow \quad \frac{1}{a+c}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{a+c}$

$\Rightarrow \quad \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$, are in A.P.

(ii) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ will be in A.P.

if $\frac{a}{b+c}+1, \frac{b}{c+a}+1, \frac{c}{a+b}+1$ are in A.P. [On adding 1 to each term]

i.e. if $\frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{c+a}$ are in A.P.

i.e. if $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P. [On dividing each term by $a+b+c$]

i.e. if $\frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a}$

i.e. if $\frac{b-a}{(c+a)(b+c)}=\frac{c-b}{(a+b)(c+a)}$

i.e. if $\frac{b-a}{b+c}=\frac{c-b}{a+b}$

i.e. if $b^2-a^2=c^2-b^2$

i.e. if $2b^2=a^2+c^2$

i.e. if a^2, b^2, c^2 are in A.P.

Thus, a^2, b^2, c^2 are in A.P. $\Rightarrow \frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

ALITER It is given that

a^2, b^2, c^2 are in A.P.

$\Rightarrow \quad b^2-a^2=c^2-b^2.$

$\Rightarrow \quad (b+a)(b-a)=(c+b)(c-b)$

$\Rightarrow \quad \frac{b-a}{b+c}=\frac{c-b}{a+b}$

$\Rightarrow \quad \frac{(b+c)-(a+c)}{b+c}=\frac{(c+a)-(b+a)}{a+b}$

$\Rightarrow \quad \frac{(b+c)-(a+c)}{(a+c)(b+c)}=\frac{(c+a)-(b+a)}{(a+b)(a+c)}$

$\Rightarrow \quad \frac{1}{a+c}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{a+c}$

$\Rightarrow \quad \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$\Rightarrow \quad \frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{a+b}$ are in A.P.

$\Rightarrow \quad 1+\frac{a}{b+c}, 1+\frac{b}{c+a}, 1+\frac{c}{a+b}$ are in A.P. $\Rightarrow \frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

EXAMPLE 3 *If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.*

SOLUTION $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.

$\Rightarrow \left\{\frac{b+c-a}{a}+2\right\}, \left\{\frac{c+a-b}{b}+2\right\}, \left\{\frac{a+b-c}{c}+2\right\}$ are in A.P. [Adding 2 to each term]

$\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in A.P.

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P [Dividing each term by $a+b+c$]

EXAMPLE 4 *If $a^2+2bc, b^2+2ac, c^2+2ab$ are in A.P., show that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.*

SOLUTION $a^2+2bc, b^2+2ac, c^2+2ab$ are in A.P.

$\Rightarrow (a^2+2bc)-(ab+bc+ca), (b^2+2ac)-(ab+bc+ca), (c^2+2ab)-(ab+bc+ca)$ are in A.P.

[On subtracting $(ab+bc+ca)$ from each term]

$\Rightarrow a^2+bc-ab-ca, b^2+ca-ab-bc, c^2+ab-bc-ca$ are in A.P.

$\Rightarrow (a-b)(a-c), (b-c)(b-a), (c-a)(c-b)$ are in A.P.

$\Rightarrow \frac{-1}{b-c}, \frac{-1}{c-a}, \frac{-1}{a-b}$ are in A.P. [On dividing each term by $(a-b)(b-c)(c-a)$]

$\Rightarrow \frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P. [On multiplying each term by -1]

EXAMPLE 5 *If $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P., prove that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.*

SOLUTION $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P.

$\Rightarrow (c-a)^2-(b-c)^2 = (a-b)^2-(c-a)^2$

$\Rightarrow (b-a)(2c-a-b) = (c-b)(2a-b-c)$...(i)

$\Rightarrow (b-a)\{(c-a)+(c-b)\} = (c-b)\{(a-b)+(a-c)\}$

$\Rightarrow (b-a)(c-a)+(b-a)(c-b) = (c-b)(a-b)+(a-c)(c-b)$

$\Rightarrow -(a-b)(c-a)+(a-b)(b-c) = -(a-b)(b-c)+(b-c)(c-a)$

$\Rightarrow -\frac{1}{b-c}+\frac{1}{c-a} = -\frac{1}{c-a}+\frac{1}{a-b}$ [Dividing throughout by $(a-b)(b-c)(c-a)$

$\Rightarrow \frac{2}{c-a} = \frac{1}{a-b}+\frac{1}{b-c} \Rightarrow \frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in AP.

Thus, if $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P., then $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.

EXAMPLE 6 *If a, b, c are in A.P., then prove that:*

(i) $(a-c)^2 = 4(b^2-ac)$ (ii) $a^3+4b^3+c^3 = 3b(a^2+c^2)$

SOLUTION (i) It is given that a, b, c are in A.P. Therefore, $2b = a+c \Rightarrow b = \frac{a+c}{2}$.

Putting $b = \frac{a+c}{2}$ on RHS, we obtain

$$\text{RHS} = 4\,(b^2 - ac) = 4\left\{\left(\frac{a+c}{2}\right)^2 - ac\right\} = 4\left\{\frac{(a+c)^2 - 4ac}{4}\right\} = (a+c)^2 - 4ac = (a-c)^2 = \text{LHS}$$

(ii) It is given that a, b, c are in A.P. Therefore, $2b = a + c \Rightarrow b = \dfrac{a+c}{2}$

$$\text{LHS} = a^3 + 4b^3 + c^3$$

$$= a^3 + 4\left(\frac{a+c}{2}\right)^3 + c^3 = (a^3 + c^3) + \frac{1}{2}(a+c)^3$$

$$= \frac{1}{2}\left\{2\,(a^3 + c^3) + (a+c)^3\right\} = \frac{1}{2}\left\{2\,(a+c)\,(a^2 - ac + c^2) + (a+c)^3\right\}$$

$$= \frac{1}{2}(a+c)\left\{2\,(a^2 - ac + c^2) + (a+c)^2\right\}$$

$$= \frac{1}{2}(a+c)\;3(a^2 + c^2) = 3\left(\frac{a+c}{2}\right)(a^2 + c^2) = 3b\,(a^2 + c^2) = \text{RHS}$$

ALITER LHS $= a^3 + 4b^3 + c^3 = (a^3 + c^3) + 4b^3 = (a+c)^3 - 3ac\,(a+c) + 4b^3$

$$= (2b)^3 - 3ac\,(2b) + 4b^3 = 12b^3 - 6abc$$

$$= 3b\,(4b^2 - 2ac) = 3b\left\{(2b)^2 - 2ac\right\} = 3b\left\{(a+c)^2 - 2ac\right\} = 3b\,(a^2 + c^2) = \text{RHS}$$

EXAMPLE 7 *If $a^2\,(b+c), b^2\,(c+a), c^2\,(a+b)$ are in A.P., show that either a, b, c are in A.P. or $ab + bc + ca = 0$.*

SOLUTION $a^2\,(b+c), b^2\,(c+a), c^2\,(a+b)$ are in A.P.

$$\Rightarrow \quad b^2\,(c+a) - a^2\,(b+c) = c^2\,(a+b) - b^2\,(c+a)$$

$$\Rightarrow \quad (b^2a - a^2b) + (b^2c - a^2c) = (c^2b - b^2c) + (c^2a - b^2a)$$

$$\Rightarrow \quad (b-a)\,(ab + bc + ca) = (c-b)\,(ab + bc + ca)$$

$$\Rightarrow \quad (ab + bc + ca)\,(2b - a - c) = 0$$

$$\Rightarrow \quad ab + bc + ca = 0 \quad \text{or} \quad 2b - a - c = 0 \Rightarrow ab + bc + ca = 0 \quad \text{or} \quad a, b, c \text{ are in A.P.}$$

EXAMPLE 8 *If $\dfrac{p-x}{ax} = \dfrac{p-y}{by} = \dfrac{p-z}{cz}$ and a, b, c are in A.P. Show that $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ are in A.P.*

SOLUTION It is given that $\dfrac{p-x}{ax} = \dfrac{p-y}{by} = \dfrac{p-z}{cz}$. Let $\dfrac{p-x}{ax} = \dfrac{p-y}{by} = \dfrac{p-z}{cz} = \lambda$. Then,

$$\frac{p-x}{ax} = \lambda, \frac{p-y}{by} = \lambda \text{ and } \frac{p-z}{cz} = \lambda$$

$$\Rightarrow \quad a = \frac{p-x}{\lambda x}, b = \frac{p-y}{\lambda y} \text{ and } c = \frac{p-z}{\lambda z} \Rightarrow a = \frac{1}{\lambda}\left(\frac{p}{x} - 1\right), b = \frac{1}{\lambda}\left(\frac{p}{y} - 1\right) \text{ and } c = \frac{1}{\lambda}\left(\frac{p}{z} - 1\right)$$

It is given that a, b, c are in A.P. Therefore,

$$2b = a + c$$

$$\Rightarrow \quad \frac{2}{\lambda}\left(\frac{p}{y}-1\right)=\frac{1}{\lambda}\left(\frac{p}{x}-1\right)+\frac{1}{\lambda}\left(\frac{p}{z}-1\right) \Rightarrow \frac{2p}{y}=\frac{p}{x}+\frac{p}{z} \Rightarrow \frac{2}{y}=\frac{1}{x}+\frac{1}{z} \Rightarrow \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \text{ are in A.P.}$$

EXERCISE 18.5

BASIC

1. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., prove that:

   (i) $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P. (ii) $a(b+c), b(c+a), c(a+b)$ are in A.P.

2. If a^2, b^2, c^2 are in A.P., prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

3. If a, b, c are in A.P., then show that:

   (i) $a^2(b+c), b^2(c+a), c^2(a+b)$ are also in A.P.

   (ii) $b+c-a, c+a-b, a+b-c$ are in A.P.

   (iii) $bc-a^2, ca-b^2, ab-c^2$ are in A.P.

4. If $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P., prove that:

   (i) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. (ii) bc, ca, ab are in A.P.

5. If a, b, c are in A.P., prove that:

   (i) $(a-c)^2 = 4(a-b)(b-c)$ (ii) $a^2+c^2+4ac = 2(ab+bc+ca)$

   (iii) $a^3+c^3+6abc = 8b^3$

6. If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P., prove that a, b, c are in A.P.

7. Show that x^2+xy+y^2, z^2+zx+x^2 and y^2+yz+z^2 are consecutive terms of an A.P., if x, y and z are in A.P. [NCERT EXEMPLAR]

HINTS TO SELECTED PROBLEMS

5. (i) Put $b=\frac{a+c}{2}$ on RHS (ii) Put $b=\frac{a+c}{2}$ on RHS (iii) Put $b=\frac{a+c}{2}$ on RHS and LHS

6. $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

$$\Rightarrow \quad a\left(\frac{1}{b}+\frac{1}{c}\right)+1, b\left(\frac{1}{c}+\frac{1}{a}\right)+1, c\left(\frac{1}{a}+\frac{1}{b}\right)+1 \text{ are A.P.} \qquad [\because \text{ Adding 1 throughout}]$$

$$\Rightarrow \quad a\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{a}{a}, b\left(\frac{1}{c}+\frac{1}{a}\right)+\frac{b}{b}, c\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{c}{c} \text{ are in A.P.}$$

$$\Rightarrow \quad a\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right), b\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right), c\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \text{ are in A.P.}$$

$$\Rightarrow \quad a, b, c \text{ are in A.P.} \qquad \left[\text{Dividing each term by } \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right]$$

7. It is given that x, y, z are in A.P. Therefore, $y-x=z-y=d$ (say)

Now, $(x^2+zx+z^2)-(x^2+xy+y^2) = (-y^2+z^2)+x(z-y) = (z-y)(x+y+z) = d(x+y+z)$

and, $(z^2+yz+y^2)-(z^2+zx+x^2) = (y^2-x^2)+z(y-x) = (y-x)(x+y+z) = d(x+y+z)$

$\therefore \quad (x^2+zx+z^2)-(x^2+xy+y^2)=(z^2+yz+y^2)-(z^2+zx+x^2)$

$\Rightarrow \quad x^2+xy+y^2, z^2+zx+x^2, y^2+yz+z^2$ are in A.P.

18.7 INSERTION OF ARITHMETIC MEANS

If between two given quantities a and b we have to insert n quantities $A_1, A_2, \ldots, A_n$ such that $a, A_1, A_2, \ldots A_n, b$ form an A.P., then we say that $A_1, A_2, \ldots, A_n$ are arithmetic means between a and b.

ILLUSTRATION *Since 15, 11, 7, 3, –1, –5 are in A.P., it follows that 11, 7, 3, –1 are four arithmetic means between 15 and –5.*

If a, A, b are in A.P., we say that A is the arithmetic mean of a and b.

18.7.1 INSERTION OF ARITHMETIC MEANS

Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between two quantities a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ is an A.P. Let d be the common difference of this A.P. Clearly, it contains $(n+2)$ terms.

$$\therefore \quad b = (n+2)^{\text{th}} \text{ term} \Rightarrow b = a+(n+1)d \Rightarrow d = \frac{b-a}{n+1}$$

Now,
$$A_1 = a+d = a+\frac{b-a}{n+1}$$
$$A_2 = a+2d = a+\frac{2(b-a)}{n+1}$$
$$A_3 = a+3d = a+\frac{3(b-a)}{n+1}$$
$$\vdots \qquad \vdots \quad \vdots$$
$$A_n = a+nd = a+\frac{n(b-a)}{n+1}$$

These are the required arithmetic means between a and b.

18.7.2 INSERTION OF A SINGLE ARITHMETIC MEAN BETWEEN TWO NUMBERS

Let a and b be two numbers and A be the single arithmetic mean between them. Then,

$$a, A, b \text{ are in A.P.} \Rightarrow A-a=b-A \Rightarrow 2A=a+b \Rightarrow A=\frac{a+b}{2}$$

Thus, the arithmetic mean of a and b is $\frac{a+b}{2}$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Insert three arithmetic means between 3 and 19.*

SOLUTION Let A_1, A_2, A_3 be 3 A.M.'s between 3 and 19. Then $3, A_1, A_2, A_3, 19$ are in A.P. whose common difference d is given by $d=\frac{19-3}{3+1}=4$.

$$\therefore \quad A_1 = 3+d = 3+4=7,\ A_2 = 3+2d = 3+2\times 4 = 11,\ A_3 = 3+3d = 3+3\times 4 = 15.$$

Hence, the required A.M.'s are 7, 11, 15.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 2 *For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the arithmetic mean of a and b?* [NCERT]

SOLUTION The A.M. of a and b is $\frac{a+b}{2}$. Therefore, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ will be the A.M. of a and b, if

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{a+b}{2}$$

$$\Rightarrow \quad 2(a^{n+1}+b^{n+1}) = (a^n+b^n)(a+b)$$

$$\Rightarrow \quad 2a^{n+1}+2b^{n+1} = a^{n+1}+a^n b+b^n a+b^{n+1}$$

$$\Rightarrow \quad a^{n+1}+b^{n+1} = a^n b+b^n a$$

$$\Rightarrow \quad a^n(a-b) = b^n(a-b) \Rightarrow a^n = b^n \Rightarrow \frac{a^n}{b^n} = 1 \Rightarrow \left(\frac{a}{b}\right)^n = 1 \Rightarrow \left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0 \Rightarrow n = 0.$$

EXAMPLE 3 *If n arithmetic means are inserted between 20 and 80 such that the ratio of first mean to the last mean is 1 : 3, then find the value of n.*

SOLUTION Let $A_1, A_2, \dots, A_n$ be n arithmetic means between 20 and 80 and let d be the common difference of the A.P. 20, $A_1, A_2, \dots A_n$, 80. Then,

$$d = \frac{80-20}{n+1} = \frac{60}{n+1} \qquad \left[\text{Using: } d = \frac{b-a}{n+1}\right]$$

Now, $A_1 = 20+d \Rightarrow A_1 = 20+\dfrac{60}{n+1} = 20\left(\dfrac{n+4}{n+1}\right)$

And, $A_n = 20+nd \Rightarrow A_n = 20+\dfrac{60n}{n+1} = 20\left(\dfrac{4n+1}{n+1}\right)$

It is given that

$$\frac{A_1}{A_n} = \frac{1}{3} \Rightarrow \frac{\frac{20(n+4)}{n+1}}{\frac{20(4n+1)}{n+1}} = \frac{1}{3} \Rightarrow \frac{n+4}{4n+1} = \frac{1}{3} \Rightarrow 4n+1 = 3n+12 \Rightarrow n = 11$$

EXAMPLE 4 *Between 1 and 31 are inserted m arithmetic means so that the ratio of the 7th and (m – 1)th means is 5 : 9. Find the value of m.* [NCERT]

SOLUTION Let $A_1, A_2, \dots, A_m$ be m arithmetic means between 1 and 31. Then 1, $A_1, A_2, \dots, A_m$, 31 is an A.P. with common difference d given by

$$d = \frac{31-1}{m+1} = \frac{30}{m+1} \qquad \left[\text{Using: } d = \frac{b-a}{n+1}\right]$$

Now, $A_7 = 1+7d = 1+\dfrac{7\times 30}{m+1} = \dfrac{m+211}{m+1}$

and, $A_{m-1} = 1+(m-1)d = 1+\dfrac{30(m-1)}{m+1} = \dfrac{31m-29}{m+1}$

It is given that

$$\frac{A_7}{A_{m-1}} = \frac{5}{9} \Rightarrow \frac{m+211}{31m-29} = \frac{5}{9} \Rightarrow 9m+1899 = 155m-145 \Rightarrow 146m = 2044 \Rightarrow m = 14$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 5 *Prove that the sum of n arithmetic means between two numbers is n times the single A.M. between them.*

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ is an A.P. with common difference d given by $d = \frac{b-a}{n+1}$.

Now, $A_1 + A_2 + \ldots + A_n = \frac{n}{2}\left(A_1 + A_n\right)$ $\left[\because S_n = \frac{n}{2}(a+l)\right]$

$= \frac{n}{2}(a+b)$ $[\because a, A_1, A_2, \ldots, A_n, b$ is an A.P. $\therefore a + b = A_1 + A_n]$

$= n\left(\frac{a+b}{2}\right) = n \times$ (A.M. between a and b)

EXAMPLE 6 *The sum of two numbers is $\frac{13}{6}$. An even number of arithmetic means are being inserted between them and their sum exceeds their number by 1. Find the number of means inserted.*

SOLUTION Let a and b be two numbers such that $a + b = \frac{13}{6}$ ...(i)

Let $A_1, A_2, \ldots, A_{2n}$ be $2n$ arithmetic means between a and b. Then,

$A_1 + A_2 + \ldots + A_{2n} = 2n\left(\frac{a+b}{2}\right)$ [Using result of Example 5]

$\Rightarrow A_1 + A_2 + \ldots + A_{2n} = n(a+b) = \frac{13}{6}n$ [Using (i)]

$\Rightarrow 2n + 1 = \frac{13}{6}n$ $[\because A_1 + A_2 + \ldots + A_{2n} = 2n+1$ (given)$]$

$\Rightarrow 12n + 6 = 13n \Rightarrow n = 6$

Hence, 126 arithmetic means are inserted between the numbers.

EXAMPLE 7 *If the A.M. between pth and qth terms of an A.P. be equal to the A.M. between rth and sth terms of the A.P., then show that $p + q = r + s$.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then

a_p = pth term $= a + (p-1)d$; a_q = qth term $= a + (q-1)d$

a_r = rth term $= a + (r-1)d$ and, a_s = sth term $= a + (s-1)d$

It is given that

A.M. between a_p and a_q = A.M. between a_r and a_s

$\Rightarrow \frac{1}{2}(a_p + a_q) = \frac{1}{2}(a_r + a_s)$

$\Rightarrow a_p + a_q = a_r + a_s$

$\Rightarrow \{a + (p-1)d\} + \{a + (q-1)d\} = \{a + (r-1)d\} + \{a + (s-1)d\}$

$\Rightarrow (p + q - 2)d = (r + s - 2)d \Rightarrow p + q = r + s$

EXAMPLE 8 *Suppose x and y are two real numbers such that the rth mean between x and 2y is equal to the rth mean between 2x and y when n arithmetic means are inserted between them in both the cases. Show that $\frac{n+1}{r} - \frac{y}{x} = 1$.*

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between x and $2y$. Then, $x, A_1, A_2, \ldots, A_n, 2y$ are in AP with common difference d_1 given by $d_1 = \frac{2y-x}{n+1}$.

$\therefore$ r^{th} mean $= A_r = x + r\,d_1 = x + r\left(\frac{2y-x}{n+1}\right)$

Let $A_1', A_2', \ldots, A_n'$ be n arithmetic means between $2x$ and y. Then, $2x, A_1', A_2', \ldots, A_n', y$ are in A.P. with common difference d_2 given by $d_2 = \dfrac{y-2x}{n+1}$.

$$\therefore \qquad r^{\text{th}} \text{ mean} = A_r' = 2x + r\,d_2 = 2x + r\left(\frac{y-2x}{n+1}\right)$$

It is given that :

$$A_r = A_r'$$

$$\Rightarrow \qquad x + r\left(\frac{2y-x}{n+1}\right) = 2x + r\left(\frac{y-2x}{n+1}\right)$$

$$\Rightarrow \qquad (n+1)\,x + r\,(2y-x) = (n+1)\,2x + r\,(y-2x)$$

$$\Rightarrow \qquad (n+1)\,x - ry = rx \;\Rightarrow\; \frac{n+1}{r} - \frac{y}{x} = 1$$

EXERCISE 18.6

BASIC

1. Find the A.M. between:
   (i) 7 and 13 (ii) 12 and -8 (iii) $(x-y)$ and $(x+y)$.
2. Insert 4 A.M.s between 4 and 19.
3. Insert 7 A.M.s between 2 and 17.
4. Insert six A.M.s between 15 and -13.

BASED ON LOTS

5. There are n A.M.s between 3 and 17. The ratio of the last mean to the first mean is 3 : 1. Find the value of n.
6. Insert A.M.s between 7 and 71 in such a way that the 5th A.M. is 27. Find the number of A.M.s.
7. If n A.M.s are inserted between two numbers, prove that the sum of the means equidistant from the beginning and the end is constant.
8. If x, y, z are in A.P. and A_1 is the A.M. of x and y and A_2 is the A.M. of y and z, then prove that the A.M. of A_1 and A_2 is y.
9. Insert five numbers between 8 and 26 such that the resulting sequence is an A.P. [NCERT]

ANSWERS

1. (i) 10 (ii) 2 (iii) x
2. 7, 10, 13, 16
3. $\frac{31}{8}, \frac{23}{4}, \frac{61}{8}, \frac{19}{2}, \frac{91}{8}, \frac{53}{4}, \frac{121}{8}$
4. 11, 7, 3, -1, -5, -9
5. 6
6. 15
9. 11, 14, 17, 20, 23

HINTS TO SELECTED PROBLEMS

9. Let a_1, a_2, a_3, a_4, a_5 be five natural numbers between 8 and 26 such that $8, a_1, a_2, a_3, a_4, a_5, 26$ is an A.P. Let d be the common difference. Then,

$$d = \frac{26-8}{5+1} = 3 \qquad \left[\because\ d = \frac{b-a}{n+1}\right]$$

$\therefore \quad a_1 = 8 + 3 = 11,\ a_2 = a_1 + 3 = 14,\ a_3 = 17,\ a_4 = 20$ and $a_5 = 23$

Hence, five numbers are 11, 14, 17, 20 and 23.

18.8 APPLICATIONS OF A.P.

In this section, we shall discuss some problems based upon the applications of arithmetic progressions.

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *The digits of a positive integer, having three digits, are in A.P. and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.*

SOLUTION Let the digits at ones, tens and hundreds place be $(a - d)$, a and $(a + d)$ respectively. Then the number is

$$(a + d) \times 100 + a \times 10 + (a - d) = 111a + 99d$$

The number obtained by reversing the digits is

$$(a - d) \times 100 + a \times 10 + (a + d) = 111a - 99d$$

It is given that $(a - d) + a + (a + d) = 15$ and, $111a - 99d = 111a + 99d - 594$

$\Rightarrow \quad 3a = 15$ and $198d = 594 \Rightarrow a = 5$ and $d = 3$

So, the number is $111a + 99d = 111 \times 5 + 99 \times 3 = 852$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 2 *Two cars start together in the same direction from the same place. The first goes with uniform speed of 10 km/h. The second goes at a speed of 8 km/h in the first hour and increases the speed by 1/2 km each succeeding hour. After how many hours will the second car overtake the first car if both cars go non-stop ?*

SOLUTION Suppose the second car overtakes the first car after t hours. Then the two cars travel the same distance in t hours.

Distance travelled by the first car in t hours = $10\,t$ km.

Distance travelled by the second car in t hours

= Sum of t terms of an A.P. with first term 8 and common difference 1/2.

$$= \frac{t}{2}\left\{2 \times 8 + (t - 1) \times \frac{1}{2}\right\} = \frac{t\,(t + 31)}{4}$$

When the second car overtakes the first car. The distance travelled by both cars is same.

$$\therefore \quad 10t = \frac{t\,(t + 31)}{4} \Rightarrow t\,(t - 9) = 0 \Rightarrow t = 9 \qquad [\because t \neq 0]$$

Thus, the second car will overtake the first car in 9 hours.

EXAMPLE 3 *A man repays a loan of ₹ 3250 by paying ₹ 20 in the first month and then increases the payment by ₹ 15 every month. How long will it take him to clear the loan ?*

SOLUTION Suppose the loan is cleared in n months. Clearly, the amounts form an A.P. with first term 20 and the common difference 15.

$\therefore \quad$ Sum of the amounts = 3250

$$\Rightarrow \quad \frac{n}{2}\left\{2 \times 20 + (n - 1) \times 15\right\} = 3250$$

$$\Rightarrow \quad 3n^2 + 5n - 1300 = 0 \Rightarrow (n - 20)(3n + 65) = 0 \Rightarrow n = 20 \qquad \left[\because\ 3n + 65 \neq 0\right]$$

Thus, the loan is cleared in 20 months.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *150 workers were engaged to finish a piece of work in a certain number of days. Four workers dropped the second day, four more workers dropped the third day and so on. It takes 8 more days to finish the work now. Find the number of days in which the work was completed.*

SOLUTION Suppose the work is completed in n days when the workers started dropping. Since 4 workers are dropped on every day except the first day. Therefore, the total number of workers who worked all the n days is the sum of n terms of an A.P. with first term 150 and common difference -4.

i.e. $$\frac{n}{2}\left\{2 \times 150 + (n-1) \times -4\right\} = n(152 - 2n)$$

Had the workers not dropped then the work would have finished in $(n-8)$ days with 150 workers working on each day. Therefore, the total number of workers who would have worked all the n days is $150(n-8)$.

$$\therefore \quad n(152 - 2n) = 150(n-8) \Rightarrow n^2 - n - 600 = 0 \Rightarrow (n-25)(n+24) = 0 \Rightarrow n = 25.$$

Thus, the work is completed in 25 days.

EXAMPLE 5 *Along a road lie an odd number of stones placed at intervals of 10 metres. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man carried the job with one of the end stones by carrying them in succession. In carrying all the stones he covered a distance of 3 km. Find the number of stones.*

SOLUTION Let there be $(2n+1)$ stones. Clearly, one stone lies in the middle and n stones on each side of it in a row. Let P be the mid-stone and let A and B be the end stones on the left and right of P respectively. Clearly, there are n intervals each of length 10 metres on both the sides of P. Now, suppose the man starts from A. He picks up the end stone on the left of mid-stone and goes to the mid-stone, drops it and goes to $(n-1)$th stone on left, picks it up, goes to the mid-stone and drops it. This process is repeated till he collects all stones on the left of the mid-stone at the mid-stone. So, distance covered in collecting stones on the left of the mid-stones

$$= 10 \times n + 2[10 \times (n-1) + 10 \times (n-2) + \ldots + 10 \times 2 + 10 \times 1].$$

After collecting all stones on left of the mid-stone the man goes to the stone B on the right side of the mid-stone, picks it up, goes to the mid-stone and drops it. Then he goes to $(n-1)^{th}$ stone on the right and the process is repeated till he collects all stones at the mid-stone.

Distance covered in collecting the stones on the right side of the mid-stone

$$= 2[10 \times n + 10 \times (n-1) + 10 \times (n-2) + \ldots + 10 \times 2 + 10 \times 1]$$

$\therefore$ Total distance covered

$$= 10 \times n + 2[10 \times (n-1) + 10 \times (n-2) + \ldots + 10 \times 2 + 10 \times 1] + 2[10 \times n + 10 \times (n-1) + \ldots + 10 \times 2 + 10 \times 1]$$

$$= 4[10 \times n + 10 \times (n-1) + \ldots + 10 \times 2 + 10 \times 1] - 10 \times n$$

$$= 40\left[1 + 2 + 3 + \ldots + n\right] - 10n = 40\left\{\frac{n}{2}(1+n)\right\} - 10n = 20n^2 + 10n.$$

But, the total distance covered is 3 km = 3000 m.

$$\therefore \quad 20n^2 + 10n = 3000 \Rightarrow 2n^2 + n - 300 = 0 \Rightarrow (n-12)(2n+25) = 0 \Rightarrow n = 12$$

Hence, the number of stones $= 2n + 1 = 25$.

EXERCISE 18.7

BASIC

1. A man saved ₹ 16500 in ten years. In each year after the first he saved ₹ 100 more than he did in the receding year. How much did he save in the first year ?
2. A man saves ₹ 32 during the first year, ₹ 36 in the second year and in this way he increases his savings by ₹ 4 every year. Find in what time his saving will be ₹ 200.
3. A man arranges to pay off a debt of ₹ 3600 by 40 annual instalments which form an arithmetic series. When 30 of the instalments are paid, he dies leaving one-third of the debt unpaid, find the value of the first instalment.
4. A manufacturer of radio sets produced 600 units in the third year and 700 units in the seventh year. Assuming that the product increases uniformly by a fixed number every year, find (i) the production in the first year (ii) the total product in 7 years and (iii) the product in the 10th year.
5. There are 25 trees at equal distances of 5 metres in a line with a well, the distance of the well from the nearest tree being 10 metres. A gardener waters all the trees separately starting from the well and he returns to the well after watering each tree to get water for the next. Find the total distance the gardener will cover in order to water all the trees.

BASED ON LOTS

6. A man is employed to count ₹ 10710. He counts at the rate of ₹ 180 per minute for half an hour. After this he counts at the rate of ₹ 3 less every minute than the preceding minute. Find the time taken by him to count the entire amount.
7. A piece of equipment cost a certain factory ₹ 600,000. If it depreciates in value, 15% the first, 13.5% the next year, 12% the third year, and so on. What will be its value at the end of 10 years, all percentages applying to the original cost ?
8. A farmer buys a used tractor for ₹ 12000. He pays ₹ 6000 cash and agrees to pay the balance in annual instalments of ₹ 500 plus 12% interest on the unpaid amount. How much the tractor cost him?
9. Shamshad Ali buys a scooter for ₹ 22000. He pays ₹ 4000 cash and agrees to pay the balance in annual instalments of ₹ 1000 plus 10% interest on the unpaid amount. How much the scooter will cost him.
10. The income of a person is ₹ 300,000 in the first year and he receives an increase of ₹ 10000 to his income per year for the next 19 years. Find the total amount, he received in 20 years. [NCERT]
11. A man starts repaying a loan as first instalment of ₹ 100. If he increases the instalments by ₹ 5 every month, what amount he will pay in the 30th instalment? [NCERT]
12. A carpenter was hired to build 192 window frames. The first day he made five frames and each day thereafter he made two more frames than he made the day before. How many days did it take him to finish the job? [NCERT EXEMPLAR]

BASED ON HOTS

13. We know that the sum of the interior angles of a triangle is 180°. Show that the sums of the interior angles of polygons with 3, 4, 5, 6,... sides form an arithmetic progression. Find the sum of the interior angles for a 21 sided polygon. [NCERT EXEMPLAR]
14. In a potato race 20 potatoes are placed in a line at intervals of 4 meters with the first potato 24 metres from the starting point. A contestant is required to bring the potatoes back to the starting place one at a time. How far would he run in bringing back all the potatoes? [NCERT EXEMPLAR]

15. A man accepts a position with an initial salary of ₹ 5200 per month. It is understood that he will receive an automatic increase of ₹320 in the very next month and each month thereafter.
    (i) Find his salary for the tenth month.
    (ii) What is his total earnings during the first year?
16. A man saved ₹66000 in 20 years. In each succeeding year after the first year he saved ₹ 200 more than what he saved in the previous year. How much did he save in the first year?
17. In a cricket team tournament 16 teams participated. A sum of ₹ 8000 is to be awarded among themselves as prize money. If the last place team is awarded ₹ 275 in prize money and the award increases by the same amount for successive finishing places, how much amount will the first place team receive?

ANSWERS

1. ₹ 1200 2. 5 yrs 3. ₹ 51 4. (i) 550 (ii) 4375 (iii) 775
5. 3500 m 6. 89 minutes 7. ₹ 105000 8. ₹ 16680
9. ₹ 39100 10. ₹7900,000 11. ₹ 245 12. 12 days
13. 3420° 14. 2480 m 15. (i) ₹ 8080 (ii) ₹ 83520
16. ₹1400 17. ₹ 725

HINTS TO SELECTED PROBLEMS

10. Here, $a = 300{,}000$, $d = 10{,}000$ and $n = 20$. Let S be the total amount received in 20 years. Then,

$$S = ₹\ \frac{20}{2}\{2\times 300{,}000 + (20-1)\times 10{,}000\} = ₹10\,(600{,}000 + 190{,}000) = ₹\,7900{,}000$$

11. Here, $a = 100$, $d = 5$ and $n = 30$.

$\therefore$ Amount to be paid in 30th instalment $= a_{30} = a + 29d = 100 + 29\times 5 = 245$

12. Suppose the carpenter takes n days to make 192 window frames. the numbers of window frames made by the carpenter on various days form an AP with first term $a = 5$ and common difference $d = 2$.

$$192 = \frac{n}{2}\left\{2\times 5 + (n-1)\times 2\right\}$$

$\Rightarrow \quad 182 = 5n + n^2 - n \Rightarrow n^2 + 4n - 192 = 0 \Rightarrow (n+16)(n-12) = 0 \Rightarrow n = 12$

13. The sum of the interior angles of an n sided polygon is $(2n-4)\times 90° = 180°n - 360°$. Putting $n - 3, 4, 5, 6, \ldots,$ we obtain: $180°, 360°, 540°, 720° \ldots\ldots$

Clearly, it is an A.P. with common difference 180°.

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The sum of the terms equidistant from the beginning and end in an A.P. is always same and is equal to the sum of .... and .... terms.
2. The minimum value of $4^x + 4^{1-x}$, $x \in R$, is ....
3. If the first, second and last terms of an A.P. are a, b and $2a$ respectively, then the sum of its terms is ....
4. The number of terms in an A.P. whose first term is 10, last term is 50 and the sum of all terms is 300, is ....
5. The arithmetic mean of first n natural numbers is ....

6. The sum of first n odd natural numbers is ....
7. The sum of first n even natural numbers is ....
8. If n is even, then the sum of first n terms of the series $1 - 2 + 3 - 4 + 5 - 6 + \ldots$, is............................ .
9. If twice the 11th term of an A.P is equal to 7 times of its 21st terms, then the value of 25th term is . ...
10. If the sums of n terms of two arithmetic progressions are the ratio $(2n+3):(6n+5)$, then the ratio of their 13th terms is ....
11. The sum of n arithmetic means between a and b is ....
12. If the sum of n arithmetic means between 9 and 51 is 270, then the value of n is............................ .
13. The sum of 4 arithmetic means between 3 and 23 is ....
14. If $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the A.M. of a and b, then $n =$ ....

ANSWERS

1. First, Last	2. 4	3. $\dfrac{3ab}{2(b-a)}$	4. 10	5. $\dfrac{n+1}{2}$
6. n^2	7. $n(n+1)$	8. $-\dfrac{n}{2}$	9. 0	10. 53 : 155
11. $\dfrac{n}{2}(a+b)$	12. 9	13. 52	14. 0	

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the common difference of an A.P. whose nth term is $xn + y$.
2. Write the common difference of an A.P. the sum of whose first n terms is $\dfrac{P}{2}n^2 + Qn$.
3. If the sum of n terms of an AP is $2n^2 + 3n$, then write its nth term.
4. If $\log 2$, $\log(2^x - 1)$ and $\log(2^x + 3)$ are in A.P., write the value of x.
5. If the sums of n terms of two arithmetic progressions are in the ratio $2n+5 : 3n+4$, then write the ratio of their mth terms.
6. Write the sum of first n odd natural numbers.
7. Write the sum of first n even natural numbers.
8. Write the value of n for which nth terms of the A.P.s 3, 10, 17, ... and 63, 65, 67,.... are equal.
9. If $\dfrac{3+5+7+\ldots+\text{upto } n \text{ terms}}{5+8+11+\ldots \text{upto } 10 \text{ terms}} = 7$, then find the value of n.
10. If mth term of an A.P. is n and nth term is m, then write its pth term.
11. If the sums of n terms of two A.P.'s are in the ratio $(3n+2):(2n+3)$, find the ratio of their 12th terms.

ANSWERS

1. x
2. P
3. $4n+1$
4. $\log_2 5$
5. $(4m+3):(6m+1)$
6. n^2
7. $n(n+1)$
8. 13
9. 35
10. $m+n-p$
11. $71:49$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If 7th and 13th terms of an A.P. be 34 and 64 respectively, then its 18th term is
   (a) 87 (b) 88 (c) 89 (d) 90
2. If the sum of p terms of an A.P. is q and the sum of q terms is p, then the sum of $p+q$ terms will be
   (a) 0 (b) $p-q$ (c) $p+q$ (d) $-(p+q)$
3. If the sum of n terms of an A.P. be $3n^2-n$ and its common difference is 6, then its first term is
   (a) 2 (b) 3 (c) 1 (d) 4
4. Sum of all two digit numbers which when divided by 4 yield unity as remainder is
   (a) 1200 (b) 1210 (c) 1250 (d) none of these.
5. In A.M.'s are introduced between 3 and 17 such that the ratio of the last mean to the first mean is 3 : 1, then the value of n is
   (a) 6 (b) 8 (c) 4 (d) none of these.
6. If S_n denotes the sum of first n terms of an A.P. $<a_n>$ such that $\frac{S_m}{S_n}=\frac{m^2}{n^2}$, then $\frac{a_m}{a_n}=$
   (a) $\frac{2m+1}{2n+1}$ (b) $\frac{2m-1}{2n-1}$ (c) $\frac{m-1}{n-1}$ (d) $\frac{m+1}{n+1}$.
7. The first and last terms of an A.P. are 1 and 11. If the sum of its terms is 36, then the number of terms will be
   (a) 5 (b) 6 (c) 7 (d) 8
8. If the sum of n terms of an A.P., is $3n^2+5n$ then which of its terms is 164?
   (a) 26th (b) 27th (c) 28th (d) none of these.
9. If the sum of n terms of an A.P. is $2n^2+5n$, then its nth term is
   (a) $4n-3$ (b) $3n-4$ (c) $4n+3$ (d) $3n+4$
10. If $a_1, a_2, a_3, \ldots, a_n$ are in A.P. with common difference d, then the sum of the series $\sin d\,[\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \ldots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n]$ is
    (a) $\sec a_1 - \sec a_n$ (b) $\operatorname{cosec} a_1 - \operatorname{cosec} a_n$
    (c) $\cot a_1 - \cot a_n$ (d) $\tan a_1 - \tan a_n$
11. In the arithmetic progression whose common difference is non-zero, the sum of first $3n$ terms is equal to the sum of next n terms. Then the ratio of the sum of the first $2n$ terms to the next $2n$ terms is
    (a) 1/5 (b) 2/3 (c) 3/4 (d) none of these

12. If $a_1, a_2, a_3, \ldots, a_n$ are in A.P. with common difference d, then the sum of the series

$$\sin d\,[\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \ldots + \sec a_{n-1} \sec a_n], \text{ is}$$

(a) $\sec a_1 - \sec a_n$ (b) $\operatorname{cosec} a_1 - \operatorname{cosec} a_n$
(c) $\cot a_1 - \cot a_n$ (d) $\tan a_n - \tan a_1$

13. If four numbers in A.P. are such that their sum is 50 and the greatest number is 4 times the least, then the numbers are

(a) 5, 10, 15, 20 (b) 4, 10, 16, 22 (c) 3, 7, 11, 15 (d) none of these

14. If n arithmetic means are inserted between 1 and 31 such that the ratio of the first mean and nth mean is 3 : 29, then the value of n is

(a) 10 (b) 12 (c) 13 (d) 14

15. Let S_n denote the sum of n terms of an A.P. whose first term is a. If the common difference d is given by $d = S_n - k\,S_{n-1} + S_{n-2}$, then $k =$

(a) 1 (b) 2 (c) 3 (d) none of these

16. The first and last term of an A. P. are a and l respectively. If S is the sum of all the terms of the A.P. and the common difference is given by $\dfrac{l^2 - a^2}{k - (l + a)}$, then $k =$

(a) S (b) $2S$ (c) $3S$ (d) none of these

17. If the sum of first n even natural numbers is equal to k times the sum of first n odd natural numbers, then $k =$

(a) $\dfrac{1}{n}$ (b) $\dfrac{n-1}{n}$ (c) $\dfrac{n+1}{2n}$ (d) $\dfrac{n+1}{n}$

18. If the first, second and last term of an A.P are a, b and $2a$ respectively, then its sum is

(a) $\dfrac{ab}{2(b-a)}$ (b) $\dfrac{ab}{b-a}$ (c) $\dfrac{3ab}{2(b-a)}$ (d) none of these

19. If, S_1 is the sum of an arithmetic progression of 'n' odd number of terms and S_2 the sum of the terms of the series in odd places, then $\dfrac{S_1}{S_2} =$

(a) $\dfrac{2n}{n+1}$ (b) $\dfrac{n}{n+1}$ (c) $\dfrac{n+1}{2n}$ (d) $\dfrac{n+1}{n}$

20. If in an A.P., $S_n = n^2 p$ and $S_m = m^2 p$, where S_r denotes the sum of r terms of the A.P., then S_p is equal to

(a) $\dfrac{1}{2}p^3$ (b) $mn\,p$ (c) p^3 (d) $(m+n)\,p^2$

21. If in an A.P., the pth term is q and $(p+q)^{\text{th}}$ term is zero, then the q^{th} term is

(a) $-p$ (b) p (c) $p+q$ (d) $p-q$

[NCERT EXEMPLAR]

22. The 10th common term between the A.P.s 3, 7, 11, 15, … and 1, 6, 11, 16, … is

(a) 191 (b) 193 (c) 211 (d) none of these

[NCERT EXEMPLAR]

23. If in an A.P. $S_n = n^2 q$ and $S_m = m^2 q$, where S_r denotes the sum of r terms of the A.P., then S_q equals

(a) $\dfrac{q^3}{2}$ (b) mnq (c) q^3 (d) $(m^2 + n^2)\,q$

[NCERT EXEMPLAR]

24. Let S_n denote the sum of first n terms of an A.P. If $S_{2n} = 3\, S_n$, then $S_{3n} : S_n$ is equal to
(a) 4 (b) 6 (c) 8 (d) 10
[NCERT EXEMPLAR]

25. If the sum of n terms of an A.P. is given by $S_n = 3n + 2n^2$, then the common difference of the A.P. is
(a) 3 (b) 2 (c) 6 (d) 4
[NCERT EXEMPLAR]

26. If 9 times the 9th term of an A.P. is equal to 13 times the 13th term, then the 22nd term of the A.P. is
(a) 0 (b) 22 (c) 220 (d) 198
[NCERT EXEMPLAR]

ANSWERS

1. (c)	2. (d)	3. (a)	4. (b)	5. (a)	6. (b)	7. (b)	8. (b)
9. (c)	10. (c)	11. (a)	12. (d)	13. (a)	14. (d)	15. (b)	16. (b)
17. (d)	18. (c)	19. (a)	20. (c)	21. (b)	22. (a)	23. (c)	24. (b)
25. (d)	26. (a)						

ACTIVITY

OBJECTIVE To demonstrate the concept of arithmetic progression and its sum.

MATERIALS REQUIRED Plastic strips, chart papers, thermocol sheets, adhesive etc.

STEPS OF CONSTRUCTION

Step I Take a thermocol sheet in the shape of the rectangle $ABCD$.

Step II Take some plastic strips each of equal fixed length denoted by a and some plastic strips, each of equal fixed length denoted by b.

Step III Cover the thermocol sheet by a chart paper.

Step IV On the chart paper arrange and paste both types of strips so as to get terms $a, a+b, a+2b, \ldots, a+9b$ placed at unit distance apart and arrange along the rectangle as shown in the figure. 18.1. The last strip ends in F along BC, extend F to C by a fixed length a so as to cut rectangle $ABCD$.

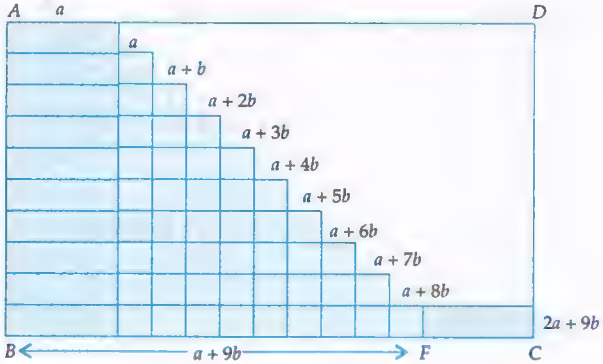

Fig. 18.1

STEPS OF DEMONSTRATION

Step I Length of the first strip $= a$

Length of the second strip $= a + b$

Length of the third strip $= a + 2b$

$\vdots \qquad \vdots$

Length of the 10^{th} strip $= a + 9b$

Clearly, lengths of the strips from an A.P. $a, a+b, a+2b, \ldots, a+9b$.

Step II The sum of the A.P. in step I is given by

$$S = a + (a+b) + (a+2b) + \ldots + (a+9b)$$

$$\Rightarrow \quad S = 10a + (b + 2b + 3b + \ldots + 9b)$$

$$\Rightarrow \quad S = 10a + 45b$$

$$\Rightarrow \quad S = \frac{10}{2}(2a+9b)$$

$$\Rightarrow \quad S = \frac{1}{2}\{10 \times (2a+9b\}$$

$\Rightarrow \quad S = \frac{1}{2}\{$ Area of the rectangle $ABCD$ whose length and breadth are $BC = 2a + 9b$ and 10 units respectively$\}$

OBSERVATIONS (i) The n^{th} term of the A.P. $a, a+b, a+2b, a+3b, \ldots$ is $a + (n-1)b$.

(ii) The sum of the arithmetic progression $a, a+b, a+2b, \ldots, a+(n-1)b$ is given by

$$S = \frac{10}{2}\{2a + (n-1)b\} = \frac{1}{2}n[2a + (n-1)b]$$

= Half of the area of the rectangle whose length is $2a + (n-1)b$ and breadth is n units.

SUMMARY

1. A sequence is a function whose domain is the set N of all natural numbers or some subsets of the type $\{1, 2, 3, \ldots, n\}$.

   A sequence containing a finite number of terms is called a finite sequence.

   A sequence is called an infinite sequence if it is not a finite sequence.

2. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a sequence, then the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called a series.

   A series is called a finite series if it has got finite number of terms, otherwise, it is called an infinite series.

3. Those sequences whose terms follow certain patterns are called progressions.

4. A sequence is called an arithmetic progression if the difference of a term and the previous term is always same, i.e. $a_{n+1} - a_n$ = constant $(= d)$ for all $n \in N$

   The constant difference 'd' is called the common difference.

5. A sequence is an arithmetic progression if and only if its nth terms is a linear expression in n and in such a case the common difference is equal to the coefficient of n.
6. If a is the first term and d is the common difference of an A.P., then its nth term is given by
$$a_n = a + (n-1)\, d$$
7. If an A.P. consists of m terms, then nth term from the end is equal to $(m-n+1)^{th}$ term from the beginning.
8. The following ways of selecting terms of an A.P. are generally very convenient:

Number of terms	*Terms*	*Common difference*
3	$a-d, a, a+d$	d
4	$a-3d, a-d, a+d, a+3d$	$2d$
5	$a-2d, a-d, a, a+d, a+2d$	d
6	$a-5d, a-3d, a-d, a+d, a+3d, a+5d$	$2d$

9. The sum S_n of n terms of an A.P. with first term 'a' and common difference 'd' is given by
$$S_n = \frac{n}{2}\{2a+(n-1)\, d\} \text{ or, } S_n = \frac{n}{2}(a+l), \text{ where } l = \text{last term} = a+(n-1)\, d.$$
10. If the sum S_n of n terms of a sequence is given, then n^{th} term a_n of the sequence can be determined by using the formula $a_n = S_n - S_{n-1}$
11. A sequence is an A.P. iff the sum of its n terms is of the form $An^2 + Bn$ i.e. a quadratic expression in n and in such a case the common difference is twice the coefficient of n^2.
12. If the ratio of the sums of n terms of two A.P.'s is given, then the ratio of their n^{th} terms is obtained by replacing n by $(2n-1)$ in the given ratio
13. Three numbers a, b, c are in A.P. iff $2b = a+c$. In such a case b is called the arithmetic mean of a and c.
14. The arithmetic mean of a and b is $\frac{a+b}{2}$.
15. If n numbers $A_1, A_2, \ldots, A_n$ are inserted between two given numbers a and b such that $a, A_1, A_2, \ldots, A_n, b$ is an arithmetic progression, then $A_1, A_2, \ldots, A_n$ are known as n arithmetic means between a and b and the common difference of the A.P. is $d = \frac{b-a}{n+1}$.

   Also, $A_1 + A_2 + \ldots + A_n = n\left(\frac{a+b}{2}\right)$.
16. In an A.P. the sum of the terms equidistant from the beginning and the end is always same and is equal to the sum of first and last term.

CHAPTER 19

GEOMETRIC PROGRESSIONS

19.1 GEOMETRIC PROGRESSION

A sequence of non-zero numbers is called a geometric progression (abbreviated as G.P.) if the ratio of a term and the term preceding to it is always a constant quantity.

The constant ratio is called the common ratio of the G.P.

In other words, a sequence, $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called a geometric progression if $\frac{a_{n+1}}{a_n}$ = constant for all $n \in N$.

ILLUSTRATION 1 *The sequence 4, 12, 36, 108, ... is a G.P., because $\frac{12}{4} = \frac{36}{12} = \frac{108}{36} = \ldots = 3$, which is constant.*

Clearly, this sequence is a G.P. with first term 4 and common ratio 3.

ILLUSTRATION 2 *The sequence $\frac{1}{3}, -\frac{1}{2}, \frac{3}{4}, -\frac{9}{8}, \ldots$ is a G.P. with first term $\frac{1}{3}$ and common ratio equal to $\left(-\frac{1}{2}\right) \div \left(\frac{1}{3}\right) = -\frac{3}{2}$.*

ILLUSTRATION 3 *Show that the sequence given by $a_n = 3(2^n)$, for all $n \in N$, is a G.P. Also, find its common ratio.*

SOLUTION We have, $a_n = 3(2^n)$. Therefore, $a_{n+1} = 3(2^{n+1})$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{3(2^{n+1})}{3(2^n)} = 2, \text{ which is constant for all } n \in N.$$

So, the given sequence is a G.P. with common ratio 2.

GEOMETRIC SERIES If $a_1, a_2, a_3, \ldots a_n, \ldots$ is a G.P., then the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ *is called a geometric series.*

Note that the geometric series is finite or infinite according as the corresponding G.P. consists of finite or infinite number of terms.

19.2 GENERAL TERM OF A G.P.

THEOREM *Prove that the nth term of a G.P. with first term a and common ratio r is given by $a_n = ar^{n-1}$.*

PROOF Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be the given G.P. Then, $a_1 = a \Rightarrow a_1 = ar^{1-1}$

Since $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a G.P. with common ratio r. Therefore,

$$\frac{a_2}{a_1} = r \Rightarrow a_2 = a_1 r \Rightarrow a_2 = ar \Rightarrow a_2 = ar^{2-1}$$

$$\frac{a_3}{a_2} = r \Rightarrow a_3 = a_2 r \Rightarrow a_3 = (ar) r \Rightarrow a_3 = ar^2 \Rightarrow a_3 = ar^{3-1}$$

$$\frac{a_4}{a_3} = r \Rightarrow a_4 = a_3 r \Rightarrow a_4 = (ar^2)\, r \Rightarrow a_4 = ar^3 \Rightarrow a_4 = ar^{4-1}$$

Continuing in this manner, we get $a_n = ar^{n-1}$ Q.E.D.

NOTE *It follows from the above discussion that if a is the first term and r is the common ratio of a G.P., then the G.P. can be written as* $a, ar, ar^2, \ldots, ar^{n-1}$ *or,* $a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}, \ldots$ *according as it is finite or infinite.*

19.2.1 *n*th TERM FROM THE END OF A FINITE G.P.

THEOREM 1 *Prove that the nth term from the end of a finite G.P. consisting of m terms is* ar^{m-n}, *where a is the first term and r is the common ratio of the G.P.*

PROOF Since the G.P. consists of m terms.

$\therefore$ nth term from the end $=$ $(m - n + 1)$th term from the beginning $= ar^{m-n}$

THEOREM 2 *Prove that the nth term from the end of a G.P. with last term l and common ratio r is given by* $a_n = l\left(\frac{1}{r}\right)^{n-1}$.

PROOF Clearly, when we look at the terms of a G.P. from the last term and move towards the beginning we find that the progression is a G.P. with common ratio $1/r$.

So, nth term from the end $= l\left(\frac{1}{r}\right)^{n-1}$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I FINDING THE INDICATED TERM OF A G.P. WHEN ITS FIRST TERM AND THE COMMON RATIO ARE GIVEN

EXAMPLE 1 *Find the 9th term and the general term of the progression:* $\frac{1}{4}, -\frac{1}{2}, 1, -2, \ldots$

SOLUTION The given progression is clearly a G.P. with first term $a = 1/4$ and common ratio $r = -2$.

$\therefore$ 9th term $= a_9 = ar^{(9-1)} = ar^8 = \frac{1}{4}(-2)^8 = 64$

and, General term $= a_n = ar^{(n-1)} = \frac{1}{4}(-2)^{n-1} = (-1)^{n-1}\, 2^{n-3}$

EXAMPLE 2 *Find the 5th term of the progression* $1, \frac{(\sqrt{2}-1)}{2\sqrt{3}}, \left(\frac{3-2\sqrt{2}}{12}\right), \left(\frac{5\sqrt{2}-7}{24\sqrt{3}}\right), \ldots\ldots$

SOLUTION Clearly, the given progression is a G.P. with first term $a = 1$ and common ratio $\frac{\sqrt{2}-1}{2\sqrt{3}}$. So, its 5th term is given by

$$a_5 = ar^{(5-1)} = 1 \times \left(\frac{\sqrt{2}-1}{2\sqrt{3}}\right)^4 = \frac{(\sqrt{2}-1)^4}{144}$$

EXAMPLE 3 *Find 4th term from the end of the G.P.* 3, 6, 12, 24, ..., 3072.

SOLUTION Clearly, the given progression is a G.P. with common ratio $r = 2$.

$\therefore$ 4th term from the end $= l\left(\frac{1}{r}\right)^{4-1} = (3072)\left(\frac{1}{2}\right)^{4-1} = 384$

Type II ON FINDING THE POSITION OF A GIVEN TERM IN A GIVEN G.P.

EXAMPLE 4 *Which term of the G.P.* $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$ *is* $\frac{1}{128}$?

SOLUTION Clearly, the given progression is a G.P. with first term $a = 2$ and common ratio $r = 1/2$. Let the nth term be $\frac{1}{128}$. Then,

$$a_n = \frac{1}{128} \Rightarrow ar^{n-1} = \frac{1}{128} \Rightarrow 2\left(\frac{1}{2}\right)^{n-1} = \frac{1}{128} \Rightarrow \left(\frac{1}{2}\right)^{n-2} = \left(\frac{1}{2}\right)^7 \Rightarrow n - 2 = 7 \Rightarrow n = 9$$

Thus, 9th term of the given G.P. is $\frac{1}{128}$.

EXAMPLE 5 *Which term of the G.P. 5, 10, 20, 40, ... is 5120 ?*

SOLUTION Clearly, the given G.P. has first term $a = 5$ and the common ratio $r = 2$. Let the nth term be 5120. Then,

$$a_n = 5120$$

$$\Rightarrow ar^{n-1} = 5120 \Rightarrow 5(2^{n-1}) = 5120 \Rightarrow 2^{n-1} = 1024 \Rightarrow 2^{n-1} = 2^{10} \Rightarrow n - 1 = 10 \Rightarrow n = 11$$

Thus, 11th term of the given G.P. is 5120.

EXAMPLE 6 *Which term of the G.P. 2, 8, 32, ... is 131072 ?* [NCERT]

SOLUTION Here, $a = 2$ and $r = 4$. Let the nth term be 131072. Then,

$$a_n = 131072$$

$$\Rightarrow ar^{n-1} = 131072 \Rightarrow 2 \times 4^{n-1} = 131072 \Rightarrow 4^{n-1} = 65536 \Rightarrow 4^{n-1} = 4^8 \Rightarrow n - 1 = 8 \Rightarrow n = 9$$

Hence, 131072 is the 9th term of the given G.P.

Type III PROBLEMS BASED ON THE DEFINITION OF A G.P. AND THE FORMULA $a_n = ar^{n-1}$

EXAMPLE 7 *The fourth, seventh and the last term of a G.P. are 10, 80 and 2560 respectively. Find the first term and the number of terms in the G.P.*

SOLUTION Let a be the first term and r be the common ratio of the given G.P. Then,

$$a_4 = 10,\ a_7 = 80 \Rightarrow ar^3 = 10 \text{ and } ar^6 = 80 \Rightarrow \frac{ar^6}{ar^3} = \frac{80}{10} \Rightarrow r^3 = 8 \Rightarrow r = 2.$$

Putting $r = 2$ in $ar^3 = 10$, we get: $a(2)^3 = 10 \Rightarrow a = \frac{10}{8} = \frac{5}{4}$.

Let there be n terms in the given G.P. Then,

$$a_n = 2560$$

$$\Rightarrow ar^{n-1} = 2560 \Rightarrow \frac{5}{4}(2^{n-1}) = 2560 \Rightarrow 2^{n-4} = 256 \Rightarrow 2^{n-4} = 2^8 \Rightarrow n - 4 = 8 \Rightarrow n = 12.$$

EXAMPLE 8 *The first term of a G.P. is 1. The sum of the third and fifth terms is 90. Find the common ratio of the G.P.* [NCERT]

SOLUTION Let r be the common ratio of the G.P. It is given that the first term $a = 1$.

Now, $a_3 + a_5 = 90$

$\Rightarrow$ $ar^2 + ar^4 = 90 \Rightarrow r^2 + r^4 = 90 \Rightarrow r^4 + r^2 - 90 = 0$

$\Rightarrow$ $r^4 + 10r^2 - 9r^2 - 90 = 0 \Rightarrow (r^2 + 10)(r^2 - 9) = 0 \Rightarrow r^2 - 9 = 0 \Rightarrow r = \pm 3.$

Hence, the common ratio of the given G.P. is 3 or -3.

EXAMPLE 9 *If the 4th and 9th terms of a G.P. be 54 and 13122 respectively, find the G.P.*

SOLUTION Let a be the first term and r the common ratio of the given G.P. Then,

$$a_4 = 54 \quad \text{and} \quad a_9 = 13122$$

$$\Rightarrow \quad ar^3 = 54 \text{ and } ar^8 = 13122 \Rightarrow \frac{ar^8}{ar^3} = \frac{13122}{54} \Rightarrow r^5 = 243 \Rightarrow r^5 = 3^5 \Rightarrow r = 3$$

Putting $r = 3$ in $ar^3 = 54$, we get: $a(3)^3 = 54 \Rightarrow a = 2$

Thus, the given G.P. is $a, ar, ar^2, ar,^3 \ldots$ i.e. $2, 6, 18, 54, \ldots$

EXAMPLE 10 *Find a G.P. for which the sum of first two terms is –4 and the fifth term is 4 times the third term.* [NCERT]

SOLUTION Let a be the first term and r be the common ratio of the given G.P. It is given that The sum of first two terms$= -4$. $\Rightarrow a_1 + a_2 = -4 \Rightarrow a + ar = -4$ ...(i)

It is also given that

$$a_5 = 4\,a_3 \Rightarrow ar^4 = 4\,ar^2 \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$$

Putting $r = 2$ and -2 respectively in (i), we get $a = -\frac{4}{3}$ and $a = 4$ respectively.

Thus, the required G.P. is $-\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}, \ldots$ or $4, -8, 16, -32, \ldots$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 11 *The third term of a G.P. is 4. Find the product of its first five terms.*

SOLUTION Let a be the first term and r the common ratio. Then,

$$a_3 = 4 \Rightarrow ar^2 = 4 \qquad \text{...(i)}$$

$\therefore$ Product of first five terms $= a_1\, a_2\, a_3\, a_4\, a_5 = a\,(ar)\,(ar^2)\,(ar^3)\,(ar^4)$

$$= a^5\, r^{10} = (ar^2)^5 = 4^5 \qquad \text{[Using (i)]}$$

EXAMPLE 12 *If the pth, qth and rth terms of a G.P. are a, b, c respectively, prove that:* $a^{(q-r)} \cdot b^{(r-p)} \cdot c^{(p-q)} = 1.$ [NCERT]

SOLUTION Let A be the first term and R be the common ratio of the given G.P. Then,

$$a = pth \text{ term} = AR^{(p-1)}, \quad b = qth \text{ term} = AR^{(q-1)}, \text{ and } \quad c = rth \text{ term} = AR^{(r-1)}$$

Substituting the values of a, b and c, we get

$$a^{(q-r)} \cdot b^{(r-p)} \cdot c^{(p-q)}$$

$$= \left\{AR^{(p-1)}\right\}^{(q-r)} \cdot \left\{AR^{(q-1)}\right\}^{(r-p)} \cdot \left\{AR^{(r-1)}\right\}^{(p-q)}$$

$$= A^{(q-r)}\, R^{(p-1)(q-r)} \cdot A^{(r-p)}\, R^{(q-1)(r-p)} \cdot A^{(p-q)}\, R^{(r-1)(p-q)}$$

$$= A^{(q-r+r-p+p-q)}\, R^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$$

$$= A^0\, R^{p(q-r)+q(r-p)+r(p-q)-(q-r)-(r-p)-(p-q)} = A^0 R^0 = 1.$$

EXAMPLE 13 *If a, b, c are respectively the p^{th}, q^{th} and r^{th} terms of a G.P., show that* $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0.$

SOLUTION Let A be the first term and R the common ratio of the given G.P. Then,

$$a = p\text{th term} \Rightarrow a = AR^{p-1} \Rightarrow \log a = \log A + (p-1)\log R \qquad \text{...(i)}$$

$b = q$th term $\Rightarrow b = AR^{q-1} \Rightarrow \log b = \log A + (q-1)\log R$ ...(ii)

$c = r$th term $\Rightarrow c = AR^{r-1} \Rightarrow \log c = \log A + (r-1)\log R$ ...(iii)

Substituting the values of $\log a$, $\log b$ and $\log c$, we get

$$(q-r)\log a + (r-p)\log b + (p-q)\log c$$

$$= (q-r)\left\{\log A + (p-1)\log R\right\} + (r-p)\left\{\log A + (q-1)\log R\right\} + (p-q)\left\{\log A + (r-1)\log R\right\}$$

$$= \log A\left\{(q-r) + (r-p) + (p-q)\right\} + \log R\left\{(p-1)(q-r) + (q-1)(r-p) + (r-1)(p-q)\right\}$$

$$= (\log A)\,0 + \left\{p(q-r) + q(r-p) + r(p-q) - (q-r) - (r-p) - (p-q)\right\}\log R$$

$$= (\log A)\,0 + (\log R)\,0 = 0.$$

EXAMPLE 14 *Find four numbers forming a geometric progression in which the third term is greater than the first terms by 9, and second term is greater than the 4th by 18.* [NCERT]

SOLUTION Let the four numbers in G.P. be a, ar, ar^2 and ar^3. It is given that

$$ar^2 = a + 9 \quad\text{and}\quad ar = ar^3 + 18$$

$$\Rightarrow a(r^2-1) = 9 \text{ and } ar(1-r^2) = 18 \Rightarrow \frac{ar(1-r^2)}{a(r^2-1)} = \frac{18}{9} \Rightarrow -r = 2 \Rightarrow r = -2$$

Putting $r = -2$ in $a(r^2-1) = 9$, we get: $a(4-1) = 9 \Rightarrow a = 3$

Hence, the numbers are: $3, 3(-2), 3(-2)^2, 3(-2)^3$ or, $3, -6, 12, -24$.

EXAMPLE 15 *The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and nth hour?* [NCERT]

SOLUTION Clearly, number of bacteria at the end of different hours forms a G.P. with first term $a = 30$ and common ratio $r = 2$.

Number of bacteria present at the end of 2nd hour

= (Third term of the G.P. with first term $a = 30$ and common ratio $r = 2$) $= ar^2 = 30 \times 2^2 = 120$

Number of bacteria present at the end of 4th hour

= (5th term of the G.P. with first term $a = 30$ and common ratio $r = 2$) $= ar^4 = 30 \times 2^4 = 480$

Number of bacteria present at the end of nth hour

$= \{(n+1)^{\text{th}}$ term of the G.P. with first term $a = 30$ and common ratio $r = 2\} = ar^n = 30 \times 2^n$

EXAMPLE 16 *What will ₹ 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?* [NCERT]

SOLUTION We have, P = Principal = ₹ 500, R = Rate of interest = 10%

$\therefore$ Amount at the end of one year $= ₹\left(P + \frac{PR}{100}\right) = ₹\,P\left(1 + \frac{R}{100}\right)$

Amount at the end of second year $= ₹\left\{P\left(1 + \frac{R}{100}\right) + P\left(1 + \frac{R}{100}\right)\frac{R}{100}\right\}$

$$= ₹\,P\left(1 + \frac{R}{100}\right)\left(1 + \frac{R}{100}\right) = ₹\,P\left(1 + \frac{R}{100}\right)^2$$

Amount at the end of the third year $= ₹\left\{P\left(1+\frac{R}{100}\right)^2 + P\left(1+\frac{R}{100}\right)^2 \cdot \frac{R}{100}\right\}$

$$= ₹\, P\left(1+\frac{R}{100}\right)^3$$

and so on.

We find that amounts at the end of various year form a G.P. with first term and common ratio $\left(1+\frac{R}{100}\right)$.

$\therefore$ Amount at the end of 10th year $= (11^{\text{th}}$ term of the G.P.$) = ₹\, P\left(1+\frac{R}{100}\right)^{10}$

$$= ₹\, 500\left(1+\frac{10}{100}\right)^{10} = ₹\, 500 \times \left(\frac{11}{10}\right)^{10} = ₹\, 500 \times (1.1)^{10}$$

EXAMPLE 17 *A manufacturer reckons that the value of a machine, which costs him ₹ 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.* [NCERT]

SOLUTION We have,

Initial value of the machine $= V_0 = ₹\, 15625$ and, R = Rate of depreciation = 20%

$\therefore$ Depreciated value at the end of first year $= V_0 - \frac{V_0 R}{100} = V_0\left(1-\frac{R}{100}\right)$

Depreciated value at the end of second year $= V_1 - \frac{V_1 R}{100} = V_1\left(1-\frac{R}{100}\right) = V_0\left(1-\frac{R}{100}\right)^2$

ans so on.

Clearly, depreciated values at the end of different years form a G.P. with first term V_0 and common ratio $\left(1-\frac{R}{100}\right)$.

$\therefore$ Depriciated value at the end of 5 years

$= $ 6th term of the G.P. with first term V_0 (= ₹ 15625) and common ratio $r = \left(1-\frac{R}{100}\right)$

$$= V_0\left(1-\frac{R}{100}\right)^5 = ₹\left\{15625\left(1-\frac{20}{100}\right)^5\right\} = ₹\left\{15625 \times \left(\frac{4}{5}\right)^5\right\} = ₹\, 5120$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 18 *In a G.P. of positive terms, if any term is equal to the sum of next two terms, find the common ratio of the G.P.*

SOLUTION Let a be the first term and r be the common ratio of the G.P. By hypothesis

$$a_n = a_{n+1} + a_{n+2}$$

$$\Rightarrow \quad ar^{n-1} = ar^n + ar^{n+1} \Rightarrow 1 = r + r^2 \Rightarrow r^2 + r - 1 = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

But, $r > 0$. Therefore, $r = \frac{-1+\sqrt{5}}{2} = 2\left(\frac{\sqrt{5}-1}{4}\right) = 2 \sin 18°$

EXAMPLE 19 *In a finite G.P. the product of the terms equidistant from the beginning and the end is always same and equal to the product of first and last term.*

SOLUTION Let $a_1, a_2, a_3, \dots, a_{n-1}, a_n$ be a finite G.P. with common ratio r.

Now, a_k = kth term from the beginning $= a_1 r^{k-1}$

and, a_{n-k+1} = kth term from the end $= a_n \left(\frac{1}{r}\right)^{k-1}$, where $1 < k < n$

$$\therefore \quad a_k\, a_{n-k+1} = \left(a_1 r^{k-1}\right) a_n \left(\frac{1}{r}\right)^{k-1} = a_1 a_n \quad \text{for all } k \text{ satisfying } 1 < k < n.$$

Hence, the product of terms equidistant from the beginning and the end is always equal to the product of first and last term.

EXAMPLE 20 *If the first and the nth terms of a G.P. are a and b respectively and if P is the product of the first n terms, prove that $P^2 = (ab)^n$.* [NCERT]

SOLUTION Let r be the common ratio of the given G.P. Then,

$$b = n\text{th term} = ar^{n-1} \Rightarrow r^{n-1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n-1}}$$

Now,

$$P = \text{Product of the first } n \text{ terms} = a \cdot ar \cdot ar^2 \dots ar^{n-1} = a^n r^{1+2+3+\dots+(n-1)}$$

$$\Rightarrow \quad P = a^n r^{\frac{n(n-1)}{2}} \qquad \left[\because 1+2+3+\dots+(n-1) = \left(\frac{n-1}{2}\right)\left(1+(n-1)\right) = \frac{n(n-1)}{2}\right]$$

$$\Rightarrow \quad P = a^n \left\{\left(\frac{b}{a}\right)^{\frac{1}{n-1}}\right\}^{\frac{n(n-1)}{2}} = a^n \left(\frac{b}{a}\right)^{n/2} = a^{n/2} b^{n/2} = (ab)^{n/2}$$

$$\therefore \quad P^2 = \left\{(ab)^{n/2}\right\}^2 = (ab)^n$$

ALITER Let $a_1 = a, a_2, a_3 \dots\dots, a_{n-1}, a_n = b$ be the given G.P. the product of whose terms is P. Then,

$$P = a_1 a_2 a_3 \dots\dots a_{n-2}\, a_{n-1} = (a_1 a_n)(a_2 a_{n-1})(a_3 a_{n-2}) \dots\dots \left(a_{\frac{n}{2}}\, a_{\frac{n}{2}+1}\right)$$

$$\Rightarrow \quad P = \underbrace{(a\,b)\,(a\,b)\,(a\,b)\dots\dots(a\,b)}_{\frac{n}{2}-\text{times}} \qquad [\because a_k\, a_{n-k+1} = a_1 a_n \text{ for all } k = 1, 2, \dots\dots, n]$$

$$\Rightarrow \quad P = (ab)^{n/2} \Rightarrow P^2 = (ab)^n$$

EXAMPLE 21 *The $(m+n)$th and $(m-n)$th terms of a G.P. are p and q respectively. Show that the mth and nth terms are $\sqrt{pq}$ and $p\left(\frac{q}{p}\right)^{m/2n}$ respectively.*

SOLUTION Let a be the first term and r be the common ratio. Then,

$$a_{m+n} = p \quad \text{and} \quad a_{m-n} = q$$

$$\Rightarrow \quad ar^{m+n-1} = p \quad \text{and} \quad ar^{m-n-1} = q$$

$$\Rightarrow \quad \frac{ar^{m+n-1}}{ar^{m-n-1}} = \frac{p}{q} \Rightarrow r^{2n} = \frac{p}{q} \Rightarrow r = \left(\frac{p}{q}\right)^{1/2n} \Rightarrow \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}$$

Now, $a_m = ar^{m-1}$

$$\Rightarrow \quad a_m = ar^{(m+n-1)}\left(\frac{1}{r}\right)^n = a_{m+n}\left(\frac{1}{r}\right)^n \qquad [\because a_{m+n} = ar^{m+n-1}]$$

$$\Rightarrow \quad a_m = p\left(\frac{q}{p}\right)^{n/2n} \qquad \left[\because a_{m+n} = p \text{ and } \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}\right]$$

$$\Rightarrow \quad a_m = p\left(\frac{q}{p}\right)^{1/2} = \sqrt{pq}$$

and, $a_n = ar^{n-1}$

$$\Rightarrow \quad a_n = ar^{m+n-1}\left(\frac{1}{r}\right)^m = a_{m+n}\left(\frac{1}{r}\right)^m \qquad [\because a_{m+n} = ar^{m+n-1}]$$

$$\Rightarrow \quad a_n = p\left(\frac{q}{p}\right)^{m/2n} \qquad \left[\because a_{m+n} = p \text{ and } \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}\right]$$

EXAMPLE 22 *If pth, qth and rth terms of an A.P. as well as a G.P. are a, b and c respectively. Prove that* $a^{b-c}\, b^{c-a}\, c^{a-b} = 1.$ [NCERT EXEMPLAR]

SOLUTION Let A be the first term and d be the common difference of the A.P. It is given that a, b and c are p^{th}, q^{th} and r^{th} terms of the A.P. Therefore,

$$a = A + (p-1)\,d,\; b = A + (q-1)\,d,\; c = A + (r-1)\,d$$

$$\therefore \quad b - c = (q-r)\,d,\; c - a = (r-p)\,d \text{ and } a - b = (p-q)\,d.$$

Let α be the first term and R be the common ratio of the G.P. Then,

$$a = \alpha R^{p-1},\; b = \alpha R^{q-1}, \text{ and } c = \alpha R^{r-1}$$

$$a^{b-c}\, b^{c-a}\, c^{a-b} = \left\{\alpha R^{p-1}\right\}^{(q-r)d} \times \left\{\alpha R^{q-1}\right\}^{(r-p)d} \times \left\{\alpha R^{r-1}\right\}^{(p-q)d}$$

$$= \alpha^{(q-r)d + (r-p)d + (p-q)d}\, R^{(p-1)(q-r)d + (q-1)(r-p)d + (r-1)(p-q)d}$$

$$= \alpha^{(q-r+r-p+p-q)d}\, R^{\{p(q-r) + q(r-p) + r(p-q) - (q-r) - (r-p) - (p-q)\}d}$$

$$= \alpha^0 R^0 = 1.$$

EXAMPLE 23 *Find all sequences which are simultaneously A.P. and G.P.*

SOLUTION Let $a_1, a_2, a_3, \ldots, a_n \ldots$ be a sequence which is both an A.P. as well as a G.P. Let a_n, a_{n+1}, a_{n+2} be three consecutive terms of the A.P. Then,

$$2\,a_{n+1} = a_n + a_{n+2},\; n \in N \qquad \text{...(i)}$$

Let r be the common ratio of the sequence when it is considered a G.P. Then,

$$a_n = a_1 r^{n-1},\; a_{n+1} = a_1 r^n \text{ and } a_{n+2} = a_1 r^{n+1}$$

Putting these values in (i), we get

$$2\,a_1 r^n = a_1 r^{n-1} + a_1 r^{n+1} \Rightarrow 2r = 1 + r^2 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$$

Putting $r = 1$ in $a_1, a_2 = a_1 r, a_3 = a_1 r^2, a_4 = a_1 r^3, \ldots,$ we obtain

$$a_1,\; a_2 = a_1,\; a_3 = a_1,\; a_4 = a_1 \ldots, \text{ which is a constant squance.}$$

Hence, the constant sequence is the only sequence which is both an A.P. as well as G.P.

EXAMPLE 24 *Show that the products of the corresponding terms of the sequences $a, ar, ar^2, \dots, ar^{n-1}$ and $A, AR, AR^2, \dots, AR^{n-1}$ form a G.P., and find the common ratio.* [NCERT]

SOLUTION The sequence formed by the products of the corresponding terms of the given sequences is

$$aA, aA\,rR, aA\,r^2 R^2, \dots, aA\,r^{n-1} R^{n-1}$$

or, $$aA, aA\,(rR), aA\,(rR)^2, aA\,(rR)^3, \dots, aA\,(rR)^{n-1}$$

Clearly, the ratio of any term and preceding term in the above squance is same equal to rR. So, it is a G.P. with common ratio rR.

EXAMPLE 25 *If p^{th}, q^{th}, r^{th} and s^{th} terms of an A.P. are in G.P., then show that $(p-q), (q-r), (r-s)$ are also in G.P.* [NCERT]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Further, let a_p, a_q, a_r and a_s be its p^{th}, q^{th}, r^{th} and s^{th} terms respectively. Then,

$$a_p = a + (p-1)\,d,\ a_q = a + (q-1)\,d,\ a_r = a + (r-1)\,d \text{ and } a_s = a + (s-1)\,d.$$

$$\Rightarrow \quad a_p - a_q = (p-q)\,d,\ a_q - a_r = (q-r)\,d \text{ and } a_r - a_s = (r-s)\,d \quad \dots \text{(i)}$$

It is given that a_p, a_q, a_r and a_s are in G.P. Let A be the first term and R be the common ratio of the G.P. Then,

$$A = a_p,\ AR = a_q,\ AR^2 = a_r \text{ and } AR^3 = a_s.$$

$$\therefore \quad A - AR = a_p - a_q,\ AR - AR^2 = a_q - a_r \text{ and } AR^2 - AR^3 = a_r - a_s$$

$$\Rightarrow \quad A(1-R) = a_p - a_q,\ AR(1-R) = a_q - a_r \text{ and } AR^2(1-R) = a_r - a_s$$

$$\Rightarrow \quad (a_q - a_r)^2 = \left\{AR\,(1-R)\right\}^2 = \left\{A\,(1-R)\right\}\left\{AR^2(1-R)\right\} = (a_p - a_q)(a_r - a_s)$$

$$\Rightarrow \quad (q-r)^2 d^2 = \left\{(p-q)\,d\right\}\left\{(r-s)\,d\right\} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad (q-r)^2 = (p-q)(r-s) \Rightarrow p-q,\ q-r,\ r-s \text{ are in G.P.}$$

EXERCISE 19.1

BASIC

1. Show that each one of the following progressions is a G.P. Also, find the common ratio in each case:
   (i) $4, -2, 1, -1/2, \dots$
   (ii) $-2/3, -6, -54, \dots$
   (iii) $a, \frac{3a^2}{4}, \frac{9a^3}{16}, \dots$
   (iv) $1/2, 1/3, 2/9, 4/27, \dots$
2. Show that the sequence defined by $a_n = \frac{2}{3^n}, n \in N$ is a G.P.
3. Find:
   (i) the ninth term of the G.P. $1, 4, 16, 64, \dots$
   (ii) the 10th term of the G.P. $-\frac{3}{4}, \frac{1}{2}, -\frac{1}{3}, \frac{2}{9}, \dots$
   (iii) the 8th term of the G.P. $0.3, 0.06, 0.012, \dots$
   (iv) the 12th term of the G.P. $\frac{1}{a^3 x^3}, ax, a^5 x^5, \dots$

(v) nth term of the G.P. $\sqrt{3}, \frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}}, \ldots$

(vi) the 10th term of the G.P. $\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \ldots$

4. Find the 4th term from the end of the G.P. $\frac{2}{27}, \frac{2}{9}, \frac{2}{3}, \ldots, 162$.

5. Which term of the progression 0.004, 0.02, 0.1, ... is 12.5 ?

6. Which term of the G.P. :

(i) $\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{4\sqrt{2}}, \ldots$ is $\frac{1}{512\sqrt{2}}$?

(ii) $2, 2\sqrt{2}, 4, \ldots$ is 128 ? [NCERT]

(iii) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729? [NCERT]

(iv) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27} \ldots$ is $\frac{1}{19683}$? [NCERT]

7. Which term of the progression 18, -12, 8, ... is $\frac{512}{729}$?

8. Find the 4th term from the end of the G.P. $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \ldots, \frac{1}{4374}$.

BASED ON LOTS

9. The fourth term of a G.P. is 27 and the 7th term is 729, find the G.P.

10. The seventh term of a G.P. is 8 times the fourth term and 5th term is 48. Find the G.P.

11. If the G.P.'s 5, 10, 20, ... and 1280, 640, 320, ... have their nth terms equal, find the value of n.

12. If 5^{th}, 8^{th} and 11^{th} terms of a G.P. are p, q and s respectively, prove that $q^2 = ps$. [NCERT]

13. The 4th term of a G.P. is square of its second term, and the first term is -3. Find its 7^{th} term. [NCERT]

14. In a GP the 3^{rd} term is 24 and the 6^{th} term is 192. Find the 10^{th} term. [NCERT]

BASED ON HOTS

15. If a, b, c, d and p are different real numbers such that:
$(a^2 + b^2 + c^2)\, p^2 - 2\,(ab + bc + cd)\, p + (b^2 + c^2 + d^2) \leq 0$, then show that a, b, c and d are in G.P. [NCERT]

16. If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$ $(x \neq 0)$, then show that a, b, c and d are in G.P. [NCERT]

17. If the p^{th} and q^{th} terms of a G.P. are q and p respectively, show that $(p+q)^{th}$ term is $\left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$.

[NCERT EXEMPLAR]

ANSWERS

1. (i) $-\frac{1}{2}$ (ii) 9 (iii) $\frac{3a}{4}$ (iv) $\frac{2}{3}$

3. (i) 4^8 (ii) $\frac{1}{2}\left(\frac{2}{3}\right)^8$ (iii) $(0.3)(0.2)^7$ (iv) $(ax)^{41}$ (v) $\sqrt{3}\left(\frac{1}{3}\right)^{n-1}$ (vi) $\frac{1}{\sqrt{2}} \times \frac{1}{2^8}$

4. 6 5. 6 6. (i) 11^{th} (ii) 13^{th} (iii) 12^{th} (iv) 9^{th} 7. 9 8. $\frac{1}{162}$
9. 1, 3, 9, ... 10. 3, 6, 12, .. 11. 5 13. -2187 14. 3072

HINTS TO SELECTED PROBLEMS

6. (ii) Let n^{th} term of the G.P. $2, 2\sqrt{2}, 4, \dots$ be 128. Then,

$$2(\sqrt{2})^{n-1} = 128 \Rightarrow 2^{\frac{n+1}{2}} = 2^7 \Rightarrow \frac{n+1}{2} = 7 \Rightarrow n = 13$$

Thus, 13th term of the G.P. $2, 2\sqrt{2}, 4, \dots$ is 128.

(iii) Let n^{th} term of the G.P. $\sqrt{3}, 3, 3\sqrt{3}, \dots$ be 729. Then,

$$\sqrt{3} \times (\sqrt{3})^{n-1} = 729 \Rightarrow 3^{\frac{n}{2}} = 3^6 \Rightarrow \frac{n}{2} = 6 \Rightarrow n = 12$$

Hence, 12^{th} term of the given G.P. is 729.

(iv) Let nth term of the G.P. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$ be $\frac{1}{19683}$. Then,

$$\frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{1}{19683} \Rightarrow n^3 = 3^9 \Rightarrow n = 9$$

12. Let a be the first term and r the common ratio of the given G.P. It is given that

$p = 5^{th}$ term, $q = 8^{th}$ term, $s = 11^{th}$ term $\Rightarrow p = ar^4,\ q = ar^7,\ s = ar^{10}$

$\therefore \quad q^2 = a^2 r^{14}$ and $ps = a^2 r^{14} \Rightarrow q^2 = ps.$

13. Let the common ratio of the given G.P. be r. It is given that the fourth term is square of its second term.

$\therefore \quad (-3) r^3 = (-3r)^2 \Rightarrow -3r^3 = 9r^2 \Rightarrow r = -3$

Hence, 7^{th} term $= (-3) r^6 = -3(-3)^6 = -2187.$

14. Let the first term and common ratio of the given G.P. be a and r respectively.

It is given that 3^{rd} term $= 24$ and 6^{th} term $= 192$

$$\Rightarrow ar^2 = 24 \text{ and } ar^5 = 192 \Rightarrow \frac{ar^5}{ar^2} = \frac{192}{24} \Rightarrow r^3 = 8 \Rightarrow r = 2$$

Putting $r = 2$ in $ar^2 = 24$, we get $a = 6$.

$\therefore \quad 10^{th}$ term $= ar^9 = 6 \times 2^9 = 3072$

15. It is given that

$$(a^2 + b^2 + c^2) p^2 - 2(ab + bc + cd) p + (b^2 + c^2 + d^2) \le 0$$

$$\Rightarrow (a^2 p^2 - 2abp + b^2) + (b^2 p^2 - 2bcp + c^2) + (c^2 p^2 - 2cdp + d^2) \le 0$$

$$\Rightarrow (ap - b)^2 + (bp - c)^2 + (cp - d)^2 \le 0$$

$$\Rightarrow (ap - b)^2 + (bp - c)^2 + (cp - d)^2 = 0 \quad [\because (ap - b)^2 + (bp - c)^2 + (cp - d)^2 \text{ cannot be negative}]$$

$$\Rightarrow ap - b = 0,\ bp - c = 0,\ cp - d = 0$$

$$\Rightarrow \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = p \Rightarrow a, b, c, d \text{ are in G.P. with common ratio } p.$$

16. We have,

$$\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}$$

$$\Rightarrow \quad \frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}$$

$$\Rightarrow \quad \frac{(a+bx)+(a-bx)}{(a+bx)-(a-bx)}=\frac{(b+cx)+(b-cx)}{(b+cx)-(b-cx)} \qquad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \quad \frac{a}{bx}=\frac{b}{cx} \Rightarrow \frac{b}{a}=\frac{c}{b}$$

Similarly, $\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx} \Rightarrow \frac{c}{b}=\frac{d}{c}$

$$\therefore \quad \frac{b}{a}=\frac{c}{b}=\frac{d}{c} \Rightarrow a, b, c, d \text{ are in G.P.}$$

19.3 SELECTION OF TERMS IN G.P.

Sometimes it is required to select a finite number of terms in G.P. It is always convenient if we select the terms in the following manner:

No. of terms	*Terms*	*Common ratio*
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r

If the product of the numbers is not given, then the numbers are taken as $a, ar, ar^2, ar^3, \ldots$

The following examples illustrate the application of the above selections.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *If the sum of three numbers in G.P. is 38 and their product is 1728, find them.*

SOLUTION Let the numbers be $\frac{a}{r}, a, ar$. It is given that the product and sum of these numbers are 38 and 1728 respectively.

Now, Product = 1728 $\Rightarrow \frac{a}{r}(a)(ar)=1728 \Rightarrow a^3=1728 \Rightarrow a=12$ and, Sum = 38

$$\Rightarrow \quad \frac{a}{r}+a+ar = 38 \Rightarrow a\left(\frac{1}{r}+1+r\right) = 38 \Rightarrow 12\left(\frac{1+r+r^2}{r}\right)=38$$

$$\Rightarrow \quad 6+6r+6r^2=19r \Rightarrow 6r^2-13r+6=0 \Rightarrow (3r-2)(2r-3) = 0 \Rightarrow r=3/2 \text{ or, } r=2/3$$

Putting the values of a and r in $\frac{a}{r}, a, ar$, we find that the required numbers are 8, 12, 18 or 18, 12, 8.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 2 *If the continued product of three numbers in G.P. is 216 and the sum of their products in pairs is 156, find the numbers.*

SOLUTION Let the three numbers be $a/r, a, ar$. Then,

Product = 216 $\Rightarrow (a/r) \cdot (a) \cdot (ar) = 216 \Rightarrow a^3 = 6^3 \Rightarrow a = 6.$

Sum of the products in pairs = 156

$$\Rightarrow \quad \frac{a}{r} \cdot a + a \cdot ar + \frac{a}{r} \cdot ar = 156 \Rightarrow a^2\left(\frac{1}{r} + r + 1\right) = 156 \Rightarrow 36\left(\frac{1 + r^2 + r}{r}\right) = 156$$

$$\Rightarrow \quad 3(r^2 + r + 1) = 13r \Rightarrow 3r^2 - 10r + 3 = 0 \Rightarrow (3r - 1)(r - 3) = 0 \Rightarrow r = \frac{1}{3} \text{ or, } r = 3$$

Putting the values of a and r, the required numbers are 18, 6, 2 or 2, 6, 18.

EXAMPLE 3 *Three numbers are in G.P. whose sum is 70. If the extremes be each multiplied by 4 and the means by 5, they will be in A.P. Find the numbers.*

SOLUTION Let the numbers be a, ar, ar^2. It is given that the sum of these numbers is 70.

$$\therefore \quad a(1 + r + r^2) = 70 \quad \text{...(i)}$$

It is also given that $4a, 5ar, 4ar^2$ are in A.P.

$$\therefore \quad 2(5ar) = 4a + 4ar^2$$

$$\Rightarrow \quad 5r = 2 + 2r^2 \Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow (2r - 1)(r - 2) = 0 \Rightarrow r = 2 \text{ or, } r = 1/2$$

Putting $r = 2$ in (i), we obtain $a = 10$. So, the numbers are 10, 20, 40

Putting $r = 1/2$ in (i), we get $a = 40$. So, the numbers are 40, 20, 10.

EXAMPLE 4 *Find three numbers in G.P. whose sum is 52 and the sum of whose products in pairs is 624.*

SOLUTION Let the required numbers be a, ar, ar^2. Then,

$$\text{Sum} = 52 \Rightarrow a + ar + ar^2 = 52 \Rightarrow a(1 + r + r^2) = 52 \quad \text{...(i)}$$

Sum of the products in pairs = 624

$$\Rightarrow \quad a \cdot ar + ar \cdot ar^2 + a \cdot ar^2 = 624$$

$$\Rightarrow \quad a^2 r(1 + r + r^2) = 624 \quad \text{...(ii)}$$

Dividing (ii) by (i), we get $ar = 12 \Rightarrow a = \frac{12}{r}$ ....(iii)

Putting $a = \frac{12}{r}$ in (i), we get

$$\frac{12}{r}(1 + r + r^2) = 52 \Rightarrow 3r^2 - 10r + 3 = 0 \Rightarrow (3r - 1)(r - 3) = 0 \Rightarrow r = \frac{1}{3} \text{ or, } r = 3$$

Putting $r = 3$ in (iii), we obtain $a = 4$. So, the numbers are 4, 12, 36.

Putting $r = \frac{1}{3}$ in (iii), we get $a = 36$. So, the numbers are 36, 12, 4.

EXAMPLE 5 *The product of first three terms of a G.P. is 1000. If 6 is added to its second term and 7 added to its third term, the terms become in A.P. Find the G.P.*

SOLUTION Let first three terms of the given G.P. be $\frac{a}{r}, a, ar$. Then,

$$\text{Product} = 1000 \Rightarrow a^3 = 1000 \Rightarrow a = 10.$$

It is given that $\frac{a}{r}, a + 6, ar + 7$ are in A.P.

$$\therefore \quad 2(a + 6) = \frac{a}{r} + ar + 7$$

$$\Rightarrow \quad 32 = \frac{10}{r} + 10r + 7$$

$\Rightarrow \quad 25 = \frac{10}{r} + 10r \Rightarrow 5 = \frac{2}{r} + 2r \Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow (2r-1)(r-2) = 0 \Rightarrow r = 2, \frac{1}{2}$

Hence, the G.P. is 5, 10, 20, ... or 20, 10, 5, ...

EXAMPLE 6 *The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.* [NCERT]

SOLUTION Let the numbers in G.P. be a, ar, ar^2. It is given that the sum of these numbers is 56.

$\therefore \quad a + ar + ar^2 = 56$ ...(i)

It is also given that

$a-1, ar-7$ and $ar^2 - 21$ are in A.P.

$2(ar-7) = (a-1) + (ar^2 - 21) \Rightarrow 2ar = a + ar^2 - 8 \Rightarrow a + ar^2 = 2ar + 8$ ...(ii)

From (i), we obtain

$a + ar^2 = 56 - ar$ ...(iii)

Substituting $a + ar^2 = 56 - ar$ on the LHS of (ii), we get

$2ar + 8 = 56 - ar \Rightarrow 3ar = 48 \Rightarrow ar = 16 \Rightarrow r = \frac{16}{a}$

Putting $r = \frac{16}{a}$ in (i), we get

$a + 16 + \frac{256}{a} = 56 \Rightarrow a^2 + 16a + 256 = 56a \Rightarrow a^2 - 40a + 256 = 0 \Rightarrow (a-32)(a-8) = 0 \Rightarrow a = 8, 32$

Putting $a = 8$, in $r = \frac{16}{a}$ we get: $r = \frac{16}{8} = 2$. Putting $a = 32$, in $r = \frac{16}{a}$ we get: $r = \frac{16}{32} = \frac{1}{2}$

When $a = 8$ and $r = 2$, we obtain 8, 16 and 32 as the numbers in G.P.

When $a = 32$ and $r = \frac{1}{2}$, we obtain 32, 16, 8 as the numbers in G.P.

Hence, the numbers, in order, are 8, 16 and 32 or 32, 16 and 8.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 7 *Find three numbers in G.P. whose sum is 13 and the sum of whose squares is 91.*

SOLUTION Let the numbers be a, ar, ar^2. Then,

Sum $= 13 \Rightarrow a + ar + ar^2 = 13 \Rightarrow a(1 + r + r^2) = 13$ ...(i)

Sum of the squares = 91

$\Rightarrow \quad a^2 + a^2 r^2 + a^2 r^4 = 91 \Rightarrow a^2(1 + r^2 + r^4) = 91$ ...(ii)

Now, $a(1 + r + r^2) = 13$

$\Rightarrow \quad a^2(1 + r + r^2)^2 = 169$ [From (i)]

$\Rightarrow \quad a^2(1 + r^2 + r^4) + 2a^2 r(1 + r + r^2) = 169$

$\Rightarrow \quad 91 + 2ar\left\{a(1 + r + r^2)\right\} = 169$

$\Rightarrow \quad 91 + 2ar \times 13 = 169$ [Using (ii)]

$\Rightarrow \quad ar = 3$ [Using (i)]

$\Rightarrow \quad a = \frac{3}{r}$ ...(iii)

Putting $a = \frac{3}{r}$ in (i), we get

$$\frac{3}{r}(1+r+r^2) = 13$$

$$\Rightarrow \quad \frac{3}{r} + 3 + 3r = 13 \Rightarrow 3r^2 - 10r + 3 = 0 \Rightarrow (3r-1)(r-3) = 0 \Rightarrow r = 3 \text{ or, } r = \frac{1}{3}$$

Putting $r = 3$ in (iii), we get $a = 1$. So, the numbers are 1, 3, 9. Putting $r = \frac{1}{3}$ in (iii), we get $a = 9$. So, the nmbers are 9, 3, 1. Hence, the numbers are 1, 3, 9 or 9, 3, 1.

EXAMPLE 8 *Find four numbers in G.P. whose sum is 85 and product is 4096.*

SOLUTION Let the four numbers in G.P. be $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.

It is given that

$$\text{Product} = 4096 \Rightarrow a^4 = 4096 \Rightarrow a^4 = 8^4 \Rightarrow a = 8$$

and, $\quad \text{Sum} = 85$

$$\Rightarrow \quad a\left(\frac{1}{r^3} + \frac{1}{r} + r + r^3\right) = 85 \Rightarrow 8\left(r^3 + \frac{1}{r^3}\right) + 8\left(r + \frac{1}{r}\right) = 85$$

$$\Rightarrow \quad 8\left\{\left(r + \frac{1}{r}\right)^3 - 3\left(r + \frac{1}{r}\right)\right\} + 8\left(r + \frac{1}{r}\right) = 85 \Rightarrow 8\left(r + \frac{1}{r}\right)^3 - 16\left(r + \frac{1}{r}\right) - 85 = 0$$

$$\Rightarrow \quad 8x^3 - 16x - 85 = 0, \text{ where } r + \frac{1}{r} = x$$

$$\Rightarrow \quad (2x-5)(4x^2 + 10x + 17) = 0$$

$$\Rightarrow \quad 2x - 5 = 0 \qquad [\because 4x^2 + 10x + 17 = 0 \text{ has imaginary roots}]$$

$$\Rightarrow \quad x = \frac{5}{2} \Rightarrow r + \frac{1}{r} = \frac{5}{2} \Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow (r-2)(2r-1) = 0 \Rightarrow r = 2 \text{ or, } r = \frac{1}{2}$$

Putting $a = 8$ and $r = 2$ or $r = \frac{1}{2}$, we obtain that the four numbers are either 1, 4, 16, 64 or, 64, 16, 4, 1.

EXERCISE 19.2

BASIC

1. Find three numbers in G.P. whose sum is 65 and whose product is 3375.
2. Find three numbers in G.P. whose sum is 38 and their product is 1728.
3. The sum of first three terms of a G.P. is 13/12 and their product is – 1. Find the G.P.

BASED ON LOTS

4. The product of three numbers in G.P. is 125 and the sum of their products taken in pairs is 87½. Find them.
5. The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms. [NCERT]

6. The sum of three numbers in G.P. is 14. If the first two terms are each increased by 1 and the third term decreased by 1, the resulting numbers are in A.P. Find the numbers.
7. The product of three numbers in G.P. is 216. If 2, 8, 6 be added to them, the results are in A.P. Find the numbers.
8. Find three numbers in G.P. whose product is 729 and the sum of their products in pairs is 819.

BASED ON HOTS

9. The sum of three numbers in G.P. is 21 and the sum of their squares is 189. Find the numbers. [NCERT]

ANSWERS

1. 45, 15, 5 or 5, 15, 45
2. 8, 12, 18
3. $\frac{4}{3}, -1, \frac{3}{4}, \ldots$ or $\frac{3}{4}, -1, \frac{4}{3}, \ldots$
4. $10, 5, \frac{5}{2}$ or $\frac{5}{2}, 5, 10$
5. $\frac{2}{5}, 1, \frac{5}{2}$
6. 2, 4, 8 or 8, 4, 2
7. 18, 6, 2 or 2, 6, 18
8. 1, 9, 81 or 81, 9,
9. 3, 6, 12

HINTS TO SELECTED PROBLEMS

3. Let the terms of the G.P. be $\frac{a}{r}$, a, ar. It is given that

$$\frac{a}{r} + a + ar = \frac{13}{12} \text{ and } \frac{a}{r} \times a \times ar = -1$$

$$\Rightarrow \quad a\left(\frac{r^2 + r + 1}{r}\right) = \frac{13}{12} \text{ and } a^3 = -1$$

$$\Rightarrow \quad a = -1 \text{ and } a(r^2 + r + 1) = \frac{13r}{12} \Rightarrow r^2 + r + 1 = -\frac{13r}{12} \Rightarrow 12r^2 + 25r + 12 = 0$$

$$\Rightarrow \quad 12r^2 + 16r + 9r + 12 = 0 \Rightarrow (3r + 4)(4r + 3) = 0 \Rightarrow r = -\frac{4}{3} \text{ or } -\frac{3}{4}$$

Hence, three numbers are $\frac{3}{4}, -1, \frac{4}{3}$ or, $\frac{4}{3}, -1, \frac{3}{4}$.

5. Let the terms of the G.P. be $\frac{a}{r}$, a, ar. It is given that

$$\frac{a}{r} + a + ar = \frac{39}{10} \text{ and } \frac{a}{r} \times a \times ar = 1$$

$$\Rightarrow \quad a\left(\frac{r^2 + r + 1}{r}\right) = \frac{39}{10} \text{ and } a^3 = 1 \Rightarrow a = 1 \text{ and } a(r^2 + r + 1) = \frac{39r}{10} \Rightarrow 10(r^2 + r + 1) = 39r$$

$$\Rightarrow \quad 10r^2 - 29r + 10 = 0 \Rightarrow (2r - 5)(5r - 2) = 0 \Rightarrow r = \frac{5}{2} \text{ or } r = \frac{2}{5}$$

Hence, the numbers are $\frac{2}{5}, 1, \frac{5}{2}$ or $\frac{5}{2}, 1, \frac{2}{5}$

19.4 SUM OF THE TERMS OF A G.P.

THEOREM *Prove that the sum of n terms of a G.P. with first term 'a' and common ratio 'r' is given by*

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right) \quad \text{or,} \quad S_n = a\left(\frac{1 - r^n}{1 - r}\right), r \neq 1$$

PROOF Let S_n denote the sum of n terms of the G.P. with first term 'a' and common ratio r. Then,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \qquad \ldots(i)$$

Multiplying both sides by r, we get

$$r\,S_n = ar + ar^2 + \ldots + ar^{n-1} + a\,r^n \qquad \ldots(ii)$$

On subtracting (ii) from (i), we get

$$S_n - r\,S_n = a - a\,r^n$$

$$\Rightarrow \quad S_n(1 - r) = a(1 - r^n)$$

$$\Rightarrow \quad S_n = a\left(\frac{1 - r^n}{1 - r}\right) \text{ or, } S_n = a\left(\frac{r^n - 1}{r - 1}\right), \text{ provided that } r \neq 1,$$

Hence, $S_n = a\left(\dfrac{1 - r^n}{1 - r}\right)$ or, $S_n = a\left(\dfrac{r^n - 1}{r - 1}\right), r \neq 1$

Q.E.D

NOTE *Some authors state two different formulas for S_n viz.,*

$$S_n = a\left(\frac{1 - r^n}{1 - r}\right) \text{ for } r < 1 \quad \text{and} \quad S_n = a\left(\frac{r^n - 1}{r - 1}\right) \text{ for } r > 1.$$

In fact these two are exactly identical. The only thing which must be noted is that the above formulae do not hold for $r = 1$. For $r = 1$, the sum of n terms of the G.P. is $S_n = a + a + a + \ldots + a$ (n times) $= na$.

REMARK 1 *If l is the last term of the G.P., then $l = a\,r^{n-1}$.*

$$\therefore \quad S_n = a\left(\frac{1 - r^n}{1 - r}\right) = \frac{a - ar^n}{1 - r} = \frac{a - (a\,r^{n-1})\,r}{1 - r} = \frac{a - lr}{1 - r}$$

Thus, $S_n = \dfrac{a - lr}{1 - r}$ or, $S_n = \dfrac{lr - a}{r - 1}, \; r \neq 1$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I FINDING THE SUM OF GIVEN NUMBER OF TERMS OF A GIVEN G.P.

EXAMPLE 1 *Find the sum of 7 terms of the G.P. 3, 6, 12, ...*

SOLUTION Here, $a = 3, r = 2$ and $n = 7$.

$$\therefore \quad S_7 = a\left(\frac{r^7 - 1}{r - 1}\right) = 3\left(\frac{2^7 - 1}{2 - 1}\right) = 3(128 - 1) = 381$$

EXAMPLE 2 *Find the sum of 10 terms of the G.P.* $1, 1/2, 1/4, 1/8 \ldots$

SOLUTION Here, $a=1, r=1/2$ and $n=10$.

$$\therefore \quad S_{10} = a\left(\frac{r^{10}-1}{r-1}\right)$$

$$\Rightarrow \quad S_{10} = 1\left\{\frac{(1/2)^{10}-1}{(1/2)-1}\right\} = 2\left(1-\frac{1}{2^{10}}\right) = 2\left(\frac{2^{10}-1}{2^{10}}\right) = \frac{(1024-1)}{512} = \frac{1023}{512}$$

EXAMPLE 3 *Find the sum to 7 terms of the sequence*

$$\left(\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}\right), \left(\frac{1}{5^4}+\frac{2}{5^5}+\frac{3}{5^6}\right), \left(\frac{1}{5^7}+\frac{2}{5^8}+\frac{3}{5^9}\right), \ldots$$

SOLUTION The given sequence is

$$\left(\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}\right), \frac{1}{5^3}\left(\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}\right), \frac{1}{5^6}\left(\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}\right), \ldots$$

Clearly, this is a G.P. with first term $a = \frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3} = \frac{38}{125}$ and common ratio $r = \frac{1}{5^3}$.

$$\therefore \quad S_7 = a\left(\frac{1-r^7}{1-r}\right) \Rightarrow S_7 = \frac{38}{125}\left\{\frac{1-(1/5^3)^7}{1-(1/5^3)}\right\} = \frac{38}{125}\left\{\frac{1-1/5^{21}}{1-\frac{1}{125}}\right\} = \frac{19}{62}\left(1-\frac{1}{5^{21}}\right)$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Sum the series:* $x(x+y) + x^2(x^2+y^2) + x^3(x^3+y^3) + \ldots$ *to n terms*

SOLUTION Let S_n denote the sum to n terms of the given series. Then,

$$S_n = x(x+y) + x^2(x^2+y^2) + x^3(x^3+y^3) + \ldots + x^n(x^n+y^n)$$

$$\Rightarrow \quad S_n = (x^2+x^4+x^6+\ldots+x^{2n}) + (xy + x^2y^2 + x^3y^3 + \ldots + x^ny^n)$$

$$\Rightarrow \quad S_n = x^2\left\{\frac{(x^2)^n-1}{x^2-1}\right\} + xy\left\{\frac{(xy)^n-1}{xy-1}\right\} = x^2\left\{\frac{x^{2n}-1}{x^2-1}\right\} + xy\left\{\frac{(xy)^n-1}{xy-1}\right\}$$

EXAMPLE 5 *Find the sum of the series* $2+6+18+\ldots+4374$.

SOLUTION The given series is a geometric series in which $a=2, r=3$ and $l=4374$.

$$\therefore \quad \text{Required sum} = \frac{(lr-a)}{(r-1)} = \frac{4374\times 3-2}{3-1} = 6560.$$

EXAMPLE 6 *Find the sum of the following series:*

(i) $5+55+555+\ldots$ to n terms (ii) $0.7+0.77+0.777+\ldots$ to n terms

(iii) $5+55+555+5555+\ldots$ to n terms

SOLUTION (i) Let S be the sum of the series $5+55+555+\ldots$ to n terms. Then,

$$S = 5\left\{1+11+111+\ldots+\text{to } n \text{ terms}\right\} = \frac{5}{9}\left\{9+99+999+\ldots+\text{to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S = \frac{5}{9}\left\{(10-1)+(10^2-1)+(10^3-1)+\ldots+(10^n-1)\right\}$$

$$\Rightarrow \quad S = \frac{5}{9}\left\{(10+10^2+10^3+\ldots+10^n) - \underbrace{(1+1+1+\ldots+1)}_{n \text{ times}}\right\}$$

$$\Rightarrow \quad S=\frac{5}{9}\left\{10\times\frac{(10^n-1)}{10-1}-n\right\}=\frac{5}{9}\left\{\frac{10}{9}(10^n-1)-n\right\}=\frac{5}{81}\left\{10^{n+1}-10-9n\right\}$$

(ii) Let S be the sum $0.7 + 0.77 + 0.777 + \dots$ to n terms. Then,

$$S = 7\times 0.1 + 7\times 0.11 + 7\times 0.111 + \dots \text{ to } n \text{ terms}$$

$$\Rightarrow \quad S = 7\left\{0.1 + 0.11 + 0.111 + \dots \text{ to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S=\frac{7}{9}\left\{0.9+0.99+0.999+\dots \text{ to } n \text{ terms}\right\}=\frac{7}{9}\left\{\frac{9}{10}+\frac{99}{100}+\frac{999}{1000}+\dots \text{ to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S=\frac{7}{9}\left\{\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{100}\right)+\left(1-\frac{1}{1000}\right)+\dots \text{ to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S=\frac{7}{9}\left\{\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+\dots+\left(1-\frac{1}{10^n}\right)\right\}$$

$$\Rightarrow \quad S=\frac{7}{9}\left\{n-\left(\frac{1}{10}+\frac{1}{10^2}+\dots+\frac{1}{10^n}\right)\right\}=\frac{7}{9}\left[n-\frac{1}{10}\frac{\left\{1-\left(\frac{1}{10}\right)^n\right\}}{\left(1-\frac{1}{10}\right)}\right]$$

$$\Rightarrow \quad S=\frac{7}{9}\left\{n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right\}=\frac{7}{81}\left\{9n-1+\frac{1}{10^n}\right\}$$

(iii) Let S be the sum of the series $5 + 5.5 + 5.55 + 5.555 +$ to n terms. Then,

$$S = 5 + 5.5 + 5.55 + 5.555 + \dots \text{to } n \text{ terms}$$

$$\Rightarrow S = 5 + (5 + 0.5) + (5 + 0.55) + (5 + 0.555) + \text{ to } n \text{ terms}$$

$$\Rightarrow S = (5 + 5 + 5 + \dots n \text{ terms}) + \{0.5 + 0.55 + 0.555 + \dots \text{to } (n-1) \text{ terms}\}$$

$$\Rightarrow S = 5n + 5\{0.1 + 0.11 + 0.11 + \dots \text{to } (n-1) \text{ terms}\}$$

$$\Rightarrow S=5n+\frac{5}{9}\{0.9+0.99+0.999+\text{to } (n-1) \text{ terms}\}=5n+\frac{5}{9}\left\{\frac{9}{10}+\frac{99}{100}+\frac{999}{1000}+\dots\text{to } (n-1) \text{ terms}\right\}$$

$$\Rightarrow S=5n+\frac{5}{9}\left\{\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{100}\right)+\left(1-\frac{1}{1000}\right)+\dots+\left(1-\frac{1}{10^{n-1}}\right)\right\}$$

$$\Rightarrow S=5n+\frac{5}{9}\left\{(n-1)-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\dots+\frac{1}{10^{n-1}}\right)\right\}$$

$$\Rightarrow S=5n+\frac{5}{9}\left[(n-1)-\frac{1}{10}\left\{\frac{1-\left(\frac{1}{10}\right)^{n-1}}{\left(1-\frac{1}{10}\right)}\right\}\right]=5n+\frac{5}{9}\left\{(n-1)-\frac{1}{9}\left(1-\frac{1}{10^{n-1}}\right)\right\}$$

EXAMPLE 7 *The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Find the sum of n terms of the G.P.* [NCERT]

SOLUTION Let a be the first term and r the common ratio of the G.P. It is given that

$$a + ar + ar^2 = 16 \quad \dots\text{(i)} \quad \text{and,} \quad ar^3 + ar^4 + ar^5 = 128 \quad \dots\text{(ii)}$$

$$\Rightarrow \quad a(1+r+r^2) = 16 \text{ and, } ar^3(1+r+r^2) = 128$$

$$\Rightarrow \quad \frac{ar^3(1+r+r^2)}{a(1+r+r^2)} = \frac{128}{16} \Rightarrow r^3 = 8 \Rightarrow r = 2$$

Putting $r = 2$ in (i), we get: $a = \frac{16}{7}$.

$$\therefore \quad S_n = a\left(\frac{r^n - 1}{r-1}\right) = \frac{16}{7}\left(\frac{2^n - 1}{2-1}\right) = \frac{16}{7}(2^n - 1)$$

EXAMPLE 8 *Find a G.P. for which the sum of the first two terms is – 4 and the fifth term is 4 times the third term.* [NCERT]

SOLUTION Let a be the first term and r be the common ratio of the G.P.

We have,

$$a_1 + a_2 = -4 \text{ and } a_5 = 4a_3$$

$$\Rightarrow a + ar = -4 \text{ and } ar^4 = 4ar^2 \Rightarrow a(1+r) = -4 \text{ and } r^2 = 4 \Rightarrow a(1+r) = -4 \text{ and } r = \pm 2$$

When $r = 2: a(1+r) = -4 \Rightarrow a = -\frac{4}{3}$

When $r = -2: a(1+r) = -4 \Rightarrow a = 4$

Hence, required G.P. is $-\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}, \ldots$ or, $4, -8, 16, \ldots$

Type II FINDING VALUE(S) OF n, r AND a WHEN THE SUM OF n TERMS OF A G.P. IS GIVEN

EXAMPLE 9 *Determine the number of terms in G.P.* $< a_n >$, *if* $a_1 = 3$, $a_n = 96$ *and* $S_n = 189$.

SOLUTION Let r be the common ratio of the given G.P. Then,

$$a_n = 96 \Rightarrow a_1 r^{n-1} = 96 \Rightarrow 3r^{n-1} = 96 \Rightarrow r^{n-1} = 32 \qquad \text{...(i)}$$

Now, $S_n = 189$

$$\Rightarrow \quad a_1\left(\frac{r^n - 1}{r-1}\right) = 189 \Rightarrow 3\left\{\frac{(r^{n-1})r - 1}{r-1}\right\} = 189 \Rightarrow \frac{32r-1}{r-1} = 63 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 32r - 1 = 63r - 63 \Rightarrow 31r = 62 \Rightarrow r = 2$$

Putting $r = 2$ in (i), we get

$$2^{n-1} = 32 \Rightarrow 2^{n-1} = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6.$$

EXAMPLE 10 *How many terms of the geometric series* $1 + 4 + 16 + 64 + \ldots$ *will make the sum 5461 ?*

SOLUTION Let the sum of n terms of the given series 5461. Here, $a = 1, r = 4$ and $S_n = 5461$.

$$\therefore \quad S_n = 5461$$

$$\Rightarrow \quad a\left(\frac{r^n - 1}{r-1}\right) = 5461 \Rightarrow \frac{4^n - 1}{4-1} = 5461 \qquad [\because a = 1 \text{ and } r = 4]$$

$$\Rightarrow \quad 4^n - 1 = 16383 \Rightarrow 4^n = 16384 \Rightarrow 4^n = 4^7 \Rightarrow n = 7$$

EXAMPLE 11 *The sum of some terms of a G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.* [NCERT]

SOLUTION Let there be n terms in the G.P. with first term $a = 5$ and common ratio $r = 2$. Then, Sum of n terms $= 315$

$\Rightarrow \quad a\left(\dfrac{r^n - 1}{r - 1}\right) = 315 \Rightarrow 5\left(\dfrac{2^n - 1}{2 - 1}\right) = 315$ $\quad [\because\ a = 5 \text{ and } r = 2]$

$\Rightarrow \quad 2^n - 1 = 63 \Rightarrow 2^n = 64 = 2 \Rightarrow n = 6$

$\therefore \quad$ Last term $= ar^{n-1} = 5 \times 2^{6-1} = 160$

EXAMPLE 12 *In an increasing G.P., the sum of the first and the last term is 66, the product of the second and the last but one is 128 and the sum of the terms is 126. How many terms are there in the progression ?*

SOLUTION Let a be the first term and r the common ratio of the given G.P. Further, let there be n terms in the given G.P. It is given that the sum of the first and last term is 66.

i.e. $\quad a_1 + a_n = 66 \Rightarrow a + ar^{n-1} = 66$ $\quad$ ...(i)

It is also given that the product of second and the second last term is 128.

i.e. $a_2\ a_{n-1} = 128 \Rightarrow ar\ ar^{n-2} = 128 \Rightarrow a^2\, r^{n-1} = 128 \Rightarrow a\,(ar^{n-1}) = 128 \Rightarrow ar^{n-1} = \dfrac{128}{a}$

Putting this value of ar^{n-1} in (i), we get

$$a + \frac{128}{a} = 66 \Rightarrow a^2 - 66a + 128 = 0 \Rightarrow (a-2)(a-64) = 0 \Rightarrow a = 2, 64$$

Putting $a = 2$ in (i), we get: $2 + 2\,.\,r^{n-1} = 66 \Rightarrow r^{n-1} = 32.$

Putting $a = 64$ in (i), we get: $64 + 64r^{n-1} = 66 \Rightarrow r^{n-1} = \dfrac{1}{32}.$

We reject the second value as the G.P. is an increasing G.P. and therefore $r > 1$. Thus, we obtain $a = 2$ and $r^{n-1} = 32.$

Now, $\quad S_n = 126$

$\Rightarrow \quad 2\left(\dfrac{r^n - 1}{r - 1}\right) = 126 \Rightarrow \dfrac{r^n - 1}{r - 1} = 63 \Rightarrow \dfrac{r^{n-1}\, r - 1}{r - 1} = 63 \Rightarrow \dfrac{32r - 1}{r - 1} = 63 \Rightarrow r = 2$

$\therefore \quad r^{n-1} = 32 \Rightarrow 2^{n-1} = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6$

Hence, there are 6 terms in the progression.

EXAMPLE 13 *Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$.* **[NCERT]**

SOLUTION If $a, ar, ar^2, \ldots$ and $A, AR, AR^2, \ldots$ are two geometric sequences, then the sequence having terms as the product of corresponding terms of the two sequences is also a geometric sequence with first term aA and common ratio rR.

Given sequences are geometric sequences with first terms 2 and 128 respectively and common ratios 2 and $\frac{1}{4}$ respectively. Therefore, the sequence formed by multiplying the corresponding terms of the given sequences is a G.P. with first term $a = 2 \times 128 = 256$ and common ratio $r = 2 \times \dfrac{1}{4} = \dfrac{1}{2}.$

Since each sequence contains 5 terms. Therefore, the sequence formed by the products of the corresponding terms has 5 terms.

Hence, required sum $= 256\left\{\dfrac{1-\left(\frac{1}{2}\right)^5}{1-\frac{1}{2}}\right\} = 256\left\{\dfrac{\left(1-\frac{1}{32}\right)}{\frac{1}{2}}\right\} = 512\left(1-\dfrac{1}{32}\right) = 512\times\dfrac{31}{32} = 496$

ALITER Required sum $= 2\times 128 + 4\times 32 + 8\times 8 + 16\times 2 + 32\times\dfrac{1}{2}$

$$= 256 + 128 + 64 + 32 + 16$$

$$= 256\left\{\dfrac{1-\left(\frac{1}{2}\right)^5}{1-\frac{1}{2}}\right\} = 512\left(1-\dfrac{1}{32}\right) = 512\times\dfrac{31}{32} = 496$$

Type III ON PROVING RESULTS BASED UPON THE FORMULA FOR THE SUM OF n TERMS OF A G.P.

EXAMPLE 14 *If S_1, S_2 and S_3 be respectively the sum of n, $2n$ and $3n$ terms of a G.P., prove that*

$$S_1(S_3 - S_2) = (S_2 - S_1)^2$$

SOLUTION Let a be the first term and r the common ratio of the G.P. Then,

$$S_1 = a\left(\frac{r^n-1}{r-1}\right),\ S_2 = a\left(\frac{r^{2n}-1}{r-1}\right) \text{ and } S_3 = a\left(\frac{r^{3n}-1}{r-1}\right)$$

Now,

$$S_1(S_3 - S_2) = a\left(\frac{r^n-1}{r-1}\right)\left\{a\left(\frac{r^{3n}-1}{r-1}\right) - a\left(\frac{r^{2n}-1}{r-1}\right)\right\}$$

$$\Rightarrow \quad S_1(S_3 - S_2) = \frac{a^2}{(r-1)^2}(r^n-1)\left\{(r^{3n}-1)-(r^{2n}-1)\right\} = \frac{a^2}{(r-1)^2}(r^n-1)(r^{3n}-r^{2n})$$

$$\Rightarrow \quad S_1(S_3 - S_2) = \frac{a^2}{(r-1)^2}(r^n-1)\, r^{2n}(r^n-1) = \left\{ar^n\left(\frac{r^n-1}{r-1}\right)\right\}^2 \quad \ldots \text{(i)}$$

and, $$(S_2 - S_1)^2 = \left\{a\left(\frac{r^{2n}-1}{r-1}\right) - a\left(\frac{r^n-1}{r-1}\right)\right\}^2 = \frac{a^2}{(r-1)^2}\left\{(r^{2n}-1)-(r^n-1)\right\}^2$$

$$\Rightarrow \quad (S_2 - S_1)^2 = \frac{a^2}{(r-1)^2}\left\{r^n(r^n-1)\right\}^2 = \left\{ar^n\left(\frac{r^n-1}{r-1}\right)\right\}^2 \quad \ldots \text{(ii)}$$

From (i) and (ii) we obtain: $S_1(S_3 - S_1) = (S_2 - S_1)^2$

EXAMPLE 15 *If S be the sum, P the product and R the sum of the reciprocals of n terms of a G.P., prove that* $\left(\dfrac{S}{R}\right)^n = P^2$. [NCERT]

SOLUTION Let a be the first term and r the common ratio of the G.P. Then,

$$S = a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\frac{r^n - 1}{r - 1}\right) \quad \ldots\text{(i)}$$

$$P = a \,.\, ar\, ar^2 \ldots ar^{n-1} = a^n\, r^{1+2+3+\ldots+(n-1)} = a^n\, r^{\frac{n(n-1)}{2}} \quad \ldots\text{(ii)}$$

and, $$R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \ldots + \frac{1}{ar^{n-1}} = \frac{1}{a}\frac{\left\{(1/r)^n - 1\right\}}{\left\{(1/r) - 1\right\}} = \frac{1}{a}\left(\frac{1-r^n}{1-r}\right)\frac{1}{r^{n-1}}$$

$$\Rightarrow \quad R = \frac{1}{a}\left(\frac{r^n - 1}{r - 1}\right)\frac{1}{r^{n-1}} \quad \ldots\text{(iii)}$$

$$\therefore \quad \frac{S}{R} = a\left(\frac{r^n - 1}{r - 1}\right) . \, a\left(\frac{r - 1}{r^n - 1}\right) r^{n-1} = a^2\, r^{n-1}$$

$$\Rightarrow \quad \left(\frac{S}{R}\right)^n = a^{2n}\, r^{n(n-1)} = \left\{a^n\, r^{\frac{n(n-1)}{2}}\right\}^2 = P^2 \quad \text{[Using (ii)]}$$

Hence, $\left(\frac{S}{R}\right)^n = P^2$

EXAMPLE 16 *A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8th set of letter is mailed* [NCERT]

SOLUTION Amount spent on mailing one letter = ₹ $\frac{1}{2}$

$\therefore$ Amount spent when first set of 4 letters is mailed = ₹ 2

Amount spent when second set of $4 \times 4 = 16$ letters is mailed = ₹ $(2 \times 4) = 8$

Amount spent when third set of $4 \times 4 \times 4 = 64$ letters is mailed = ₹ $(8 \times 4) = 32$

Clearly, 2, 8, 32, ... is a G.P. with first term 2 and common ratio 4.

$\therefore$ Total amount spent when 8th set of letters is mailed = Sum of 8 terms of the G.P.

$$= a\left(\frac{r^8 - 1}{r - 1}\right)$$

$$= ₹\left\{2\left(\frac{4^8 - 1}{4 - 1}\right)\right\} \quad [\because\ a = ₹\ 2 \text{ and } r = 4]$$

$$= ₹\left\{2 \times \left(\frac{65536 - 1}{3}\right)\right\} = ₹\ (2 \times 21845)$$

$$= ₹\ 43690$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 17 *Find the sum to n terms of the sequence* $\left(x+\frac{1}{x}\right)^2, \left(x^2+\frac{1}{x^2}\right)^2, \left(x^3+\frac{1}{x^3}\right)^2, \dots$

SOLUTION Let S_n denote the sum to n terms of the given sequence. Then,

$$S_n = \left(x+\frac{1}{x}\right)^2 + \left(x^2+\frac{1}{x^2}\right)^2 + \left(x^3+\frac{1}{x^3}\right)^2 + \dots + \left(x^n+\frac{1}{x^n}\right)^2$$

$$\Rightarrow \quad S_n = \left(x^2+\frac{1}{x^2}+2\right)+\left(x^4+\frac{1}{x^4}+2\right)+\left(x^6+\frac{1}{x^6}+2\right)+\dots+\left(x^{2n}+\frac{1}{x^{2n}}+2\right)$$

$$\Rightarrow \quad S_n = (x^2+x^4+x^6+\dots+x^{2n})+\left(\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+\dots+\frac{1}{x^{2n}}\right)+\underbrace{(2+2+\dots)}_{n \text{ times}}$$

$$\Rightarrow \quad S_n = x^2\left\{\frac{(x^2)^n-1}{x^2-1}\right\}+\frac{1}{x^2}\left\{\frac{(1/x^2)^n-1}{(1/x^2)-1}\right\}+2n = x^2\left(\frac{x^{2n}-1}{x^2-1}\right)+\frac{1}{x^{2n}}\left(\frac{1-x^{2n}}{1-x^2}\right)+2n$$

$$\Rightarrow \quad S_n = x^2\left(\frac{x^{2n}-1}{x^2-1}\right)+\frac{1}{x^{2n}}\left(\frac{x^{2n}-1}{x^2-1}\right)+2n = \left(\frac{x^{2n}-1}{x^2-1}\right)\left(x^2+\frac{1}{x^{2n}}\right)+2n$$

EXAMPLE 18 *Find the sum to n terms of the sequence given by* $a_n = 2^n + 3n, n \in N$.

SOLUTION Let S_n denote the sum to terms of the given sequence. Then,

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$\Rightarrow \quad S_n = (2^1+3\times 1)+(2^2+3\times 2)+(2^3+3\times 3)+\dots+(2^n+3\times n)$$

$$\Rightarrow \quad S_n = (2^1+2^2+2^3+\dots+2^n)+(3\times 1+3\times 2+3\times 3+\dots+3\times n)$$

$$\Rightarrow \quad S_n = (2^1+2^2+2^3+\dots+2^n)+3(1+2+3+\dots+n)$$

$$\Rightarrow \quad S_n = 2\left(\frac{2^n-1}{2-1}\right)+3\left\{\frac{n}{2}(1+n)\right\} = 2(2^n-1)+\frac{3n}{2}(n+1)$$

EXAMPLE 19 *Prove that the sum to n terms of the series:* $11+103+1005+\dots$ *is,* $\frac{10}{9}(10^n-1)+n^2$.

SOLUTION Let S_n denote the sum to n terms of the given series. Then,

$$S_n = 11+103+1005+\dots \text{ to } n \text{ terms}$$

$$\Rightarrow \quad S_n = (10+1)+(10^2+3)+(10^3+5)+\dots+\{10^n+(2n-1)\}$$

$$\Rightarrow \quad S_n = (10+10^2+\dots+10^n)+\left\{1+3+5+\dots+(2n-1)\right\}$$

$$\Rightarrow \quad S_n = \frac{10(10^n-1)}{(10-1)}+\frac{n}{2}(1+2n-1) = \frac{10}{9}(10^n-1)+n^2$$

EXAMPLE 20 *Find the least value of n for which the sum* $1+3+3^2+\dots$ *to n terms is greater than 7000.*

SOLUTION We have,

$$S_n = 1+3+3^2+\dots \text{ to } n \text{ terms} \Rightarrow S_n = 1\times\left(\frac{3^n-1}{3-1}\right) = \frac{3^n-1}{2}$$

Now, $S_n > 7000 \Rightarrow \frac{3^n-1}{2} > 7000 \Rightarrow 3^n - 1 > 14000$

$$\Rightarrow \qquad 3^n > 14001 \Rightarrow n \log 3 > \log 14001 \Rightarrow n > \frac{\log 14001}{\log 3} \Rightarrow n > \frac{4.1461}{0.4771} = 8.69$$

Hence, the least value of n is 9.

EXAMPLE 21 *If f is a function satisfying $f(x + y) = f(x) f(y)$ for all $x, y \in N$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$, find the value of n.* [NCERT]

SOLUTION We have, $f(x + y) = f(x) f(y)$ for all $x, y \in N$

$$\therefore \qquad f(x) = f(\underbrace{1 + 1 + 1 + \ldots + 1}_{x-\text{times}}) = \underbrace{f(1) f(1) \ f(1) \ \ldots f(1)}_{x\text{-times}} = [f(1)]^x \text{ for all } x \in N$$

$$\Rightarrow \qquad f(x) = 3^x \text{ for all } x \in N \qquad [\because f(1) = 3]$$

Now,

$$\sum_{x=1}^{n} f(x) = 120 \Rightarrow \sum_{x=1}^{n} 3^x = 120 \Rightarrow 3 + 3^2 + \ldots + 3^n = 120$$

$$\Rightarrow \qquad 3\left(\frac{3^n - 1}{3 - 1}\right) = 120 \Rightarrow 3^n - 1 = 80 \Rightarrow 3^n = 81 \Rightarrow 3^n = 3^4 \Rightarrow n = 4.$$

EXAMPLE 22 *Find the natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function f satisfies $f(x + y) = f(x) \cdot f(y)$ for all natural numbers x, y and further $f(1) = 2$.*

[NCERT, NCERT EXEMPLAR]

SOLUTION Proceding as in Example 20, we obtain $f(x) = (f(1))^x = 2^x$ for all $x \in N$.

$$\therefore \qquad \sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$$

$$\Rightarrow \qquad \sum_{k=1}^{n} 2^{a+k} = 16(2^n - 1)$$

$$\Rightarrow \qquad 2^a\left(\sum_{k=1}^{n} 2^k\right) = 16(2^n - 1)$$

$$\Rightarrow \qquad 2^a(2 + 2^2 + 2^3 + \ldots + 2^n) = 16(2^n - 1) \Rightarrow 2^a\left\{2\left(\frac{2^n - 1}{2 - 1}\right)\right\} = 16(2^n - 1)$$

$$\Rightarrow \qquad 2^{a+1}(2^n - 1) = 16(2^n - 1) \Rightarrow 2^{a+1} = 2^4 \Rightarrow a + 1 = 4 \Rightarrow a = 3.$$

EXERCISE 19.3

BASIC

1. Find the sum of the following geometric progressions:
   (i) 2, 6, 18, ... to 7 terms (ii) 1, 3, 9, 27, ... to 8 terms
   (iii) $1, -1/2, 1/4, -1/8, \ldots$ (iv) $(a^2 - b^2), (a - b), \left(\frac{a-b}{a+b}\right)$, ... to n terms
   (v) 4, 2, 1, 1/2 ... to 10 terms.
2. Find the sum of the following geometric series:
   (i) $0.15 + 0.015 + 0.0015 + \ldots$ to 8 terms; (ii) $\sqrt{2} + \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} + \ldots$ to 8 terms;

(iii) $\frac{2}{9}-\frac{1}{3}+\frac{1}{2}-\frac{3}{4}+$ ... to 5 terms; (iv) $\sqrt{7}$, $\sqrt{21}$, $3\sqrt{7}$, ... to n terms [NCERT]

(v) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\ldots$ to $2n$ terms; (vi) $\frac{a}{1+i}+\frac{a}{(1+i)^2}+\frac{a}{(1+i)^3}+\ldots+\frac{a}{(1+i)^n}$.

(vii) $1, -a, a^2, -a^3, \ldots$ to n terms $(a \neq 1)$ [NCERT]

(viii) $x^3, x^5, x^7, \ldots$ to n terms [NCERT]

(ix) $(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\ldots$ to n terms;

BASED ON LOTS

3. Evaluate the following:

(i) $\sum_{n=1}^{11}(2+3^n)$ [NCERT] (ii) $\sum_{k=1}^{n}(2^k+3^{k-1})$ (iii) $\sum_{n=2}^{10}4^n$

4. Find the sum of the following series:

(i) $5+55+555+\ldots$ to n terms. [NCERT] (ii) $7+77+777+\ldots$ to n terms. [NCERT]

(iii) $9+99+999+\ldots$ to n terms. (iv) $0.5+0.55+0.555+\ldots$ to n terms.

(v) $0.6+0.66+0.666+\ldots$. to n terms. [NCERT]

5. How many terms of the G.P. 3, 3/2, 3/4, ... be taken together to make $\frac{3069}{512}$?

6. How many terms of the series $2+6+18+\ldots$ must be taken to make the sum equal to 728 ?

7. How many terms of the sequence $\sqrt{3}$, 3, $3\sqrt{3}$, ... must be taken to make the sum $39+13\sqrt{3}$?

8. The sum of n terms of the G.P. 3, 6, 12, ... is 381. Find the value of n.

9. The common ratio of a G.P. is 3 and the last term is 486. If the sum of these terms be 728, find the first term.

10. The ratio of the sum of first three terms is to that of first 6 terms of a G.P. is 125 : 152. Find the common ratio.

11. The 4th and 7th terms of a G.P. are $\frac{1}{27}$ and $\frac{1}{729}$ respectively. Find the sum of n terms of the G.P.

12. Find the sum : $\sum_{n=1}^{10}\left\{\left(\frac{1}{2}\right)^{n-1}+\left(\frac{1}{5}\right)^{n+1}\right\}$.

13. The fifth term of a G.P. is 81 whereas its second term is 24. Find the series and sum of its first eight terms.

BASED ON HOTS

14. If S_1, S_2, S_3 be respectively the sums of n, $2n$, $3n$ terms of a G.P., then prove that $S_1^2+S_2^2 = S_1(S_2+S_3)$.

15. Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n+1)^{\text{th}}$ to $(2n)^{\text{th}}$ term is $\frac{1}{r^n}$. [NCERT]

16. If a and b are the roots of $x^2-3x+p=0$ and c, d are the roots $x^2-12x+q=0$, where a, b, c, d form a G.P. Prove that $(q+p):(q-p) = 17:15$. [NCERT]

17. How many terms of the G.P. $3, \frac{3}{2}, \frac{3}{4}$ .....are needed to give the sum $\frac{3069}{512}$? [NCERT]

18. A person has 2 parents, 4 grandparents, 8 great grand parents, and so on. Find the number of his ancestors during the ten generations preceding his own. [NCERT]

19. If $S_1, S_2, \dots, S_n$ are the sums of n terms of n G.P.'s whose first term is 1 in each and common ratios are 1, 2, 3, ... , n respectively, then prove that

$$S_1 + S_2 + 2S_3 + 3S_4 + \dots (n-1) S_n = 1^n + 2^n + 3^n + \dots + n^n.$$

20. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of the terms occupying the odd places. Find the common ratio of the G.P. [NCERT]

21. Let a_n be the nth term of the G.P. of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$, such that $\alpha \neq \beta$. Prove that the common ratio of the G.P. is α/β.

22. Find the sum of $2n$ terms of the series whose every even term is 'a' times the term before it and every odd term is 'c' times the term before it, the first term being unity.

ANSWERS

1. (i) 2186 (ii) 3280 (iii) $\frac{171}{256}$ (iv) $\frac{a-b}{(a+b)^{n-2}}\left\{\frac{(a+b)^n-1}{(a+b)-1}\right\}$ (v) $8\left(1-\frac{1}{1024}\right)$

2. (i) $\frac{1}{6}\left(1-\frac{1}{10^8}\right)$ (ii) $\frac{255\sqrt{2}}{128}$ (iii) $\frac{55}{72}$ (iv) $\frac{1}{x-y}\left\{x^2\left(\frac{x^n-1}{x-1}\right)-y^2\left(\frac{y^n-1}{y-1}\right)\right\}$

(v) $\frac{19}{24}\left(1-\frac{1}{5^{2n}}\right)$ (vi) $-ai\{1-(1+i)^{-n}\}$ (vii) $\frac{1-(-a)^n}{1+a}$ (viii) $x^3\frac{(x^{2n}-1)}{x^2-1}$

(xi) $\sqrt{7}\left(\frac{3^{n/2}-1}{\sqrt{3}-1}\right)$

3. (i) 265741 (ii) $\frac{1}{2}(2^{n+2}+3^n-5)$ (iii) $\frac{16}{3}(4^9-1)$

4. (i) $\frac{5}{81}[10^{n+1}-9n-10]$ (ii) $\frac{7}{81}[10^{n+1}-9n-10]$ (iii) $\frac{1}{9}[10^{n+1}-9n-10]$

(iv) $\frac{5}{9}\left\{n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right\}$ (v) $\frac{6}{9}\left\{n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right\}$

5. 10 6. 6

7. 6 8. 7 9. 2 10. $\frac{3}{5}$ 11. $\frac{3}{2}\left(1-\frac{1}{3^n}\right)$

12. $\frac{2^{10}-1}{2^9}+\frac{5^{10}-1}{4\times 5^{11}}$ 13. $a=16,\ r=\frac{3}{2},\ S_8=\frac{65\times 97}{8}$ 17. 10

18. 2046 20. 4 22. $(a+1)\left\{\frac{(ac)^n-1}{ac-1}\right\}$

HINTS TO SELECTED PROBLEMS

2. (vii) Let S_n denote the sum of n terms of the G.P. $1 - a, a^2, -a^3, \ldots$ Then,

$$S_n = 1\left\{\frac{(-a)^n - 1}{-a-1}\right\} = \frac{1-(-1)^n a^n}{1+a}$$

(viii) Let S_n be the sum of n terms of the G.P. $x^3, x^5, x^7, \ldots$. Then,

$$S_n = x^3\left\{\frac{(x^2)^n - 1}{x^2-1}\right\} = x^3\left(\frac{x^{2n}-1}{x^2-1}\right).$$

(ix) Let S_n denote the sum of n terms of the G.P. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$. Then,

$$S_n = \sqrt{7}\left\{\frac{(\sqrt{3})^n - 1}{\sqrt{3}-1}\right\} = \sqrt{7}\left(\frac{3^{n/2}-1}{3^{1/2}-1}\right)$$

3. (i) $\sum\limits_{n=1}^{11}(2+3^n) = \sum\limits_{n=1}^{11} 2 + \sum\limits_{n=1}^{11} 3^n = 2\times 11 + 3\left(\frac{3^{11}-1}{3-1}\right) = 22 + \frac{3}{2}(3^{11}-1) = 265741.$

4. (i) Let $S_n = 5 + 55 + 555 + \ldots$ to n terms. Then,

$S_n = 5(1 + 11 + 111 + \ldots \text{ to } n \text{ terms})$

$\Rightarrow S_n = \frac{5}{9}(9 + 99 + 999 + \ldots \text{ to } n \text{ terms})$

$\Rightarrow S_n = \frac{5}{9}\left\{(10-1) + (10^2-1) + (10^3-1) + \ldots + (10^n - 1)\right\}$

$\Rightarrow S_n = \frac{5}{9}\left\{(10 + 10^2 + 10^3 + \ldots + 10^n) - n\right\}$

$\Rightarrow S_n = \frac{5}{9}\left\{10\left(\frac{10^n-1}{10-1}\right) - n\right\} \Rightarrow S_n = \frac{5}{9}\left\{\frac{10}{9}(10^n-1) - n\right\} = \frac{5}{81}(10^{n+1} - 9n - 10).$

(ii) Proceed as in (i)

(v) Let $S_n = 0.6 + 0.66 + 0.666 + \ldots$ to n terms. Then,

$S_n = 0.6 + 0.66 + 0.666 + \ldots$ to n terms

$\Rightarrow S_n = \frac{6}{9}(0.1 + 0.11 + 0.111 + \ldots \text{ to } n \text{ terms})$

$\Rightarrow S_n = \frac{6}{9}\left\{0.9 + 0.99 + 0.999 + \ldots \text{to } n \text{ terms}\right\}$

$\Rightarrow S_n = \frac{6}{9}\left\{\left(1-\frac{1}{10}\right) + \left(1-\frac{1}{10^2}\right) + \left(1-\frac{1}{10^3}\right) + \ldots + \left(1-\frac{1}{10^n}\right)\right\}$

$\Rightarrow S_n = \frac{6}{9}\left\{n - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots + \frac{1}{10^n}\right)\right\}$

$\Rightarrow S_n = \frac{6}{9}\left\{n - \frac{1}{10}\frac{\left(1-\left(\frac{1}{10}\right)^n\right)}{1-\frac{1}{10}}\right\} = \frac{6}{9}\left\{n - \frac{1}{9}\left(1-\frac{1}{10^n}\right)\right\}$

15. Required ratio $= \dfrac{a_1 + a_2 + \ldots + a_n}{a_{n+1} + a_{n+2} + \ldots + a_{2n}} = \dfrac{a + ar + \ldots + ar^{n-1}}{ar^n + ar^{n+1} + \ldots + ar^{2n-1}} = \dfrac{a\left(\dfrac{1-r^n}{1-r}\right)}{ar^n\left(\dfrac{1-r^n}{1-r}\right)} = \dfrac{1}{r^n}$

16. We have, $a + b = 3$, $ab = p$, $c + d = 12$ and $cd = q$ Let $b = ar$, $c = ar^2$ and $d = ar^3$. Then,

$a + b = 3$ and $c + d = 12$

$\Rightarrow \quad a(1 + r) = 3$ and $ar^2(1 + r) = 12 \Rightarrow \dfrac{ar^2(1+r)}{a(1+r)} = \dfrac{12}{3} \Rightarrow r = 2$

$\therefore \quad a(1 + r) = 3 \Rightarrow a = 1$

Now, $p = ab = a.ar = 2$, $q = cd = ar^2 \times ar^3 = 2^5 = 32$

$\therefore \quad \dfrac{q+p}{q-p} = \dfrac{32+2}{32-2} = \dfrac{34}{30} = \dfrac{17}{15}$

17. Let the sum of n terms of the G.P. $3, \dfrac{3}{2}, \dfrac{3}{4}, \ldots$ be $\dfrac{3069}{512}$. Then,

$$3\left\{\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right\} = \frac{3069}{512} \Rightarrow 1 - \frac{1}{2^n} = \frac{1023}{1024} \Rightarrow \frac{1}{2^n} = \frac{1}{2^{10}} \Rightarrow n = 10$$

Hence, the sum of 10 terms of the given G.P. is $\dfrac{3069}{512}$.

18. Number of ancestors during the ten generations preceding his own generation

$=$ Sum of 10 terms of the G.P. 2, 4, 8, ... $= 2\left(\dfrac{2^{10}-1}{2-1}\right) = 2046.$

20. Let there be $2n$ terms in the G.P. with first term a and common ratio r. Then,

Sum of all the terms $= 5$ (Sum of the terms occupying the odd places)

$\Rightarrow \quad a_1 + a_2 + \ldots + a_{2n} = 5(a_1 + a_3 + a_5 + \ldots + a_{2n-1})$

$\Rightarrow \quad a + ar + \ldots + ar^{2n-1} = 5(a + ar^2 + \ldots + ar^{2n-2})$

$\Rightarrow \quad a\left\{\dfrac{1-r^{2n}}{1-r}\right\} = 5a\left\{\dfrac{1-(r^2)^n}{1-r^2}\right\} \Rightarrow 1 + r = 5 \Rightarrow r = 4$

21. Let a be the first term and r be the common ratio of the G.P. Then,

$\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$

$\Rightarrow \quad a_2 + a_4 + \ldots + a_{200} = \alpha$ and $a_1 + a_3 + \ldots + a_{199} = \beta$

$\Rightarrow \quad ar + ar^3 + \ldots + ar^{199} = \alpha$ and $a + ar^2 + \ldots + ar^{198} = \beta$

$\Rightarrow \quad ar\left\{\dfrac{1-(r^2)^{100}}{1-r^2}\right\} = \alpha$, and $a\left\{\dfrac{1-(r^2)^{100}}{1-r^2}\right\} = \beta$

$\Rightarrow \quad ar\left(\dfrac{1-r^{200}}{1-r^2}\right) = \alpha$, and $a\left(\dfrac{1-r^{200}}{1-r^2}\right) = \beta \Rightarrow r = \dfrac{\alpha}{\beta}$

22. Let $a_1 + a_2 + a_3, \dots + a_{2n}$ be the given series. It is given that

$$a_1 = 1,\ a_2 = a\, a_1,\ a_3 = ca_2,\ a_4 = aa_3,\ a_5 = ca_4 \text{ and so on.}$$

$$\Rightarrow \quad a_1 = 1,\ a_2 = a,\ a_3 = ac,\ a_4 = a^2 c,\ a_5 = a^2 c^2,\ a_6 = a^3 c^2, \dots$$

$\therefore$ Required sum $= a_1 + a_2 + a_3 + \dots + a_{2n}$

$$= 1 + a + ac + a^2 c + a^2 c^2 + \dots \text{ to } 2n \text{ term}$$

$$= (1 + a) + ac\,(1 + a) + a^2 c^2 (1 + a) + \dots \text{ to } n \text{ terms}$$

$$= (1 + a) \left\{ \frac{1 - (ac)^n}{1 - ac} \right\} = (1 + a) \left\{ \frac{(ac)^n - 1}{ac - 1} \right\}$$

19.5 SUM OF AN INFINITE G.P.

THEOREM *The sum of an infinite G.P. with first term a and common ratio r* $(-1 < r < 1$ i.e., $|r| < 1)$ is $S = \dfrac{a}{1 - r}$.

PROOF Consider an infinite G.P. with first term a and common ratio r, where $-1 < r < 1$ i.e. $|r| < 1$. The sum of n terms of this G.P. is given by

$$S_n = a\left(\frac{1 - r^n}{1 - r}\right) = \frac{a}{1 - r} - \frac{ar^n}{1 - r} \qquad \text{...(i)}$$

Since $-1 < r < 1$, therefore r^n decreases as n increases and tends to zero as n tends to infinity

i.e. $\quad r^n \to 0$ as $n \to \infty$.

$$\therefore \quad \frac{a\,r^n}{1 - r} \to 0 \text{ as } n \to \infty.$$

Hence, from (i), the sum of an infinite G.P. is given by

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(\frac{a}{1 - r} - \frac{ar^n}{1 - r} \right) = \frac{a}{1 - r}, \text{ if } |r| < 1$$

NOTE *If* $r \geq 1$, *then the sum of an infinite G.P. tends to infinity.* Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I FINDING THE SUM TO INFINITY OF A G.P. OR A GEOMETRIC SERIES

EXAMPLE 1 *Find the sum to infinity of the G.P.* $-\dfrac{5}{4}, \dfrac{5}{16}, -\dfrac{5}{64}, \dots$

SOLUTION The given G.P. has first term $a = -5/4$ and the common ratio $r = -1/4$. Also, $|r| < 1$. Hence, the sum S to infinity is given by

$$S = \frac{a}{1 - r} = \frac{-5/4}{1 - (-1/4)} = -1$$

EXAMPLE 2 *Sum the following geometric series to infinity:*

(i) $(\sqrt{2} + 1) + 1 + (\sqrt{2} - 1) + \dots \infty$ (ii) $\dfrac{1}{2} + \dfrac{1}{3^2} + \dfrac{1}{2^3} + \dfrac{1}{3^4} + \dfrac{1}{2^5} + \dfrac{1}{3^6} + \dots \infty$

SOLUTION (i) The given series is a geometric series with first term $a = \sqrt{2} + 1$ and the common ratio r given by

$$r = \frac{1}{\sqrt{2} + 1} = \frac{\sqrt{2} - 1}{(\sqrt{2} + 1)(\sqrt{2} - 1)} = \sqrt{2} - 1$$

Hence, the sum S to infinity is given by

$$S = \frac{a}{1-r} = \frac{\sqrt{2}+1}{1-(\sqrt{2}-1)} = \frac{\sqrt{2}+1}{2-\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}(\sqrt{2}-1)}$$

$$\Rightarrow \quad S = \frac{(\sqrt{2}+1)^2}{\sqrt{2}(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{3+2\sqrt{2}}{\sqrt{2}} = \frac{4+3\sqrt{2}}{2}$$

(ii) We have,

$$\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\frac{1}{2^5}+\frac{1}{3^6}+\ldots\text{to }\infty$$

$$= \left(\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\ldots\right)+\left(\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\ldots\right)$$

$$= \left(\text{An infinite G.P. with } a=\frac{1}{2}, r=\frac{1}{2^2}\right)+\left(\text{An infinite G.P. with } a=\frac{1}{3^2}, r=\frac{1}{3^2}\right)$$

$$= \left\{\frac{(1/2)}{1-(1/2^2)}\right\}+\left\{\frac{(1/3^2)}{1-(1/3^2)}\right\} = \frac{2}{3}+\frac{1}{8} = \frac{19}{24}$$

EXAMPLE 3 *Prove that:* $6^{1/2}\times 6^{1/4}\times 6^{1/8}\ldots\ldots\infty = 6.$

SOLUTION Clearly,

$$6^{1/2}\times 6^{1/4}\times 6^{1/8}\ldots\infty = 6^{\{1/2+1/4+1/8+\ldots\infty\}}$$

$$= 6^{\{(1/2)/(1-1/2)\}} \quad \left[\because \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots \text{ to }\infty = \frac{1/2}{1-1/2} = 1\right]$$

$$= 6^1 = 6$$

Type II ON PROVING RESULTS BASED UPON THE FORMULA FOR THE SUM TO INFINITY OF A G.P.

EXAMPLE 4 *If* $b = a + a^2 + a^3 + \ldots\infty$, *prove that* $a = \dfrac{b}{1+b}$.

SOLUTION We have,

$$b = a + a^2 + a^3 + \ldots\infty$$

Clearly, RHS is a geometric series with first term 'a' and common ratio 'a'

$$\therefore \quad b = \frac{a}{1-a} \Rightarrow b - ab = a \Rightarrow a = \frac{b}{1+b}$$

Type III FINDING REQUIRED UNKNOWN WHEN THE SUM OF AN INFINITE G.P. IS GIVEN

EXAMPLE 5 *The first term of a G.P. is 2 and the sum to infinity is 6. Find the common ratio.*

SOLUTION Let r be the common ratio of the given G.P. It is given that, $a = 2$ and $S_\infty = 6$.

Now, $S_\infty = 6 \Rightarrow \dfrac{a}{1-r} = 6 \Rightarrow \dfrac{2}{1-r} = 6 \Rightarrow 6 - 6r = 2 \Rightarrow r = 2/3.$

EXAMPLE 6 *The sum of an infinite G.P. is 8, its second term is 2, find the first term.*

SOLUTION Let a be the first term and r the common ratio of the G.P. It is given that

$$S_\infty = 8 \quad\text{and}\quad ar = 2$$

$$\Rightarrow \quad \frac{a}{1-r} = 8 \quad\text{and}\quad r = \frac{2}{a}$$

$$\Rightarrow \quad \frac{a}{1-(2/a)} = 8 \quad \text{[Eliminating } r\text{]}$$

$$\Rightarrow \quad a^2 - 8a + 16 = 0 \Rightarrow (a-4)^2 = 0 \Rightarrow a = 4.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type I **ON PROVING RESULTS BASED UPON THE FORMULA FOR THE SUM TO INFINITY OF A G.P.**

EXAMPLE 7 *If* $x = a + \frac{a}{r} + \frac{a}{r^2} + \dots \infty$, $y = b - \frac{b}{r} + \frac{b}{r^2} - \dots \infty$ *and,* $z = c + \frac{c}{r^2} + \frac{c}{r^4} + \dots \infty$, *prove that* $\frac{xy}{z} = \frac{ab}{c}$.

SOLUTION Clearly, x, y and z are the sums of infinite geometric progressions.

$$\therefore \quad x = \frac{a}{1-\frac{1}{r}} = \frac{ar}{r-1}, \quad y = \frac{b}{1-\left(-\frac{1}{r}\right)} = \frac{br}{1+r} \quad \text{and,} \quad z = \frac{c}{1-\frac{1}{r^2}} = \frac{cr^2}{r^2-1}$$

$$\Rightarrow \quad xy = \left(\frac{ar}{r-1}\right)\left(\frac{br}{r+1}\right) = \frac{abr^2}{r^2-1} \Rightarrow \frac{xy}{z} = \left\{\left(\frac{abr^2}{r^2-1}\right) \div \frac{cr^2}{(r^2-1)}\right\} = \frac{ab}{c}$$

EXAMPLE 8 *If* $x = 1 + a + a^2 + \dots \infty$, *where* $|a| < 1$ *and* $y = 1 + b + b^2 + \dots \infty$, *where* $|b| < 1$. *Prove that:*

$$1 + ab + a^2b^2 + \dots \infty = \frac{xy}{x+y-1}$$

SOLUTION We have,

$$x = 1 + a + a^2 + \dots \infty \Rightarrow x = \frac{1}{1-a} \Rightarrow 1 - a = \frac{1}{x} \Rightarrow a = 1 - \frac{1}{x} \quad \text{...(i)}$$

and, $$y = 1 + b + b^2 + b^3 + \dots\dots \infty \Rightarrow y = \frac{1}{1-b} \Rightarrow 1 - b = \frac{1}{y} \Rightarrow b = 1 - \frac{1}{y} \quad \text{...(ii)}$$

$$\therefore \quad 1 + ab + (ab)^2 + (ab)^3 + \dots \infty = \frac{1}{1-ab} = \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)} \quad \text{[Using (i) and (ii)]}$$

$$= \frac{xy}{x+y-1}$$

EXAMPLE 9 *If* $A = 1 + r^a + r^{2a} + \dots$ *to* ∞ *and* $B = 1 + r^b + r^{2b} + \dots \infty$, *prove that*

$$r = \left(\frac{A-1}{A}\right)^{1/a} = \left(\frac{B-1}{B}\right)^{1/b}$$

SOLUTION We have, $A = 1 + r^a + r^{2a} + \dots \infty$ and, $B = 1 + r^b + r^{2b} + \dots \infty$

$$\Rightarrow \quad A = \frac{1}{1-r^a} \text{ and, } B = \frac{1}{1-r^b} \Rightarrow 1 - r^a = \frac{1}{A} \text{ and, } 1 - r^b = \frac{1}{B}$$

$$\Rightarrow \quad r^a = 1 - \frac{1}{A} \text{ and, } r^b = 1 - \frac{1}{B} \Rightarrow r = \left(\frac{A-1}{A}\right)^{1/a} \text{ and, } r = \left(\frac{B-1}{B}\right)^{1/b}$$

Hence, $$r = \left(\frac{A-1}{A}\right)^{1/a} = \left(\frac{B-1}{B}\right)^{1/b}$$

Type II ON FINDING THE REQUIRED UNKNOWN WHEN THE SUM OF AN INFINITE G.P. IS GIVEN

EXAMPLE 10 *The sum of an infinite G.P. is 57 and the sum of their cubes is 9747, find the G.P.*

SOLUTION Let a be the first term and r the common ratio of the G.P. Then,

$$\text{Sum} = 57 \Rightarrow \frac{a}{1-r} = 57 \qquad \text{...(i)}$$

$$\text{Sum of the cubes} = 9747 \Rightarrow a^3 + a^3 r^3 + a^3 r^6 + \dots = 9747 \Rightarrow \frac{a^3}{1-r^3} = 9747 \qquad \text{...(ii)}$$

Dividing the cube of (i) by (ii), we get

$$\frac{a^3}{(1-r)^3} \times \frac{(1-r^3)}{a^3} = \frac{(57)^3}{9747}$$

$$\Rightarrow \quad \frac{1-r^3}{(1-r)^3} = 19 \Rightarrow \frac{1+r+r^2}{(1-r)^2} = 19 \Rightarrow 18r^2 - 39r + 18 = 0 \Rightarrow (3r-2)(6r-9) = 0$$

$$\Rightarrow r = 2/3 \text{ or}, r = 3/2 \Rightarrow r = 2/3 \qquad [\because r \neq 3/2, \text{ because } -1 < r < 1 \text{ for an infinite G.P.}]$$

Putting $r = 2/3$ in (i), we get

$$\frac{a}{1-(2/3)} = 57 \Rightarrow a = 19$$

Hence, the G.P. is 19, 38/3, 76/9, ...

Type IV FINDING A RATIONAL NUMBER WHOSE DECIMAL EXPANSION IS GIVEN

EXAMPLE 11 *Which is the rational number having the decimal expansion* $0.3\overline{56}$?

SOLUTION We have, $0.3\overline{56} = 0.3 + 0.056 + 0.00056 + 0.0000056 + \dots \infty$

$$= 0.3 + \left\{\frac{56}{10^3} + \frac{56}{10^5} + \frac{56}{10^7} + \dots \infty\right\} = \frac{3}{10} + \frac{\frac{56}{10^3}}{1 - \frac{1}{10^2}} = \frac{3}{10} + \frac{56}{990} = \frac{353}{990}$$

EXAMPLE 12 *Use geometric series to express* $0.555\dots = 0.\overline{5}$ *as a rational number.*

SOLUTION We have, $0.\overline{5} = 0.5555\dots$

$$= 0.5 + 0.05 + 0.005 + \dots \infty = \frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \dots \infty = \frac{(5/10)}{1-(1/10)} = \frac{5}{9}$$

Type V ON APPLICATIONS OF INFINITE G.P.

EXAMPLE 13 *A square is drawn by joining the mid-points of the sides of a square. A third square is drawn inside the second square in the same way and the process is continued indefinitely. If the side of the square is 10 cm, find the sum of the areas of all the squares so formed.*

SOLUTION Let $A_1A_2A_3A_4$ be the first square with each side equal to 10 cm. Let B_1, B_2, B_3, B_4 be the mid-points of its sides. Then,

$$B_1B_2 = \sqrt{A_2B_1^2 + A_2B_2^2} = \sqrt{5^2 + 5^2} = 5\sqrt{2} \text{ cm}.$$

Let C_1, C_2, C_3, C_4 be the mid-points of the sides of the square $B_1B_2B_3B_4$. Then,

$$C_1C_2 = \sqrt{B_1C_2^2 + B_1C_1^2} = \sqrt{\left(\frac{5\sqrt{2}}{2}\right)^2 + \left(\frac{5\sqrt{2}}{2}\right)^2} = 5 \text{ cm}$$

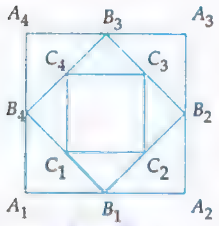

Fig. 19.1

Similarly, the side of fourth square is $\frac{5}{\sqrt{2}}$ cm and so on.

$\therefore$ Sum of the areas of all the squares so formed

$$= \left\{10^2 + (5\sqrt{2})^2 + (5)^2 + \left(\frac{5}{\sqrt{2}}\right)^2 + \dots \infty\right\} \text{ sq. cm.} \qquad [\because \text{ Area} = (\text{Side})^2]$$

$$= \left\{100 + 50 + 25 + \frac{25}{2} + \dots \infty\right\} = \frac{100}{1-(1/2)} = 200 \text{ sq. cm.}$$

EXAMPLE 14 *After striking a floor a certain ball rebounds* $\left(\frac{4}{5}\right)^{th}$ *of the height from which it has fallen. Find the total distance that it travels before coming to rest, if it is gently dropped from a height of 120 metres.*

SOLUTION Initially the ball falls from a height of 120 metres. After striking the floor it rebounds and goes to a height of $\frac{4}{5}(120)$ metres. Now, it falls from a height of $\frac{4}{5}(120)$ metres and after rebounding again it goes to a height of $\frac{4}{5}\left(\frac{4}{5}(120)\right)$ metres. This process is continued till the ball comes to rest.

$\therefore$ The total distance traveled $= 120 + 2\left\{\frac{4}{5}(120) + \left(\frac{4}{5}\right)^2(120) + \dots \infty\right\}$

$$= 120 + 2 \times \left\{\frac{\frac{4}{5}(120)}{1 - \frac{4}{5}}\right\} = 120 + 960 = 1080 \text{ metres.}$$

EXAMPLE 15 *The inventor of the chess board suggested a reward of one grain of wheat for the first square, 2 grains for the second, 4 grains for the third and so on, doubling the number of the grains for subsequent squares. How many grains would have to be given to inventor ? (There are 64 squares in the chess board).*

SOLUTION Clearly, required number of grains is the sum of an infinite G.P. with first term 1 and common ratio 2.

$\therefore$ Number of grains $= 1 + 2 + 2^2 + 2^3 + \dots$ to 64 terms $= 1\left(\frac{2^{64}-1}{2-1}\right) = 2^{64} - 1.$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

Type I ON FINDING THE SUM OF AN INFINITE G.P.

EXAMPLE 16 *Find the sum of an infinitely decreasing G.P. whose first term is equal to b + 2 and the common ratio to 2/c, where b is the least value of the product of the roots of the equation* $(m^2+1)x^2 - 3x + (m^2+1)^2 = 0$, *and c is the greatest value of the sum of its roots.*

SOLUTION Given equation is: $(m^2+1)x^2 - 3x + (m^2+1)^2 = 0$

$\therefore$ Sum of the roots $= \frac{3}{m^2+1}$ and, Product of the roots $= (m^2+1)$

Now, b = Least value of the product of roots = Least value of (m^2+1)

$\Rightarrow \quad b = 1 \qquad [\because m^2+1 > 1 \text{ for all } m]$

c = Greatest value of the sum of the roots = Greatest value of $\dfrac{3}{m^2+1}$

Clearly, $\dfrac{3}{m^2+1}$ is greatest when m^2+1 is least and the least value of m^2+1 is 1.

$$\therefore \quad c = \frac{3}{1} = 3$$

So, first term of the infinite G.P. is $b+2 = 1+2 = 3$ and, the common ratio is $\dfrac{2}{c} = \dfrac{2}{3}$.

Hence, the sum S of the infinite G.P. is given by

$$S = \frac{3}{1-\frac{2}{3}} = 9 \qquad \left[\text{Using} : S = \frac{a}{1-r}\right]$$

Type II ON PROVING RESULTS BASED UPON SUM OF AN INFINITE G.P.

EXAMPLE 17 *If* $x = \sum_{n=0}^{\infty} \cos^{2n}\theta,\ y = \sum_{n=0}^{\infty} \sin^{2n}\phi,\ z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\phi$, *where* $0 < \theta, \phi < \pi/2$ *then prove that* $xz + yz - z = xy$.

SOLUTION We have,

$$x = \sum_{n=0}^{\infty} \cos^{2n}\theta = 1 + \cos^2\theta + \cos^4\theta + \ldots \infty \Rightarrow x = \frac{1}{1-\cos^2\theta} \Rightarrow \sin^2\theta = \frac{1}{x}$$

$$\Rightarrow \quad y = \sum_{n=0}^{\infty} \sin^{2n}\phi = 1 + \sin^2\phi + \sin^4\phi + \ldots \infty \Rightarrow y = \frac{1}{1-\sin^2\phi} \Rightarrow \cos^2\phi = \frac{1}{y}$$

$$\text{and,} \quad z = \sum_{n=0}^{\infty} \cos^{2n}\theta, \sin^{2n}\phi = 1 + \cos^2\theta\sin^2\phi + \cos^4\theta\sin^4\phi + \ldots \infty$$

$$\Rightarrow \quad z = \frac{1}{1-\cos^2\theta\sin^2\phi} = \frac{1}{1-(1-\sin^2\theta)(1-\cos^2\phi)}$$

$$\Rightarrow \quad z = \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)} \Rightarrow z = \frac{1}{\frac{1}{x}+\frac{1}{y}-\frac{1}{xy}} \Rightarrow z = \frac{xy}{x+y-1} \Rightarrow xz + yz - z = xy$$

EXAMPLE 18 *If* $|x| < 1$ and $|y| < 1$, *find the sum to infinity of the following series:*

$$(x+y) + (x^2+xy+y^2) + (x^3+x^2y+xy^2+y^3) + \ldots$$

SOLUTION $(x+y) + (x^2+xy+y^2) + (x^3+x^2y+xy^2+y^3) + \ldots \infty$

$$= \frac{1}{x-y}\left\{(x^2-y^2) + (x^3-y^3) + (x^4-y^4) + \ldots \text{ to } \infty\right\}$$

$$\left[\because \frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2}a^1 + x^{n-3}a^2 + \ldots + a^{n-1}, n \in N\right]$$

$$= \frac{1}{x-y}\left\{(x^2+x^3+x^4+\ldots \text{ to } \infty) - (y^2+y^3+y^4+\ldots \text{ to } \infty)\right\}$$

$$= \frac{1}{x-y}\left\{\frac{x^2}{1-x} - \frac{y^2}{1-y}\right\}$$

$$= \frac{1}{x-y}\frac{\left\{x^2 - x^2y - y^2 + y^2x\right\}}{(1-x)(1-y)} = \frac{1}{(x-y)}\frac{\left\{(x^2-y^2) - xy(x-y)\right\}}{(1-x)(1-y)} = \frac{x+y-xy}{(1-x)(1-y)}$$

Type III ON FINDING REQUIRED UNKNOWN WHEN SUM OF AN INFINITE G.P. IS GIVEN

EXAMPLE 19 *The sum of an infinite geometric series is 15 and the sum of the squares of these terms is 45. Find the series.*

SOLUTION Let a be the first term and r be the common ratio of the infinite geometric series.

$$\text{Sum} = 15 \Rightarrow \frac{a}{1-r} = 15 \quad \text{...(i)}$$

Sum of the squares of terms $= 45 \Rightarrow (a^2 + a^2r^2 + a^2r^4 + \dots \infty) = 45 \Rightarrow \frac{a^2}{1-r^2} = 45$ ...(ii)

Dividing the square of (i), by (ii), we get

$$\frac{a^2}{(1-r)^2} \times \frac{1-r^2}{a^2} = \frac{(15)^2}{45} \Rightarrow \frac{1+r}{1-r} = 5 \Rightarrow 6r = 4 \Rightarrow r = \frac{2}{3}$$

Putting $r = \frac{2}{3}$ in (i), we get: $\frac{a}{1-2/3} = 15 \Rightarrow a = 5$

Hence, the required series is $5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \dots \infty$.

EXAMPLE 20 *If each term of an infinite G.P. is twice the sum of the terms following it, then find the common ratio of the G.P.*

SOLUTION Let a be the first term and r the common ratio of the G.P. It is given that

$$a_n = 2[a_{n+1} + a_{n+2} + a_{n+3} + \dots \infty] \text{ for all } n \in N$$

$$\Rightarrow \quad ar^{n-1} = 2[ar^n + ar^{n+1} + \dots \infty]$$

$$\Rightarrow \quad ar^{n-1} = \frac{2ar^n}{1-r} \Rightarrow 1 = \frac{2r}{1-r} \Rightarrow r = \frac{1}{3}$$

EXERCISE 19.4

BASIC

1. Find the sum of the following series to infinity:

 (i) $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} + \dots \infty$ (ii) $8 + 4\sqrt{2} + 4 + \dots \infty$

 (iii) $2/5 + 3/5^2 + 2/5^3 + 3/5^4 + \dots \infty$. (iv) $10 - 9 + 8.1 - 7.29 + \dots \infty$

 (v) $\frac{1}{3} + \frac{1}{5^2} + \frac{1}{3^3} + \frac{1}{5^4} + \frac{1}{3^5} + \frac{1}{5^6} + \dots \infty$ [NCERT]

2. Prove that: $(9^{1/3} . 9^{1/9} . 9^{1/27} \dots \infty) = 3$.

3. Prove that: $(2^{1/4} . 4^{1/8} . 8^{1/16} . 16^{1/32} \dots \infty) = 2$.

BASED ON LOTS

4. If S_p denotes the sum of the series $1 + r^p + r^{2p} + \dots$ to ∞ and s_p the sum of the series $1 - r^p + r^{2p} - \dots$ to ∞, prove that $S_p + s_p = 2\ S_{2p}$.
5. Find the sum of the terms of an infinite decreasing G.P. in which all the terms are positive, the first term is 4, and the difference between the third and fifth term is equal to 32/81.
6. Express the recurring decimal 0.125125125 ... as a rational number.
7. Find the rational number whose decimal expansion is $0.4\overline{23}$.
8. Find the rational numbers having the following decimal expansions:
   (i) $0.\overline{3}$ (ii) $0.\overline{231}$ (iii) $3.\overline{52}$ (iv) $0.6\overline{8}$ [NCERT]
9. One side of an equilateral triangle is 18 cm. The mid-points of its sides are joined to form another triangle whose mid-points, in turn, are joined to form still another triangle. The process is continued indefinitely. Find the sum of the (i) perimeters of all the triangles. (ii) areas of all triangles.

BASED ON HOTS

10. Find an infinite G.P. whose first term is 1 and each term is the sum of all the terms which follow it.
11. The sum of first two terms of an infinite G.P. is 5 and each term is three times the sum of the succeeding terms. Find the G.P.
12. Show that in an infinite G.P. with common ratio r ($|r| < 1$), each term bears a constant ratio to the sum of all terms that follow it.
13. If S denotes the sum of an infinite G.P. and S_1 denotes the sum of the squares of its terms, then prove that the first term and common ratio are respectively $\dfrac{2SS_1}{S^2 + S_1}$ and $\dfrac{S^2 - S_1}{S^2 + S_1}$.

ANSWERS

1. (i) $\frac{3}{4}$ (ii) $8(2 + \sqrt{2})$ (iii) $\frac{13}{24}$ (iv) 5.263 (v) $\frac{5}{12}$ 5. 6, $\frac{12}{3 - 2\sqrt{2}}$

6. $\frac{125}{999}$ 7. $\frac{419}{990}$ 8. (i) $\frac{1}{3}$ (ii) $\frac{231}{999}$ (iii) $\frac{317}{90}$ (iv) $\frac{31}{45}$

9. (i) 108 cm (ii) $108\sqrt{3}$ square cm 10. $1, \frac{1}{2}, \frac{1}{4}, \dots$ 11. $4, 1, \frac{1}{4}, \frac{1}{16}, \dots$

HINTS TO SELECTED PROBLEM

9. Sum of the perimeters $= 3\left\{18 + \frac{18}{2} + \frac{18}{4} + \dots \infty\right\}$

Sum of the areas $= \frac{\sqrt{3}}{4}\left\{18^2 + \left(\frac{18}{2}\right)^2 + \left(\frac{18}{4}\right)^2 + \dots \infty\right\}$

19.6 PROPERTIES OF GEOMETRIC PROGRESSIONS

In this section, we shall discuss some important properties of geometric progressions and geometric series.

PROPERTY I *If all the terms of a G.P. be multiplied or divided by the same non-zero constant, then it remains a G.P. with the same common ratio.*

PROOF Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a G.P. with common ratio r. Then,

$$\frac{a_{n+1}}{a_n} = r, \text{ for all } n \in N \quad \ldots(i)$$

Let k be a non-zero constant. Multiplying all the terms of the given G.P. by k, we obtain the new sequence: $ka_1, ka_2, ka_3, \ldots ka_n, \ldots$

Clearly, $\dfrac{k\,a_{n+1}}{k\,a_n} = \dfrac{a_{n+1}}{a_n} = r$ for all $n \in N$ [Using (i)]

Hence, the new sequence also forms a G.P. with common ratio r.

PROPERTY II *The reciprocals of the terms of a given G.P. form a G.P.*

PROOF Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a G.P. with common ratio r. Then,

$$\frac{a_{n+1}}{a_n} = r \text{ for all } n \in N \quad \ldots(i)$$

The sequence formed by the reciprocals of the terms of the given G.P. is

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots \frac{1}{a_n}, \ldots$$

For this sequence the ratio of a term and the preceding term is given by

$$\frac{1/a_{n+1}}{1/a_n} = \frac{a_n}{a_{n+1}} = \frac{1}{r} \quad \text{[Using (i)]}$$

So, the new sequence is a G.P. with common ratio $1/r$.

PROPERTY III *If each term of a G.P. be raised to the same power, the resulting sequence also forms a G.P.*

PROOF Let $a_1, a_2, a_3, \ldots a_n, \ldots$ be a G.P. with common ratio r. Then,

$$\frac{a_{n+1}}{a_n} = r \text{ for all } n \in N \quad \ldots(i)$$

Let k be a non-zero real number.

Consider the sequence whose terms are k^{th} powers of the terms of the given sequence

i.e. $a_1^k, a_2^k, a_3^k, \ldots, a_n^k, \ldots$

For this sequence, we have

$$\frac{a_{n+1}^k}{a_n^k} = \left(\frac{a_{n+1}}{a_n}\right)^k = r^k \text{ for all } n \in N \quad \text{[Using (i)]}$$

Hence, $a_1^k, a_2^k, a_3^k, \ldots a_n^k, \ldots$ is a G.P. with common ratio r^k.

PROPERTY IV *In a finite G.P the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.*

PROOF Let $a_1, a_2, a_3, \ldots, a_n$ be a finite G.P. with common ratio r. Then,

kth term from the beginning $= a_k = a_1 r^{k-1}$

kth term from the end $= (n-k+1)th$ term from the beginning $= a_{n-k+1} = a_1 r^{n-k}$

$\therefore$ (kth term from the beginning) (kth term from the end)

$$= a_k\, a_{n-k+1} = a_1 r^{k-1} a_1 r^{n-k} = a_1^2 r^{n-1} = a_1 \cdot a_1 r^{n-1} = a_1 a_n \text{ for all } k = 2, 3, \ldots, n-1$$

Hence, the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.

PROPERTY V *Three non-zero numbers a, b, c are in G.P. iff* $b^2 = ac$

PROOF Clearly,

$$a, b, c \text{ are in G.P.} \Leftrightarrow \frac{b}{a} = \frac{c}{b} = (\text{common ratio}) \Leftrightarrow b^2 = ac$$

NOTE *When a, b, c are in G.P., then b is known as the Geometric mean of a and c.*

PROPERTY VI *If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.*

PROPERTY VII *If* $a_1, a_2, a_3, \ldots, a_n, \ldots$ *is a G.P. of non-zero non-negative terms, then* $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ *is an A.P. and vice-versa.*

PROOF *Let* $a_1, a_2, a_3, \ldots, a_n, \ldots$ *be a G.P. of non-zero non-negative terms with common ratio r. Then,*

$$a_n = a_1 r^{n-1}, \text{ for all } n \in N$$

$$\Rightarrow \quad \log a_n = \log a_1 + (n-1) \log r, \quad \text{for all } n \in N$$

Let $\quad b_n = \log a_n = \log a_1 + (n-1) \log r, \quad \text{for all } n \in N$

Then, $\quad b_{n+1} - b_n = [\log a_1 + n \log r] - [\log a_1 + (n-1) \log r] = \log r \quad \text{for all } n \in N$

Clearly, $b_{n+1} - b_n = \log r = \text{Constant for all } n \in N$.

Hence, $b_1, b_2, \ldots, b_n, \ldots$ i.e. $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ is an A.P. with common difference $\log r$.

Conversely, let $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ be an A.P. with common difference d. Then,

$$\log a_{n+1} - \log a_n = d \quad \text{for all } n \in N.$$

$$\Rightarrow \quad \log\left(\frac{a_{n+1}}{a_n}\right) = d \quad \text{for all } n \in N.$$

$$\Rightarrow \quad \frac{a_{n+1}}{a_n} = e^d \text{ (a constant) for all } n \in N.$$

$$\Rightarrow \quad a_1, a_2, a_3, \ldots, a_n, \ldots \text{ is a G.P. with common ratio } e^d.$$

ILLUSTRATIVE EXAMPLES

BASED ON LOTS

Type I **PROBLEMS BASED UPON FOLLOWING RESULTS:**

(i) a, b, c are in G.P. iff $b^2 = ac$ *(ii) a, b, c are in A.P. iff* $2b = a + c$.

EXAMPLE 1 *If p, q, r are in A.P., show that the pth, qth and rth terms of any G.P. are in G.P.*

SOLUTION Let A be the first term and R the common ratio of a G.P. Then,

$$a_p = AR^{p-1}, \; a_q = AR^{q-1} \text{ and } a_r = AR^{r-1}$$

We have to prove that a_p, a_q, a_r are in G.P. For this it is sufficient to show that

$$(a_q)^2 = a_p \cdot a_r$$

Now, $\quad (a_q)^2 = (AR^{q-1})^2$

$$\Rightarrow \quad (a_q)^2 = A^2 R^{2q-2}$$

$$\Rightarrow \quad (a_q)^2 = A^2 R^{p+r-2} \qquad [\because p, q, r \text{ are in A.P.} \therefore 2q = p + r]$$

$$\Rightarrow \quad (a_q)^2 = (AR^{p-1})(AR^{r-1}) = a_p \cdot a_r$$

Hence, a_p, a_q, a_r are in G.P.

EXAMPLE 2 *If a, b, c are in G.P., then prove that* $\log a^n, \log b^n, \log c^n$ *are in A.P.*

SOLUTION It is given that a, b, c are in G.P.

$\therefore \quad b^2 = ac$

$\Rightarrow \quad (b^2)^n = (ac)^n$

$\Rightarrow \quad b^{2n} = a^n c^n$

$\Rightarrow \quad \log b^{2n} = \log (a^n c^n)$

$\Rightarrow \log (b^n)^2 = \log a^n + \log c^n \Rightarrow 2 \log b^n = \log a^n + \log c^n \Rightarrow \log a^n, \log b^n, \log c^n$ are in A.P

EXAMPLE 3 *Three numbers whose sum is 15 are in A.P. If 1, 4, 19 be added to them respectively, then they are in G.P. Find the numbers.*

SOLUTION Let the three numbers be $a-d, a, a+d$. Then,

$$\text{Sum} = 15 \Rightarrow (a-d) + a + (a+d) = 15 \Rightarrow a = 5.$$

So, the numbers are $5-d, 5, 5+d$. Adding 1, 4, 19 respectively to these numbers, we get $6-d, 9, 24+d$. These numbers are in G.P.

$\therefore \quad 9^2 = (6-d)(24+d) \Rightarrow d^2 + 18d - 63 = 0 \Rightarrow (d+21)(d-3) = 0 \Rightarrow d = -21$ or, $d = 3$.

Hence, the numbers are 26, 5, –16 or 2, 5, 8.

Type II **PROBLEMS BASED UPON PROPERTIES OF G.P.**

EXAMPLE 4 *If a, b, c, d are in G.P., show that:*

(i) $(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2$

(ii) $(ab+bc+cd)^2 = (a^2+b^2+c^2)(b^2+c^2+d^2)$ [NCERT]

SOLUTION Let r be the common ratio of the G.P. a, b, c, d. Then, $b = ar$, $c = ar^2$ and $d = ar^3$.

(i)
$$\begin{aligned} \text{LHS} &= (b-c)^2 + (c-a)^2 + (d-b)^2 \\ &= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2 \\ &= a^2r^2(1-r)^2 + a^2(r^2 - 1^2) + a^2r^2(r^2-1)^2 \\ &= a^2(r^6 - 2r^3 + 1) = a^2(1-r^3)^2 = (a - ar^3)^2 = (a-d)^2 = \text{RHS}. \end{aligned}$$

(ii)
$$\begin{aligned} \text{LHS} &= (ab+bc+cd)^2 = (a \times ar + ar \times ar^2 + ar^2 \times ar^3)^2 = a^4r^2(1+r^2+r^4)^2 \\ \text{RHS} &= (a^2+b^2+c^2)(b^2+c^2+d^2) \\ &= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6) \\ &= a^2(1+r^2+r^4)\, a^2r^2(1+r^2+r^4) = a^4r^2(1+r^2+r^4)^2 \end{aligned}$$

$\therefore \quad \text{LHS} = \text{RHS}.$

EXAMPLE 5 *If a, b, c, d are in G.P., prove that a + b, b + c, c + d are also in G.P.*

SOLUTION Let r be the common ratio of the G.P. a, b, c, d. Then, $b = ar$, $c = ar^2$ and $d = ar^3$

$\therefore \quad a + b = a + ar = a(1+r)$, $b + c = ar + ar^2 = ar(1+r)$ and $c + d = ar^2 + ar^3 = ar^2(1+r)$

Now, $(b+c)^2 = \left\{ar(1+r)\right\}^2 = a^2r^2(1+r)^2 = \left\{a(1+r)\right\}\left\{ar^2(1+r)\right\}$

$= (a+b)(c+d)$ $\quad [\because a + b = a(1+r)$, and $c + d = ar^2(1+r)]$

Hence, $a+b, b+c, c+d$ are in G.P.

EXAMPLE 6 *If a, b, c, d are in G.P., prove that* $a^n + b^n, b^n + c^n, c^n + d^n$ *are also in G.P.* [NCERT]

SOLUTION Let r be the common ratio of the G.P. a, b, c, d. Then, $b = ar$, $c = ar^2$ and $d = ar^3$.

$\therefore \quad a^n + b^n = a^n + a^n r^n = a^n (1 + r^n)$

$b^n + c^n = a^n r^n + a^n r^{2n} = a^n r^n (1 + r^n)$, $c^n + d^n = a^n r^{2n} + a^n r^{3n} = a^n r^{2n} (1 + r^n)$

Clearly, $(b^n + c^n)^2 = (a^n + b^n)(c^n + d^n)$.

Hence, $a^n + b^n, b^n + c^n, c^n + d^n$ are in G.P.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 7 *If a, b, c are in A.P. and x, y, z are in G.P., then show that $x^{b-c} \cdot y^{c-a} \cdot z^{a-b} = 1$.*

[NCERT EXEMPLAR]

SOLUTION It is given that

a, b, c are in A.P. $\Rightarrow 2b = a + c$ ...(i)

x, y, z are in G.P. $\Rightarrow y^2 = xz$ 4 ...(ii)

$$\therefore \quad x^{b-c} y^{c-a} z^{a-b} = x^{b-c} (\sqrt{xz})^{c-a} z^{a-b} \quad \text{[Using (ii)]}$$

$$= x^{b-c} x^{\frac{c-a}{2}} z^{\frac{c-a}{2}} z^{a-b}$$

$$= x^{b-c+\frac{c-a}{2}} z^{a-b+\frac{c-a}{2}} = x^{\frac{2b-(a+c)}{2}} z^{\frac{(c+a)-2b}{2}} = x^0 z^0 = 1 \quad \text{[Using (i)]}$$

EXAMPLE 8 *If mth, nth and pth terms of a G.P. form three consecutive terms of a G.P. Prove that m, n and p form three consecutive terms of an arithmetic sequence.*

SOLUTION Let a be the first term and r be the common ratio the G.P. Then,

$a_m = ar^{m-1}$, $a_n = ar^{n-1}$ and $a_p = ar^{p-1}$

It is given that a_m, a_n, a_p are in GP.

$\therefore \quad (a_n)^2 = a_m a_p$

$\Rightarrow \quad (ar^{n-1})^2 = (ar^{m-1} \times ar^{p-1})$

$\Rightarrow \quad a^2 r^{2n-2} = a^2 r^{m+p-2}$

$\Rightarrow \quad r^{2n-2} = r^{m+p-2} \Rightarrow 2n - 2 = m + p - 2 \Rightarrow 2n = m + p \Rightarrow m, n, p$ are in AP.

EXAMPLE 9 *If a, b, c are in G.P. and x, y are the arithmetic means of a, b and b, c respectively, then prove that:*

$$\frac{a}{x} + \frac{c}{y} = 2 \text{ and } \frac{1}{x} + \frac{1}{y} = \frac{2}{b}$$

SOLUTION It is given that

a, b, c are in G.P. $\Rightarrow b^2 = ac$ ...(i)

x is the A.M. of a and $b \Rightarrow x = \frac{a+b}{2}$ ...(ii)

and, y is the A.M. of b and $c \Rightarrow y = \frac{b+c}{2}$ ...(iii)

Now, $\frac{a}{x} + \frac{c}{y} = \frac{2a}{a+b} + \frac{2c}{b+c} = \frac{2a(b+c) + 2c(a+b)}{(a+b)(b+c)}$ $\left[\because x = \frac{a+b}{2} \text{ and } y = \frac{b+c}{2}\right]$

$$\Rightarrow \quad \frac{a}{x}+\frac{c}{y} = \frac{2(ab+2ac+bc)}{(ab+ac+b^2+bc)} = \frac{2(ab+2ac+bc)}{(ab+2ac+bc)} = 2 \qquad \text{[Using (i)]}$$

And, $$\frac{1}{x}+\frac{1}{y} = \frac{2}{a+b}+\frac{2}{b+c} = \frac{2(a+c+2b)}{(ab+b^2+ac+bc)} = \frac{2(a+c+2b)}{(ab+2b^2+bc)} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{1}{x}+\frac{1}{y} = \frac{2(a+c+2b)}{b(a+2b+c)} = \frac{2}{b}$$

EXAMPLE 10 *If a, b, c are in G.P. and $a^{1/x} = b^{1/y} = c^{1/z}$, prove that x, y, z are in A.P.* [NCERT]

SOLUTION We have,

$$a^{1/x} = b^{1/y} = c^{1/z} = \lambda \text{ (say)} \Rightarrow a = \lambda^x, b = \lambda^y \text{ and } c = \lambda^z$$

Now, a, b, c are in G.P.

$$\Rightarrow \quad b^2 = ac \Rightarrow (\lambda^y)^2 = \lambda^x \times \lambda^z \Rightarrow \lambda^{2y} = \lambda^{x+z} \Rightarrow 2y = x+z \Rightarrow x, y, z \text{ are in A.P.}$$

EXAMPLE 11 *If a^2+b^2, $ab+bc$ and b^2+c^2 are in G.P., prove that a, b, c are also in G.P.*

SOLUTION It is given that

$$a^2+b^2, ab+bc, b^2+c^2 \text{ are in G.P.}$$

$$\Rightarrow \quad (ab+bc)^2 = (a^2+b^2)(b^2+c^2)$$

$$\Rightarrow \quad a^2b^2+b^2c^2+2ab^2c = a^2b^2+a^2c^2+b^2c^2+b^4$$

$$\Rightarrow \quad b^4+a^2c^2-2ab^2c = 0 \Rightarrow (b^2-ac)^2 = 0 \Rightarrow b^2 = ac \Rightarrow a, b, c \text{ are in G.P.}$$

EXERCISE 19.5

BASIC

1. If a, b, c are in G.P., prove that $\log a$, $\log b$, $\log c$ are in A.P.
2. If a, b, c are in G.P., prove that $\frac{1}{\log_a m}, \frac{1}{\log_b m}, \frac{1}{\log_c m}$ are in A.P.
3. Find k such that $k+9$, $k-6$ and 4 form three consecutive terms of a G.P.
4. Three numbers are in A.P. and their sum is 15. If 1, 3, 9 be added to them respectively, they form a G.P. Find the numbers.

BASED ON LOTS

5. The sum of three numbers which are consecutive terms of an A.P. is 21. If the second number is reduced by 1 and the third is increased by 1, we obtain three consecutive terms of a G.P. Find the numbers.
6. The sum of three numbers a, b, c in A.P. is 18. If a and b are each increased by 4 and c is increased by 36, the new numbers form a G.P. Find a, b, c.
7. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an A.P. Find the numbers.
8. If a, b, c are in G.P., prove that:

   (i) $a(b^2+c^2) = c(a^2+b^2)$ (ii) $a^2b^2c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right) = a^3+b^3+c^3$

   (iii) $\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{a+b+c}{a-b+c}$ (iv) $\frac{1}{a^2-b^2}+\frac{1}{b^2} = \frac{1}{b^2-c^2}$

   (v) $(a+2b+2c)(a-2b+2c) = a^2+4c^2$.

9. If a, b, c, d are in G.P., prove that:
   (i) $\dfrac{ab-cd}{b^2-c^2}=\dfrac{a+c}{b}$ (ii) $(a+b+c+d)^2=(a+b)^2+2(b+c)^2+(c+d)^2$
   (iii) $(b+c)(b+d)=(c+a)(c+d)$
10. If a, b, c are in G.P., prove that the following are also in G.P.:
   (i) a^2, b^2, c^2 (ii) a^3, b^3, c^3 (iii) $a^2+b^2, ab+bc, b^2+c^2$
11. If a, b, c, d are in G.P., prove that:
   (i) $(a^2+b^2), (b^2+c^2), (c^2+d^2)$ are in G.P.
   (ii) $(a^2-b^2), (b^2-c^2), (c^2-d^2)$ are in G.P. [NCERT EXEMPLAR]
   (iii) $\dfrac{1}{a^2+b^2}, \dfrac{1}{b^2+c^2}, \dfrac{1}{c^2+d^2}$ are in G.P.
   (iv) $(a^2+b^2+c^2), (ab+bc+cd), (b^2+c^2+d^2)$ are in G.P.
12. If $(a-b), (b-c), (c-a)$ are in G.P., then prove that $(a+b+c)^2=3(ab+bc+ca)$
13. If a, b, c are in G.P. then prove that: $\dfrac{a^2+ab+b^2}{bc+ca+ab}=\dfrac{b+a}{c+b}$
14. If the 4^{th}, 10^{th} and 16^{th} terms of a G.P. are x, y and z respectively. Prove that x, y, z are in G.P. [NCERT]

BASED ON HOTS

15. If a, b, c are in A.P. and a, b, d are in G.P., then prove that $a, a-b, d-c$ are in G.P.
16. If pth, qth, rth and sth terms of an A.P. be in G.P., then prove that $p-q, q-r, r-s$ are in G.P. [NCERT]
17. If $\dfrac{1}{a+b}, \dfrac{1}{2b}, \dfrac{1}{b+c}$ are three consecutive terms of an A.P., prove that a, b, c are the three consecutive terms of a G.P.
18. If $x^a=x^{b/2}z^{b/2}=z^c$, then prove that $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in A.P.
19. If a, b, c are in A.P. b, c, d are in G.P. and $\dfrac{1}{c}, \dfrac{1}{d}, \dfrac{1}{e}$ are in A.P., prove that a, c, e are in G.P.
20. If a, b, c are in A.P. and a, x, b and b, y, c are in G.P., show that x^2, b^2, y^2 are in A.P.
21. If a, b, c are in A.P. and a, b, d are in G.P., show that $a, (a-b), (d-c)$ are in G.P.
22. If a, b, c are three distinct real numbers in G.P. and $a+b+c=xb$, then prove that either $x<-1$ or $x>3$.

ANSWERS

3. 0 or 16 6. $a=-2,\ b=6,\ c=14$ or $a=46,\ b=6,\ c=-34$ 7. 8, 16, 32
8. 15, 5, –5 or 3, 5, 7 9. 12, 7, 2 or 3, 7, 11 14. 2046

HINTS TO SELECTED PROBLEMS

1. a, b, c are in G.P. $\Rightarrow b^2=ac \Rightarrow \log b^2=\log ac \Rightarrow 2\log b=\log a+\log c$
2. a, b, c are in G.P.
   $\therefore \quad b^2=ac=\log_m b^2=\log_m ac$
   $\Rightarrow \quad 2\log_m b=\log_m a+\log_m c$

$$\Rightarrow \frac{2}{\log_b m} = \frac{1}{\log_a m} + \frac{1}{\log_c m} \Rightarrow \frac{1}{\log_a m}, \frac{1}{\log_b m}, \frac{1}{\log_c m} \text{ are in A.P.}$$

3. It is given that $k+9, k-6, 4$ are in G.P. $\Rightarrow (k-6)^2 = (k+9) \times 4 \Rightarrow k = 0, 16$.

14. Let the first term and common ratio of the G.P. be a and r respectively. It is given that $x = ar^3$, $y = ar^9$ and $z = ar^{15} \Rightarrow y^2 = a^2 r^{18}$ and $xz = a^2 r^{18} \Rightarrow y^2 = xz \Rightarrow x, y, z$ are in G.P.

16. Let the first term and the common difference of the AP be a and d respectively.

It is given that its p^{th}, r^{th} and s^{th} terms are in G.P. Let A be the first term and R be the common ratio of the G.P. Then,

$$a + (p-1)\, d = A \qquad \text{...(i)}$$

$$a + (q-1)\, d = AR \qquad \text{...(ii)}$$

$$a + (r-1)\, d = AR^2 \qquad \text{...(iii)}$$

$$a + (s-1)\, d = AR^3 \qquad \text{...(iv)}$$

Subtracting (ii) from (i), we get

$$\{a + (p-1)\, d\} - \{a + (q-1)\, d\} = A - AR \Rightarrow (p-q)\, d = A(1-R) \qquad \text{...(v)}$$

Subtracting (iii) from (ii), we get

$$\{a + (q-1)\, d\} - \{a + (r-1)\, d\} = AR - AR^2 \Rightarrow (q-r)\, d = AR\,(1-R) \qquad \text{...(vi)}$$

Subtracting (iv) from (iii), we get

$$\{a + (r-1)\, d\} - \{a + (s-1)\, d\} = AR^2 - AR^3 \Rightarrow (r-s)\, d = AR^2\,(1-R) \qquad \text{...(vii)}$$

From (v), (vi) and (vii), we obtain that

$$(q-r)^2\, d^2 = (p-q)\, d\, (r-s)\, d$$

$$\Rightarrow (q-r)^2 = (p-q)(r-s) \Rightarrow (p-q), (q-r), (r-s) \text{ are in G.P.}$$

17. It is given that $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ are in A.P. Therefore, $\frac{2}{2b} = \frac{1}{a+b} + \frac{1}{b+c} \Rightarrow b^2 = ac$.

19. We have,

$$2b = a + c \quad \text{...(i)} \qquad c^2 = bd \quad \text{...(ii)} \qquad \text{and,} \quad \frac{2}{d} = \frac{1}{c} + \frac{1}{e} \qquad \text{...(iii)}$$

We have to eliminate b and d from these relations. Substitute b and d obtained from (i) and (iii) in (ii) to get $c^2 = ae$.

22. Let r be the common ratio of the G.P. Then, $b = ar$ and $c = ar^2$.

Now, $a + b + c = xb \Rightarrow a + ar + ar^2 = xar \Rightarrow r^2 + (1-x)\, r + 1 = 0$.

But, r is real. Therefore, Disc $\geq 0 \Rightarrow x^2 - 2x - 3 > 0 \Rightarrow x < -1$ or $x > 3$

19.7 INSERTION OF GEOMETRIC MEANS BETWEEN TWO GIVEN NUMBERS

GEOMETRIC MEANS *Let a and b be two given numbers. If n numbers $G_1, G_2, \ldots, G_n$ are inserted between a and b such that the sequence $a, G_1, G_2, \ldots, G_n, b$ is a G.P. Then the numbers $G_1, G_2, \ldots, G_n$ are known as n geometric means (G.M.'s) between a and b.*

GEOMETRIC MEAN *If a single geometric mean G is inserted between two given numbers a and b, then G is known as the geometric mean between a and b.*

Thus,

G is the G.M. between a and b. $\Leftrightarrow$ a, G, b are in G.P. $\Leftrightarrow G^2 = ab \Leftrightarrow G = \sqrt{ab}$.

The geometric mean G between 4 and 9 is given by $G = \sqrt{4 \times 9} = 6$.

The geometric mean G between –9 and –4 is given by $G = \sqrt{-9 \times -4} = -6$.

NOTE *If a and b are two numbers of opposite signs, then geometric mean between them does not exist.*

19.7.1 INSERTION OF GEOMETRIC MEANS BETWEEN TWO GIVEN NUMBERS

Let $G_1, G_2, \dots, G_n$ be n geometric means between two given numbers a and b. Then,

$a, G_1, G_2, \dots, G_n, b$ is a G.P. consisting of $(n + 2)$ terms. Let r be the common ratio of this G.P. Then,

$$b = (n+2)th \text{ term} = ar^{n+1} \Rightarrow r^{n+1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$\therefore \quad G_1 = ar = a\left(\frac{b}{a}\right)^{1/(n+1)}, G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/(n+1)}, \dots, G_n = ar^n = a\left(\frac{b}{a}\right)^{n/n+1}.$$

THEOREM *If n geometric means are inserted between two quantities, then the product of n geometric means is the nth power of the single geometric mean between the two quantities.*

PROOF Let $G_1, G_2, G_3, \dots, G_n$ be n geometric means between two quantities a and b and let G be the single mean between a and b. Then,

$a, G_1, G_2, \dots, G_n, b$ is a G.P. Let r be the common ratio of this G.P. Then,

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} \text{ and, } G_1 = ar, G_2 = ar^2, G_3 = ar^3, \dots, G_n = ar^n.$$

$$\therefore \quad G_1 . G_2 . G_3 \dots . G_n = (ar)(ar^2)(ar^3)\dots(ar^n) = a^n r^{1+2+3+\dots+n}$$

$$= a^n r^{\frac{n(n+1)}{2}} = a^n \left\{\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right\}^{\frac{n(n+1)}{2}} = a^n \left(\frac{b}{a}\right)^{n/2} = a^{n/2} b^{n/2}$$

$$= \left\{\sqrt{ab}\right\}^n = G^n \qquad [\because G = \sqrt{ab}]$$

Q.E.D.

19.7.2 SOME IMPORTANT PROPERTIES OF ARITHMETIC AND GEOMETRIC MEANS

THEOREM 1 *If A and G are respectively arithmetic and geometric means between two positive numbers a and b, then A > G.*

PROOF We have, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

$$\therefore \quad A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 > 0 \Rightarrow A > G.$$

Q.E.D.

THEOREM 2 *If A and G are respectively arithmetic and geometric means between two positive quantities a and b, then the quadratic equation having a, b as its roots is* $x^2 - 2Ax + G^2 = 0$.

PROOF We have, $A=\frac{a+b}{2}$ and $G=\sqrt{ab}$. The equation having a and b as its roots is

$x^2-x(a+b)+ab=0$ or, $x^2-2Ax+G^2=0$

Q.E.D.

THEOREM 3 *If A and G be the A.M. and G.M. between two positive numbers, then the numbers are* $A \pm \sqrt{A^2-G^2}$. [NCERT]

PROOF The equation having its roots as the given numbers is

$$x^2-2Ax+G^2=0 \Rightarrow x=\frac{2A\pm\sqrt{4A^2-4G^2}}{2} \Rightarrow x=A\pm\sqrt{A^2-G^2}$$ Q.E.D.

Hence, the numbers are $A\pm\sqrt{A^2-G^2}$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **INSERTION OF GEOMETRIC MEANS BETWEEN TWO NUMBERS**

EXAMPLE 1 *Insert 5 geometric means between 576 and 9.*

SOLUTION Let G_1, G_2, G_3, G_4, G_5 be 5 geometric means between $a=576$ and $b=9$. Then, $576, G_1, G_2, G_3, G_4, G_5, 9$ is a G.P. with common ratio r given by

$$r=\left(\frac{9}{576}\right)^{\frac{1}{5+1}}=\left(\frac{1}{64}\right)^{\frac{1}{6}}=\frac{1}{2}. \qquad \left[\text{Using: } r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]$$

$\therefore \quad G_1=ar=576\times\frac{1}{2}=288, \quad G_2=ar^2=576\times\frac{1}{4}=144,$

$G_3=ar^3=576\times\frac{1}{8}=72, \quad G_4=ar^4=576\times\frac{1}{16}=36$ and, $G_5=ar^5=576\times\frac{1}{32}=18$

Hence, 288, 144, 72, 36, 18 are the required geometric means between 576 and 9.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type II **PROBLEMS BASED UPON ARITHMETIC AND GEOMETRIC MEANS**

EXAMPLE 2 *Find the value of n so that* $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ *may be the geometric mean between a and b.*

[NCERT]

SOLUTION It is given that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the G.M. between a and b.

$$\therefore \quad \frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\sqrt{ab}$$

$$\Rightarrow \quad a^{n+1}+b^{n+1}=a^{\left(n+\frac{1}{2}\right)}b^{1/2}+a^{1/2}b^{\left(n+\frac{1}{2}\right)}$$

$$\Leftrightarrow \quad a^{n+1}-a^{\left(n+\frac{1}{2}\right)}b^{1/2}=a^{1/2}b^{\left(n+\frac{1}{2}\right)}-b^{n+1}$$

$$\Leftrightarrow \quad a^{\left(n+\frac{1}{2}\right)}\left(a^{1/2}-b^{1/2}\right)=b^{\left(n+\frac{1}{2}\right)}\left(a^{1/2}-b^{1/2}\right)$$

$$\Leftrightarrow \quad a^{\left(n+\frac{1}{2}\right)}=b^{\left(n+\frac{1}{2}\right)} \qquad \left[\because\ a^{1/2}-b^{1/2}\neq 0, \text{ as } a\neq b\right]$$

$$\Leftrightarrow \quad \left(\frac{a}{b}\right)^{\left(n+\frac{1}{2}\right)}=1 \Leftrightarrow \left(\frac{a}{b}\right)^{\left(n+\frac{1}{2}\right)}=\left(\frac{a}{b}\right)^{0} \Leftrightarrow n+\frac{1}{2}=0 \Leftrightarrow n=-\frac{1}{2}$$

EXAMPLE 3 *Find two numbers whose arithmetic mean is 34 and the geometric mean is 16.*

SOLUTION Let the two numbers be a and b such that $a > b$. It is given that AM and GM of a and b are 34 and 16 respectively.

i.e. $\frac{a+b}{2}=34$ and $\sqrt{ab}=16 \Rightarrow a+b=68$ and $ab=256$ ...(i)

$\therefore \quad (a-b)^2=(a+b)^2-4ab=(68)^2-4\times 256=3600 \Rightarrow a-b=60 \qquad [\because\ a>b\ \therefore\ a-b>0]$

Solving $a+b=68$ and $a-b=60$ simultaneously, we get $a=64$ and $b=4$. Hence, the required numbers are 64 and 4.

ALITER Here, $A=34$ and $G=16$. So, the numbers are

$$A+\sqrt{A^2-G^2} \text{ and } A-\sqrt{A^2-G^2} \quad \text{i.e. } 34+\sqrt{34^2-16^2}=64 \text{ and } 34-\sqrt{34^2-16^2}=4.$$

EXAMPLE 4 *If the A.M. and G.M. between two numbers are in the ratio $m : n$, then prove that the numbers are in the ratio $m+\sqrt{m^2-n^2} : m-\sqrt{m^2-n^2}$.* [NCERT]

SOLUTION Let the two numbers be a and b. Let A and G be respectively the arithmetic and geometric means between a and b. Then,

$$A=\frac{a+b}{2} \text{ and } G=\sqrt{ab} \Rightarrow a+b=2A \text{ and } G^2=ab \qquad ...(i)$$

The equation having a and b as its roots is

$$x^2-(a+b)x+ab=0 \Rightarrow x^2-2Ax+G^2=0 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad x=\frac{2A\pm\sqrt{4A^2-4G^2}}{2} \Rightarrow x=A\pm\sqrt{A^2-G^2}$$

Thus the two numbers are $a=A+\sqrt{A^2-G^2}$ and $b=A-\sqrt{A^2-G^2}$.

It is given that: $A:G=m:n \Rightarrow A=\lambda m$ and $G=\lambda n$ for some λ

Substituting the values of A and G in $a=A+\sqrt{A^2-G^2}$ and $b=A-\sqrt{A^2-G^2}$, we get

$$\frac{a}{b}=\frac{\lambda m+\sqrt{\lambda^2 m^2-\lambda^2 n^2}}{\lambda m-\sqrt{\lambda^2 m^2-\lambda^2 n^2}} \Rightarrow \frac{a}{b}=\frac{m+\sqrt{m^2-n^2}}{m-\sqrt{m^2-n^2}} \Rightarrow a:b=\left\{m+\sqrt{m^2-n^2}\right\}:\left\{m-\sqrt{m^2-n^2}\right\}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

Type III ON GEOMETRIC AND ARITHMETIC MEANS

EXAMPLE 5 *Find two positive numbers whose difference is 12 and whose A.M. exceeds the G.M. by 2.*

SOLUTION Let the two numbers be a and b such that $a > b$. It is given that

$$a-b=12 \qquad ...(i)$$

It is also given that

$$AM-GM=2$$

$\Rightarrow \quad \dfrac{a+b}{2} - \sqrt{ab} = 2 \qquad \left[\because AM = \dfrac{a+b}{2} \text{ and } GM = \sqrt{ab}\right]$

$\Rightarrow \quad a + b - 2\sqrt{ab} = 4 \Rightarrow (\sqrt{a} - \sqrt{b})^2 = 4 \Rightarrow \sqrt{a} - \sqrt{b} = 2$ ...(ii)

Now, $a - b = 12 \Rightarrow (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = 12 \Rightarrow (\sqrt{a} + \sqrt{b}) \times (2) = 12 \Rightarrow \sqrt{a} + \sqrt{b} = 6$ ...(iii)

Solving (ii) and (iii), we get $a = 16$, $b = 4$. Hence, the required numbers are 16 and 4.

EXAMPLE 6 *If a, b, c are in G.P and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then show that $\dfrac{d}{a}, \dfrac{e}{b}, \dfrac{f}{c}$ are in A.P.* [NCERT]

SOLUTION It is given that a, b, c are in G.P. Therefore, $b^2 = ac$.

Now, $ax^2 + 2bx + c = 0 \Rightarrow ax^2 + 2\sqrt{ac}\, x + c = 0 \Rightarrow (\sqrt{a}x + \sqrt{c})^2 = 0 \Rightarrow \sqrt{a}x + \sqrt{c} = 0 \Rightarrow x = -\sqrt{\dfrac{c}{a}}$

It is given that the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root and the equation $ax^2 + 2bx + c = 0$ has equal roots both equal to $-\sqrt{\dfrac{c}{a}}$.

$\therefore \quad -\sqrt{\dfrac{c}{a}}$ is a root of the equation $dx^2 + 2ex + f = 0$

$\Rightarrow \quad d\dfrac{c}{a} - 2e\sqrt{\dfrac{c}{a}} + f = 0 \Rightarrow \dfrac{d}{a} - 2e\sqrt{\dfrac{1}{ac}} + \dfrac{f}{c} = 0$ [Dividing through out by c]

$\Rightarrow \quad \dfrac{d}{a} - \dfrac{2e}{b} + \dfrac{f}{c} = 0$ $[\because b^2 = ac]$

$\Rightarrow \quad 2\dfrac{e}{b} = \dfrac{d}{a} + \dfrac{f}{c} \quad \Rightarrow \dfrac{d}{a}, \dfrac{e}{b}, \dfrac{f}{c}$ are in A.P.

EXAMPLE 7 *Let x be the arithmetic mean and y, z be two geometric means between any two positive numbers. Then, prove that $\dfrac{y^3 + z^3}{xyz} = 2$.*

SOLUTION Let a and b be two positive numbers. Then,

x = A.M. of a and $b \Rightarrow x = \dfrac{a+b}{2}$ ...(i)

It is given that y and z are two geometric means between a and b. Then, a, y, z, b is a G.P. with common ratio $r = \left(\dfrac{b}{a}\right)^{\frac{1}{2+1}} = \left(\dfrac{b}{a}\right)^{1/3}$

$\therefore \quad y = ar \Rightarrow y = a\left(\dfrac{b}{a}\right)^{1/3} \Rightarrow y = b^{1/3} a^{2/3}$ and, $z = ar^2 \Rightarrow z = a\left(\dfrac{b}{a}\right)^{2/3} \Rightarrow z = b^{2/3} a^{1/3}$

$\therefore \quad y^3 + z^3 = (b^{1/3} a^{2/3})^3 + (b^{2/3} a^{1/3})^3 = ba^2 + b^2 a = ab\,(a+b)$

and, $\quad yz = (b^{1/3} a^{2/3})(b^{2/3} a^{1/3}) = ab.$

Now, $\quad y^3 + z^3 = ab(a+b)$ and $yz = ab \Rightarrow y^3 + z^3 = yz\,(a+b) \Rightarrow y^3 + z^3 = yz\,(2x)$ [Using (i)]

$\Rightarrow \quad \dfrac{y^3 + z^3}{xyz} = 2.$

EXAMPLE 8 *If a is the A.M. of b and c and the two geometric means are* G_1 and G_2, *then prove that* $G_1^3 + G_2^3 = 2\,abc$. [NCERT EXEMPLAR]

SOLUTION It is given that a is the A.M. of b and c.

$$\therefore \quad a = \frac{b+c}{2} \Rightarrow b + c = 2a \qquad \text{...(i)}$$

Since G_1 and G_2 are two geometric means between b and c. Therefore, b, G_1, G_2, c is a G.P. with common ratio $r = \left(\frac{c}{b}\right)^{1/3}$.

$$\therefore \quad G_1 = br = b\left(\frac{c}{b}\right)^{1/3} = c^{1/3} b^{2/3} \text{ and } G_2 = br^2 = b\left(\frac{c}{b}\right)^{2/3} = b^{1/3} c^{2/3}$$

$$\Rightarrow \quad G_1^3 = b^2 c \text{ and } G_2^3 = bc^2 \Rightarrow G_1^3 + G_2^3 = b^2 c + bc^2 = bc\,(b+c) = 2abc \qquad \text{[Using (i)]}$$

EXAMPLE 9 *If one geometric mean G and two arithmetic means* A_1 *and* A_2 *be inserted between two given quantities, prove that* $G^2 = (2A_1 - A_2)(2A_2 - A_1)$.

SOLUTION Let a and b be two given quantities. It is given that G is the geometric mean of a and b

$$\therefore \quad G = \sqrt{ab} \Rightarrow G^2 = ab \qquad \text{...(i)}$$

It is also given that A_1, A_2 are two arithmetic means between a and b. Therefore,, a, A_1, A_2, b is an A.P. with common difference $d = \frac{b-a}{3}$.

$$\therefore \quad A_1 = a + d = a + \frac{b-a}{3} = \frac{2a+b}{3}, \quad A_2 = a + 2d = a + \frac{2\,(b-a)}{3} = \frac{a+2b}{3}$$

So, $$2A_1 - A_2 = 2\left(\frac{2a+b}{3}\right) - \left(\frac{a+2b}{3}\right) = a \text{ and } 2A_2 - A_1 = 2\left(\frac{a+2b}{3}\right) - \left(\frac{2a+b}{3}\right) = b$$

$$\therefore \quad (2A_1 - A_2)(2A_2 - A_1) = ab \Rightarrow (2A_1 - A_2)(2A_2 - A_1) = G^2 \qquad \text{[Using (i)]}$$

Type III PROBLEMS ON A.M. > G.M.

EXAMPLE 10 *If x, y, z are distinct positive numbers, then prove that* $(x+y)(y+z)(z+x) > 8xyz$. [NCERT EXEMPLAR]

SOLUTION Using A.M. > G.M., we obtain

$$\frac{x+y}{2} > \sqrt{xy}, \ \frac{y+z}{2} > \sqrt{yz} \text{ and } \frac{z+x}{2} > \sqrt{zx}$$

$$\Rightarrow \quad x+y > 2\sqrt{xy}, \ y+z > 2\sqrt{yz} \text{ and } z+x > 2\sqrt{zx}$$

$$\Rightarrow \quad (x+y)(y+z)(z+x) > 2\sqrt{xy} \times 2\sqrt{yz} \times 2\sqrt{zx} \Rightarrow (x+y)(y+z)(z+x) > 8xyz.$$

EXAMPLE 11 *If* $x \in R$, *find the minimum value of the expression* $3^x + 3^{1-x}$. [NCERT EXEMPLAR]

SOLUTION Using A.M. ≥ G.M., we obtain

$$\frac{3^x + 3^{1-x}}{2} \geq \sqrt{3^x \times 3^{1-x}} \text{ for all } x \in R$$

$$\Rightarrow \quad \frac{3^x + 3^{1-x}}{2} \geq \sqrt{3} \text{ for all } x \in R \Rightarrow 3^x + 3^{1-x} \geq 2\sqrt{3} \text{ for all } x \in R$$

Hence, the minimum value of $3^x + 3^{1-x}$ for any $x \in R$ is $2\sqrt{3}$.

EXAMPLE 12 *If a, b, c, d are four distinct positive numbers in A.P. then show that* $bc > ad$. [NCERT EXEMPLAR]

SOLUTION It is given that a, b, c, d are in A.P. Therefore, a, b, c are in A.P. and hence b is the A.M. of a and c. But, the G.M. of a and c is $\sqrt{ac}$.

$\therefore$ A.M. of a and $c >$ G.M. of a and $c \Rightarrow b > \sqrt{ac} \Rightarrow b^2 > ac$ ...(i)

Again, a, b, c, d are in A.P. $\Rightarrow b, c, d$ are in A.P. $\Rightarrow c$ is the A.M of b and d. But, The G.M. of b and d is $\sqrt{bd}$.

$\therefore$ A.M. of b and $d >$ G.M. of b and $d \Rightarrow c > \sqrt{bd} \Rightarrow c^2 > bd$ ...(ii)

From (i) and (ii), we obtain: $b^2c^2 > (ac)(bd) \Rightarrow bc > ad$.

EXAMPLE 13 *If a, b, c, d are four distinct positive numbers in G.P. then show that* $a + d > b + c$. [NCERT EXEMPLAR]

SOLUTION It is given that a, b, c, d are in G.P. Therefore, a, b, c are in G.P. and hence, b is the G.M of a and c But, A.M. of a and c is $\frac{a+c}{2}$.

$\therefore$ A.M. of a and $c >$ G.M. of a and $c \Rightarrow \frac{a+c}{2} > b \Rightarrow a + c > 2b$ ...(i)

Again, a, b, c, d are in G.P. Therefore, b, c, d are in G.P. and hence, c is the G.M. of b and d.

But, A.M. of b and d is $\frac{b+d}{2}$.

$\therefore$ A.M. of b and $d >$ G.M. of b and $d \Rightarrow \frac{b+d}{2} > c \Rightarrow b + d > 2c$ ...(ii)

Adding (i) and (ii), we obtain

$$a + c + b + d > 2b + 2c \Rightarrow a + d > b + c$$

EXERCISE 19.6

BASIC

1. Insert 6 geometric means between 27 and $\frac{1}{81}$.
2. Insert 5 geometric means between 16 and $\frac{1}{4}$.
3. Insert 5 geometric means between $\frac{32}{9}$ and $\frac{81}{2}$.
4. If a is the G.M. of 2 and $\frac{1}{4}$, find a.
5. Find the geometric means of the following pairs of numbers:
   (i) 2 and 8 (ii) a^3b and ab^3 (iii) -8 and -2

BASED ON LOTS

6. Find the two numbers whose A.M. is 25 and GM is 20.
7. Construct a quadratic in x such that A.M. of its roots is A and G.M. is G.
8. The sum of two numbers is 6 times their geometric means, show that the numbers are in the ratio $(3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$. [NCERT]
9. If AM and GM of roots of a quadratic equation are 8 and 5 respectively, then obtain the quadratic equation. [NCERT]
10. If AM and GM of two positive numbers a and b are 10 and 8 respectively, find the numbers [NCERT]

BASED ON HOTS

11. Prove that the product of n geometric means between two quantities is equal to the nth power of a geometric mean of those two quantities.
12. If the A.M. of two positive numbers a and b $(a > b)$ is twice their geometric mean. Prove that: $a : b = (2+\sqrt{3}) : (2-\sqrt{3})$.
13. If one A.M., A and two geometric means G_1 and G_2 inserted between any two positive numbers, show that $\frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = 2A$. [NCERT EXEMPLAR]

ANSWERS

1. $9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$ 2. $8, 4, 2, 1, \frac{1}{2}$ 3. $\frac{16}{3}, 8, 12, 18, 27$ 4. $\frac{1}{\sqrt{2}}$

5.(i) 4 (ii) a^2b^2 (iii) -4 6. 40, 7. $x^2 - 2Ax + G^2 = 0$

9. $x^2 - 16x + 25 = 0$ 10. 4, 16 or 16, 4

HINTS TO SELECTED PROBLEMS

8. Let the numbers be a and b. Further, let A and G denote their arithmetic and geometric means respectively. It is given that

$$a + b = 6G \Rightarrow \frac{a+b}{2} = 3G \Rightarrow A = 3G.$$

Numbers a and b are roots of the quadratic equation

$$x^2 - x(a+b) + ab = 0 \text{ or, } x^2 - 2Ax + G^2 = 0 \text{ or, } x^2 - 6Gx + G^2 = 0 \quad [\because A = 3G]$$

$$\Rightarrow \quad x = \frac{6G \pm \sqrt{36G^2 - 4G^2}}{2} = 3G \pm 2\sqrt{2}G = 3G \pm 2\sqrt{2}G = (3 \pm 2\sqrt{2})G$$

$$\Rightarrow \quad a = (3+2\sqrt{2})G \text{ and } b = (3-2\sqrt{2})G$$

Hence, $a : b = (3+2\sqrt{2}) : (3-2\sqrt{2})$

9. The quadratic equation is $x^2 - 2Ax + G^2 = 0$ i.e. $x^2 - 16x + 25 = 0$.

10. The quadratic equation having numbers as roots is $x^2 - 2Ax + G^2 = 0$ or $x^2 - 20x + 64 = 0$. Now, $x^2 - 20x + 64 = 0 \Rightarrow (x-16)(x-4) = 0 \Rightarrow x = 16, 4$.

Hence, the numbers 4 and 16.

13. Let a and b be two numbers. Then,

$$A = \frac{a+b}{2}, G_1 = a^{2/3} b^{1/3} \text{ and } G_2 = a^{1/3} b^{2/3}$$ [See Examples 7, 8]

$$\therefore \quad \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = \frac{G_1^3 + G_2^3}{G_1 + G_2} = \frac{a^2b + ab^2}{ab} = a + b = 2A$$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The third term of a G.P. is the square of its first term. If its third term is 8, then the common ratio is ....
2. The sum of first two terms of a G.P. is 1 and every term is twice the previous term. The first term of the G.P. is ....

3. In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. Then the common ratio of the progression is ....
4. If A_1, A_2 are the two arithmetic means between two numbers a and b and G_1, G_2 are two geometric means between same two numbers, then $\dfrac{A_1 + A_2}{G_1 G_2} =$ ....
5. If A and G are the arithmetic and geometric means, respectively, of the roots of a quadratic equation. Then, the equation is ....
6. If three positive real numbers a, b, c are in A.P. and $abc = 4$, then the minimum possible value of b is ....
7. The sum of infinity of the series $9 - 3 + 1 - \dfrac{1}{3} + \ldots\ldots$, is ....
8. The value of the product $(32) \times (32)^{1/6} \times (32)^{1/36} \times \ldots\ldots$ to ∞, is ....
9. If the sum of the series $3 + 3x + 3x^2 + \ldots\ldots$ to ∞ is $\dfrac{45}{8}$, than $x =$ ....
10. The product of n geometric means between a and b is ....
11. The minimum value of the expression $3^x + 3^{1-x}$, $x \in R$, is ....
12. If a, b, c are in G.P. then the value of $\dfrac{a-b}{b-c}$ is equal to ....
13. The third term of a G.P. is 4, the product of the first five terms is ....

ANSWERS

1. 2 2. 1/3 3. $\dfrac{\sqrt{5}-1}{2}$ 4. $\dfrac{a+b}{ab}$ 5. $x^2 - 2Ax + G^2 = 0$ 6. $2^{2/3}$ 7. $\dfrac{27}{4}$

8. 64 9. $\dfrac{7}{15}$ 10. $(ab)^{n/2}$ 11. $2\sqrt{3}$ 12. $\dfrac{a}{b}$ or $\dfrac{b}{c}$ 13. 4^5

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If the fifth term of a G.P. is 2, then write the product of its 9 terms.
2. If $(p + q)^{th}$ and $(p - q)^{th}$ terms of a G.P. are m and n respectively, then write its pth term.
3. If $\log_x a$, $a^{x/2}$ and $\log_b x$ are in G.P., then write the value of x.
4. If the sum of an infinite decreasing G.P. is 3 and the sum of the squares of its term is $\dfrac{9}{2}$, then write its first term and common difference.
5. If p^{th}, q^{th} and r^{th} terms of a G.P. are x, y, z respectively, then write the value of $x^{q-r}\, y^{r-p}\, z^{p-q}$.
6. If A_1, A_2 be two AM's and G_1, G_2 be two GM's between a and b, then find the value of $\dfrac{A_1 + A_2}{G_1 G_2}$.
7. If second, third and sixth terms of an A.P. are consecutive terms of a G.P., write the common ratio of the G.P.
8. Write the quadratic equation the arithmetic and geometric means of whose roots are A and G respectively.

9. Write the product of n geometric means between two numbers a and b.
10. If $a = 1 + b + b^2 + b^3 + \ldots$ to ∞, then write b in terms of a given that $|b| < 1$.

ANSWERS

1. 512 2. $\sqrt{mn}$ 3. $\log_a(\log_b a)$ 4. $a = 2, r = \frac{1}{3}$ 5. 1 6. $\frac{a+b}{ab}$
7. 3 8. $x^2 - 2Ax + G^2 = 0$ 9. $(ab)^{n/2}$ 10. $\frac{a-1}{a}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If in an infinite G.P., first term is equal to 10 times the sum of all successive terms, then its common ratio is
(a) 1/10 (b) 1/11 (c) 1/9 (d) 1/20
2. If the first term of a G.P. $a_1, a_2, a_3, \ldots$ is unity such that $4a_2 + 5a_3$ is least, then the common ratio of G.P. is
(a) $-2/5$ (b) $-3/5$ (c) $2/5$ (d) none of these
3. If a, b, c are in A.P. and x, y, z are in G.P., then the value of $x^{b-c} y^{c-a} z^{a-b}$ is
(a) 0 (b) 1 (c) xyz (d) $x^a y^b z^c$
4. The first three of four given numbers are in G.P. and their last three are in A.P. with common difference 6. If first and fourth numbers are equal, then the first number is
(a) 2 (b) 4 (c) 6 (d) 8
5. If a, b, c are in G.P. and $a^{1/x} = b^{1/y} = c^{1/z}$, then xyz are in
(a) AP (b) GP (c) HP (d) none of these
6. If S be the sum, P the product and R be the sum of the reciprocals of n terms of a GP, then P^2 is equal to
(a) S/R (b) R/S (c) $(R/S)^n$ (d) $(S/R)^n$
7. The fractional value of $2.\dot{3}5\dot{7}$ is
(a) 2355/1001 (b) 2379/997 (d) 2355/999 (d) none of these
8. If pth, qth and rth terms of an A.P. are in G.P., then the common ratio of this G.P. is
(a) $\frac{p-q}{q-r}$ (b) $\frac{q-r}{p-q}$ (c) pqr (d) none of these
9. The value of $9^{1/3} \cdot 9^{1/9} \cdot 9^{1/27} \ldots$ to ∞, is
(a) 1 (b) 3 (c) 9 (d) none of these
10. The sum of an infinite G.P. is 4 and the sum of the cubes of its terms is 92. The common ratio of the original G.P. is
(a) 1/2 (b) 2/3 (c) 1/3 (d) $-1/2$
11. If the sum of first two terms of an infinite GP is 1 and every term is twice the sum of all the successive terms, then its first term is
(a) 1/3 (b) 2/3 (c) 1/4 (d) 3/4

12. The nth term of a G.P. is 128 and the sum of its n terms is 225. If its common ratio is 2, then its first term is
   (a) 1 (b) 3 (c) 8 (d) none of these
13. If second term of a G.P. is 2 and the sum of its infinite terms is 8, then its first term is
   (a) 1/4 (b) 1/2 (c) 2 (d) 4
14. If a, b, c are in G.P. and x, y are AM's between a, b and b, c respectively, then
   (a) $\frac{1}{x}+\frac{1}{y}=2$ (b) $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$ (c) $\frac{1}{x}+\frac{1}{y}=\frac{2}{a}$ (d) $\frac{1}{x}+\frac{1}{y}=\frac{2}{b}$
15. If A be one A.M. and p, q be two G.M.'s between two numbers, then 2 A is equal to
   (a) $\frac{p^3+q^3}{pq}$ (b) $\frac{p^3-q^3}{pq}$ (c) $\frac{p^2+q^2}{2}$ (d) $\frac{pq}{2}$
16. If p, q be two A.M.'s and G be one G.M. between two numbers, then $G^2 =$
   (a) $(2p-q)(p-2q)$ (b) $(2p-q)(2q-p)$ (c) $(2p-q)(p+2q)$ (d) none of these
17. If x is positive, the sum to infinity of the series $\frac{1}{1+x}-\frac{1-x}{(1+x)^2}+\frac{(1-x)^2}{(1+x)^3}-\frac{(1-x)^3}{(1+x)^4}+\ldots$ is
   (a) 1/2 (b) 3/4 (c) 1 (d) none of these
18. If $(4^3)(4^6)(4^9)(4^{12})\ldots(4^{3x})=(0.0625)^{-54}$, the value of x is
   (a) 7 (b) 8 (c) 9 (d) 10
19. Given that $x>0$, the sum $\sum_{n=1}^{\infty}\left(\frac{x}{x+1}\right)^{n-1}$ equals
   (a) x (b) $x+1$ (c) $\frac{x}{2x+1}$ (d) $\frac{x+1}{2x+1}$
20. In a G.P. of even number of terms, the sum of all terms is five times the sum of the odd terms. The common ratio of the G.P. is
   (a) $-\frac{4}{5}$ (b) $\frac{1}{5}$ (c) 4 (d) none of these

   [NCERT EXEMPLAR]
21. Let x be the A.M. and y, z be two G.M.s between two positive numbers. Then, $\frac{y^3+z^3}{xyz}$ is equal to
   (a) 1 (b) 2 (c) $\frac{1}{2}$ (d) none of these
22. The product $(32), (32)^{1/6}(32)^{1/36}\ldots$to ∞ is equal to
   (a) 64 (b) 16 (c) 32 (d) 0
23. The two geometric means between the numbers 1 and 64 are
   (a) 1 and 64 (b) 4 and 16 (c) 2 and 16 (d) 8 and 16
24. In a G.P. if the $(m+n)^{th}$ term is p and $(m-n)^{th}$ term is q, then its m^{th} term is

(a) 0 (b) pq (c) $\sqrt{pq}$ (d) $\frac{1}{2}(p+q)$

25. Let S be the sum, P be the product and R be the sum of the reciprocals of 3 terms of a G.P. then $P^2R^3 : S^3$ is equal to

(a) 1 : 1 (b) (Common ratio)^n : 1

(c) (First term)^2 (Common ratio)^2 (d) none of these [NCERT EXEMPLAR]

26. If x, y, z are positive integers then value of the expression $(x+y)(y+z)(z+x)$ is

(a) $= 8xyz$ (b) $> 8xyz$ (c) $< 8xyz$ (d) $= 4xyz$

[NCERT EXEMPLAR]

27. In a G.P. of positive terms, if any term is equal to the sum of the next two terms. Then the common ratio of the G.P.

(a) sin 18° (b) 2 cos 18° (c) cos 18° (d) 2 sin 18°

[NCERT EXEMPLAR]

28. The lengths of three unequal edges of a rectangular solid block are in G.P. The volume of the block is 216 cm³ and the total surface area is 252 cm². The length of the longest edge is

(a) 12 cm (b) 6 cm (c) 18 cm (d) 3 cm

[NCERT EXEMPLAR]

29. The minimum value of $4^x + 4^{1-x}$, $x \in R$, is

(a) 2 (b) 4 (c) 1 (d) 0

[NCERT EXEMPLAR]

30. If x, $2y$, $3z$ are in A.P., where the distinct numbers x, y, z are in G.P., then the common ratio of the G.P. is

(a) 3 (b) $\frac{1}{3}$ (c) 2 (d) $\frac{1}{2}$

[NCERT EXEMPLAR]

ANSWERS

1. (b)	2. (a)	3. (b)	4. (d)	5. (a)	6. (d)	7. (c)	8. (b)
9. (b)	10. (a)	11. (d)	12. (a)	13. (d)	14. (d)	15. (a)	16. (b)
17. (a)	18. (b)	19. (b)	20. (c)	21. (b)	22. (a)	23. (b)	24. (c)
25. (a)	26. (b)	27. (d)	28. (a)	29. (b)	30. (b)		

ACTIVITY

OBJECTIVE To show that the arithmetic mean of two distinct positive numbers is always greater than the geometric mean.

MATERIALS REQUIRED Cardboard, chart papers of different colours, coloured pencils, thumbpins, adhesive etc.

STEPS OF CONSTRUCTION

Step I Cut four rectangles of dimension $a \times b$, $(a > b)$ from coloured chart papers of different colours.

Step II Paste these four rectangles on the drawing board and name them as I, II, III and IV as shown in Fig. 19.2.

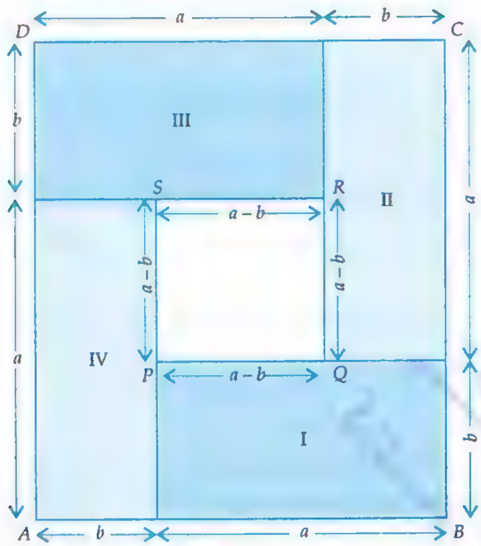

Fig. 19.2

STEPS OF DEMONSTRATION

Step I $ABCD$ is a square each side of which is of length $(a+b)$ units. So, its area is $(a+b)^2$ sq.units.

We find that :

Area of rectangle I $= ab$ sq. units

Area of rectangle II $= ab$ sq. units

Area of rectangle III $= ab$ sq. units

Area of rectangle IV $= ab$ sq. units

Clearly, $PQRS$ is a square of side $(a-b)$ units.

$\therefore$ Area of square $PQRS = (a-b)^2$ sq. units.

Step II It is evident from Fig. 19.2 that

Area of square $ABCD$ = Sum of the areas of four rectangles I, II, III, IV + Area of square $PQRS$

$\Rightarrow$ Area square $ABCD >$ Sum of the areas of four rectangles I, II, III, and IV

$$\Rightarrow (a+b)^2 > 4\,ab$$

$$\Rightarrow \frac{(a+b)^2}{4} > ab$$

$$\Rightarrow \left(\frac{a+b}{2}\right)^2 > ab \Rightarrow \frac{a+b}{2} > \sqrt{ab} \Rightarrow AM > GM.$$

SUMMARY

1. A sequence of non-zero numbers is called a geometric progression if the ratio of a term and the term preceding to it is always a constant quantity. The constant ratio is called the common ratio of the G.P.
2. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a G.P., then the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called a geometric series.
3. The nth term of a G.P. with first term 'a' and common ratio 'r' is given by $a_n = a\,r^{n-1}$.
4. If a G.P. consists of m terms, then n^{th} term from the end is $(m-n+1)^{th}$ term from the beginning and is given by ar^{m-n}.

   If l is the last term of a G.P., then nth term from the end is given by $l\left(\frac{1}{r}\right)^{n-1}$.
5. In a G.P., the product of the terms equidistant from the beginning and the end is always same and is equal to the product of first and last term.
6. It is always convenient to select the terms of a G.P. in the following manner:

No. of terms	*Terms*	*Common ratio*
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r

7. If sum of n terms of a G.P. with first term 'a' and common ratio is given by

$$S_n = a\left(\frac{r^n-1}{r-1}\right) \text{ or, } S_n = a\left(\frac{1-r^n}{1-r}\right), \text{ if } r \neq 1$$

$$S_n = n, \quad \text{if } r = 1$$

Also, $S_n = \dfrac{a-lr}{1-r}$ or, $S_n = \dfrac{lr-a}{r-1}$, where l is the last term.

8. If all the terms of G.P. be multiplied or divided by the same non-zero constant, then it remains a G.P. with the same common ratio.
9. The reciprocals of the terms of a given G.P. form a G.P.
10. If each term of a G.P. be raised to the same power the resulting sequence also forms a G.P.
11. Three numbers a, b, c are in G.P. iff $b^2 = ac$. If a, b, c are in G.P., then b is known as the geometric mean of a and c.
12. If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.
13. Let a and b be two given numbers. If n numbers $G_1, G_2, G_3, \ldots, G_n$ are inserted between a and b such that the sequence $a, G_1, G_2, \ldots, G_n, b$ is a G.P., then the numbers $G_1, G_2, G_3, \ldots, G_n$ are known as n geometric means between a and b.

The common ratio of the G.P. is given by $r = \left(\frac{b}{a}\right)^{1/n+1}$.

14. The geometric mean of a and b is given by $\sqrt{ab}$.

15. If n geometric means are inserted between two quantities, then the product of n geometric means is n^{th} power of the single geometric mean between the two quantities.

16. If A and G are respectively arithmetic and geometric means between two positive numbers a and b, then
    (i) $A > G$
    (ii) the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$
    (iii) $a : b = \left(A + \sqrt{A^2 - G^2}\right) : \left(A - \sqrt{A^2 - G^2}\right)$

17. If AM and GM between two numbers are in the ratio $m : n$, then the numbers are in the ratio $m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$.

CHAPTER 20

SOME SPECIAL SERIES

20.1 SUM TO n TERMS OF SOME SPECIAL SERIES

In this chapter, we intend to discuss the sum to n terms of some other special series viz. series of natural numbers, series of square of natural numbers, series of cubes of natural numbers etc.

20.1.1 SUM OF FIRST n NATURAL NUMBERS

THEOREM *Prove that* : $1+2+3+\ldots+n = \frac{n(n+1)}{2}$.

PROOF Let $S_n = 1+2+3+\ldots+n = \sum_{k=1}^{n} k$

Clearly, it is an arithmetic series with first term $a=1$, common difference $d=1$ and last term $l=n$.

$$\therefore \quad S_n = \frac{n}{2}(1+n) = \frac{n(n+1)}{2} \qquad \left[\text{Using: } S_n = \frac{n}{2}(a+l)\right]$$

Hence, $\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$

20.1.2 SUM OF THE SQUARES OF FIRST n NATURAL NUMBERS

THEOREM *Prove that* : $1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$.

PROOF Consider the identity $(x+1)^3 - x^3 = 3x^2+3x+1$

Putting $x = 1, 2, 3, \ldots, (n-1)$ and n successively, we get

$$\begin{aligned}
2^3 - 1^3 &= 3.1^2 + 3.1 + 1 \\
3^3 - 2^3 &= 3.2^2 + 3.2 + 1 \\
4^3 - 3^3 &= 3.3^2 + 3.3 + 1 \\
\ldots \quad \ldots & \quad \ldots \quad \ldots \quad \ldots \\
\ldots \quad \ldots & \quad \ldots \quad \ldots \quad \ldots \\
n^3 - (n-1)^3 &= 3.(n-1)^2 + 3.(n-1) + 1 \\
(n+1)^3 - n^3 &= 3.n^2 + 3.n + 1
\end{aligned}$$

Adding column wise, we obtain

$$(n+1)^3 - 1^3 = 3(1^2+2^2+\ldots+n^2) + 3(1+2+3+\ldots+n) + \underbrace{(1+1+\ldots+1)}_{n \text{ terms}}$$

$$\Rightarrow \quad (n+1)^3 - 1^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n$$

$$\Rightarrow \quad n^3 + 3n^2 + 3n = 3\sum_{k=1}^{n} k^2 + 3\frac{n(n+1)}{2} + n \qquad \left[\because \sum_{k=1}^{n} k = \frac{n(n+1)}{2}\right]$$

$$\Rightarrow \quad 3\sum_{k=1}^{n} k^2 = n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} - n$$

$$\Rightarrow \quad 3\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{2} = \frac{n(n+1)(2n+1)}{2}$$

$$\Rightarrow \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Hence, $\displaystyle\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

20.1.3 SUM OF THE CUBES OF FIRST n NATURAL NUMBERS

THEOREM *Prove that :* $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$.

PROOF Consider the identity

$$(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1 \qquad \ldots(i)$$

Putting $x = 1, 2, 3, \ldots, (n-1)$ and n successively, we get

$$2^4 - 1^4 = 4.1^3 + 6.1^2 + 4.1 + 1$$

$$3^4 - 2^4 = 4.2^3 + 6.2^2 + 4.2 + 1$$

$$4^4 - 3^4 = 4.3^3 + 6.3^2 + 4.3 + 1$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$$

$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$$

Adding column wise, we get

$$(n+1)^4 - 1^4 = 4(1^3 + 2^3 + \ldots + n^3) + 6(1^2 + 2^2 + 3^2 + \ldots + n^2) + 4(1 + 2 + 3 + \ldots + n) + \underbrace{(1 + 1 + \ldots + 1)}_{n \text{ terms}}$$

$$\Rightarrow \quad n^4 + 4n^3 + 6n^2 + 4n = 4\left(\sum_{k=1}^{n} k^3\right) + 6\left(\sum_{k=1}^{n} k^2\right) + 4\left(\sum_{k=1}^{n} k\right) + n$$

$$\Rightarrow \quad n^4 + 4n^3 + 6n^2 + 4n = 4\left(\sum_{k=1}^{n} k^3\right) + 6\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 4\left\{\frac{n(n+1)}{2}\right\} + n$$

$$\Rightarrow \quad 4\left(\sum_{k=1}^{n} k^3\right) = n^4 + 4n^3 + 6n^2 + 4n - n(n+1)(2n+1) - 2n(n+1) - n$$

$$\Rightarrow \quad 4\left(\sum_{k=1}^{n} k^3\right) = n^4 + 2n^3 + n^2 = n^2(n+1)^2$$

$$\Rightarrow \quad \sum_{k=1}^{n} k^3 = \frac{n^2 (n+1)^2}{4}$$

$$\Rightarrow \quad \sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = \left(\sum_{k=1}^{n} k\right)^2$$

Hence, $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = \left(\sum_{k=1}^{n} k\right)^2.$$

REMARK 1 *Sometimes for the sake of convenience the sum of a sequence is also denoted by putting the Greek letter Σ (Sigma) before its general term. For example,* $1 + 2 + 3 + \ldots + n$ *can be written as* Σn, $1^2 + 2^2 + \ldots + n^2$ *is denoted by* Σn^2 *and* $1^3 + 2^3 + \ldots + n^3$ *by* Σn^3.

Thus, we have

$$\Sigma n = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}, \quad \Sigma n^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Sigma n^3 = 1^3 + 2^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2, \text{ and, } \Sigma a = \underbrace{a + a + \ldots + a}_{(n \text{ terms})} = na$$

REMARK 2 *Proceeding as above, we also obtain*

$$\Sigma n^4 = 1^4 + 2^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$$

To find the sum to n terms of a given series of natural numbers, we may follow the following algorithm:

ALGORITHM

Step I *Write nth term of the given series.*

Step II *Simplify nth term and express it as a polynomial in n i.e.* $T_n = an^3 + bn^2 + cn + d$

Step III *Take the summation from 1 to n.*

$$\text{i.e. } \sum_{k=1}^{n} T_k = a\left(\sum_{k=1}^{n} k^3\right) + b\left(\sum_{k=1}^{n} k^2\right) + c\left(\sum_{k=1}^{n} k\right) + \sum_{k=1}^{n} d.$$

Step IV *Use the formulae for* $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$ *and* $\sum_{k=1}^{n} k^3$ *and obtain the sum.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the sum to n terms of the series* $1^2 + 3^2 + 5^2 + \ldots$ *to n terms.*

SOLUTION Let T_n be the nth term of this series and S_n denote the sum of its n terms. Then,

$$T_n = [1 + (n-1) \times 2]^2 = (2n-1)^2 = 4n^2 - 4n + 1$$

and, $S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (4k^2 - 4k + 1)$

$$\Rightarrow \quad S_n = 4\left(\sum_{k=1}^{n} k^2\right) - 4\left(\sum_{k=1}^{n} k\right) + \sum_{k=1}^{n} 1 = 4\,\frac{n(n+1)(2n+1)}{6} - 4\left\{\frac{n(n+1)}{2}\right\} + n$$

$$\Rightarrow \quad S_n = \frac{n}{3}\{2(n+1)(2n+1) - 6(n+1) + 3\} = \frac{n}{3}\{4n^2 + 6n + 2 - 6n - 6 + 3\} = \frac{n}{3}(4n^2 - 1)$$

EXAMPLE 2 *Find the sum of the series* $2^2 + 4^2 + 6^2 + \ldots + (2n)^2$

SOLUTION Let T_n be the nth term of this series and S_n denote the sum of its n terms. Then,

$$T_n = (2n)^2 = 4n^2$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} 4k^2 = 4\sum_{k=1}^{n} k^2 = 4\left\{\frac{n(n+1)(2n+1)}{6}\right\} = \frac{2}{3}n(n+1)(2n+1)$$

EXAMPLE 3 *Find the sum to n terms of the series* 1.2.3 + 2.3.4 + 3.45 + … [NCERT]

SOLUTION Let T_n the nth term of the given series. Then,

T_n = (nth term of the sequence formed by first digits in each term)
× (nth term of the sequence of second digits in each term)
× (nth term of the sequence of third digits in each term)

$$\Rightarrow \quad T_n = (n\text{th term of } 1, 2, 3, \ldots) \times (n\text{th term of } 2, 3, 4, \ldots) \times (n\text{th term of } 3, 4, 5, \ldots)$$

$$\Rightarrow \quad T_n = \{1 + (n-1) \times 1\} \times \{2 + (n-1) \times 1\} \times \{3 + (n-1) \times 1\}$$

$$\Rightarrow \quad T_n = n(n+1)(n+2)$$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(k+1)(k+2) = \sum_{k=1}^{n} (k^3 + 3k^2 + 2k)$$

$$\Rightarrow \quad S_n = \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right)$$

$$\Rightarrow \quad S_n = \left\{\frac{n(n+1)}{2}\right\}^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + (2n+1) + 2\right\} = \frac{n(n+1)}{4}\left\{n^2 + n + 4n + 2 + 4\right\}$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{4}(n^2 + 5n + 6) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

EXAMPLE 4 *Find the sum of n terms of the series* $1 \,.\, 2^2 + 2 \,.\, 3^2 + 3 \,.\, 4^2 + \ldots$

SOLUTION Let T_n be the nth term of the given series. Then,

T_n = (nth term of the sequence formed by first digits in each term)
× (nth term of the sequence formed by second digits in each term)

$$\Rightarrow \quad T_n = (n\text{th term of } 1, 2, 3, \ldots) \times (n\text{th term of } 2^2, 3^2, 4^2 \ldots)$$

$$\Rightarrow \quad T_n = n(n+1)^2 = n^3 + 2n^2 + n$$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (k^3 + 2k^2 + k) = \sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

$$\Rightarrow \quad S_n = \left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \left\{\frac{n(n+1)}{2}\right\}$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2}{3}(2n+1) + 1\right\} = \frac{n(n+1)}{2}\left\{\frac{3n^2 + 3n + 8n + 4 + 6}{6}\right\}$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{2}\left\{\frac{3n^2 + 11n + 10}{6}\right\} = \frac{n(n+1)(n+2)(3n+5)}{12}$$

EXAMPLE 5 *Sum the series 3.8 + 6.11 + 9.14 + ... to n terms.* [NCERT]

SOLUTION Let T_n be the nth term of the given series. Then,

$$T_n = (n\text{th term of } 3, 6, 9, \ldots) \times (n\text{th term of } 8, 11, 14, \ldots)$$

$$\Rightarrow \quad T_n = [3 + (n-1) \times 3] \times [8 + (n-1) \times 3] = 3n(3n+5) = 9n^2 + 15n$$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (9k^2 + 15k) = 9 \sum_{k=1}^{n} k^2 + 15 \sum_{k=1}^{n} k$$

$$\Rightarrow \quad S_n = 9\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 15\left\{\frac{n(n+1)}{2}\right\} = \frac{3}{2} n(n+1)[2n+1+5] = 3n(n+1)(n+3)$$

EXAMPLE 6 *Find the sum of n terms of the series whose nth term is*

(i) $2n^2 - 3n + 5$ (ii) $n^2 + 2^n$ [NCERT]

SOLUTION (i) We have, $T_n = 2n^2 - 3n + 5$.

Let S_n denote the sum of n terms of the series whose nth term is T_n. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (2k^2 - 3k + 5) = 2 \sum_{k=1}^{n} k^2 - 3 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 5$$

$$\Rightarrow \quad S_n = 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} - 3\left\{\frac{n(n+1)}{2}\right\} + 5n$$

$$\Rightarrow \quad S_n = \frac{n}{6}\left\{2(n+1)(2n+1) - 9(n+1) + 30\right\} = \frac{n}{6}\left(4n^2 + 6n + 2 - 9n - 9 + 30\right)$$

$$\Rightarrow \quad S_n = \frac{n}{6}(4n^2 - 3n + 23)$$

(ii) We have, $T_n = n^2 + 2^n$

Let S_n denote the sum of n terms of the series having T_n as its nth term. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (k^2 + 2^k) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k$$

$$\Rightarrow \quad S_n = \frac{n(n+1)(2n+1)}{6} + (2^1 + 2^2 + 2^3 + \ldots + 2^n)$$

$$\Rightarrow \quad S_n = \frac{n(n+1)(2n+1)}{6} + 2\left(\frac{2^n - 1}{2-1}\right) = \frac{n(n+1)(2n+1)}{6} + 2(2^n - 1)$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 7 *Find the sum of the following series to n terms:*

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots$$ [NCERT]

SOLUTION Let T_n be the nth term of the given series. Then,

$$T_n = \frac{1^3+2^3+3^3+\ldots+n^3}{1+3+5+\ldots+(2n-1)} = \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}\left\{1+(2n-1)\right\}} = \frac{(n+1)^2}{4} = \frac{1}{4}(n^2+2n+1)$$

Let S_n denote the sum of n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{4}(k^2+2k+1) = \frac{1}{4}\left\{\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right\}$$

$$\Rightarrow \quad S_n = \frac{1}{4}\left\{\frac{n(n+1)(2n+1)}{6} + 2\left(\frac{n(n+1)}{2}\right) + n\right\} = \frac{n}{24}(2n^2+9n+13)$$

EXAMPLE 8 *Show that:* $\dfrac{1\times 2^2+2\times 3^2+\ldots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\ldots+n^2\times(n+1)} = \dfrac{3n+5}{3n+1}$. **[NCERT]**

SOLUTION Let T_n and T_n' be the nth terms of the series in numerator and denominator of LHS. Then,

$$T_n = n\text{th term of the series } 1\times 2^2+2\times 3^2+\ldots+n\times(n+1)^2$$

$$\Rightarrow \quad T_n = n(n+1)^2 = n^3+2n^2+n$$

and, $$T_n' = n\text{th term of the series } 1^2\times 2+2^2\times 3+\ldots+n^2\times(n+1)$$

$$\Rightarrow \quad T_n' = n^2(n+1) = n^3+n^2$$

$$\therefore \quad \text{LHS} = \frac{\sum_{k=1}^{n} T_k}{\sum_{k=1}^{n} T_k'} = \frac{\sum_{k=1}^{n}(k^3+2k^2+k)}{\sum_{k=1}^{n}(k^3+k^2)} = \frac{\sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k}{\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2}$$

$$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \left\{\frac{n(n+1)}{2}\right\}}{\left\{\frac{n(n+1)}{2}\right\}^2 + \left\{\frac{n(n+1)(2n+1)}{6}\right\}}$$

$$= \frac{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2(2n+1)}{3} + 1\right\}}{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{(2n+1)}{3}\right\}}$$

$$= \frac{\frac{3n^2+3n+8n+4+6}{6}}{\frac{3n^2+3n+4n+2}{6}} = \frac{3n^2+11n+10}{3n^2+7n+2} = \frac{(3n+5)(n+2)}{(3n+1)(n+2)} = \frac{3n+5}{3n+1} = \text{RHS}$$

EXAMPLE 9 *If S_1, S_2, S_3 are the sums of first n natural numbers, their squares, their cubes respectively, show that* $9S_2^2 = S_3(1+8S_1)$. **[NCERT]**

SOLUTION We have,

$$S_1 = \sum_{k=1}^{n} k = 1+2+\ldots+n = \frac{n(n+1)}{2}$$

$$S_2 = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

and, $$S_3 = \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$\therefore$ $$9S_2^2 = 9\left\{\frac{n(n+1)(2n+1)}{6}\right\}^2 = \frac{9}{36}\left\{n(n+1)(2n+1)\right\}^2 = \frac{1}{4}\left\{n(n+1)(2n+1)\right\}^2 \quad \ldots \text{(i)}$$

and, $$S_3(1+8S_1) = \left\{\frac{n(n+1)}{2}\right\}^2 \left\{1 + 8 \times \frac{n(n+1)}{2}\right\} = \left\{\frac{n(n+1)}{2}\right\}^2 \left(4n^2 + 4n + 1\right)$$

$\Rightarrow$ $$S_3(1+8S_1) = \frac{n^2(n+1)^2(2n+1)^2}{4} = \frac{1}{4}\left\{n(n+1)(2n+1)\right\}^2 \quad \ldots \text{(ii)}$$

From (i) and (ii), we obtain $9S_2^2 = S_3(1+8S_1)$.

EXAMPLE 10 *Find the sum to n terms of the series:* $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$ [NCERT]

SOLUTION Let T_n be the nth term of the given series. Then,

$$T_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}(2n^3 + 3n^2 + n)$$

Let S_n be the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{6}(2k^3 + 3k^2 + k) = \frac{2}{6}\sum_{k=1}^{n} k^3 + \frac{3}{6}\sum_{k=1}^{n} k^2 + \frac{1}{6}\sum_{k=1}^{n} k$$

$\Rightarrow$ $$S_n = \frac{1}{3}\left(\sum_{k=1}^{n} k^3\right) + \frac{1}{2}\left(\sum_{k=1}^{n} k^2\right) + \frac{1}{6}\left(\sum_{k=1}^{n} k\right)$$

$\Rightarrow$ $$S_n = \frac{1}{3}\left\{\frac{n(n+1)}{2}\right\}^2 + \frac{1}{2}\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \frac{1}{6}\left\{\frac{n(n+1)}{2}\right\}$$

$\Rightarrow$ $$S_n = \frac{n(n+1)}{12}\left\{n(n+1) + (2n+1) + 1\right\} = \frac{n(n+1)}{12}(n^2 + 3n + 2) = \frac{n}{12}(n+1)^2(n+2).$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 11 *The sequence N of natural numbers is divided into classes as follows:*

$$\begin{array}{cccccc} & & 1 & 2 & & \\ & 3 & 4 & 5 & 6 & \\ 7 & 8 & 9 & 10 & 11 & 12 \\ \end{array}$$

....

....

Show that the sum of the numbers in nth row is $n(2n^2 + 1)$.

SOLUTION Since the first row consists of 2 natural numbers, second row 4 natural numbers, third row 6 natural numbers and so on. So, the total number of natural numbers in nth row is $2n$. Now,

Total number of natural numbers upto the end of nth row

$$= 2 + 4 + 6 + \ldots + 2n = 2(1 + 2 + \ldots + n) = \frac{2n(n+1)}{2} = n(n+1).$$

$\therefore$ Total number of natural numbers upto the end of $(n-1)$th row $= (n-1)(n-1+1) = n(n-1)$.

Let S_n denote the sum of first n natural numbers. Then,

Sum of the natural numbers in nth row

$= $ Sum of the natural numbers upto the end of nth row

$-$ Sum of the natural numbers upto the end of $(n-1)$th row

$$= S_{n(n+1)} - S_{n(n-1)} = S_m - S_p, \text{ where } m = n(n+1),\ p = n(n-1).$$

$$= \frac{m(m+1)}{2} - \frac{p(p+1)}{2} \qquad \left[\text{Using: } S_n = \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)\{n(n+1)+1\}}{2} - \frac{n(n-1)\{n(n-1)+1\}}{2}$$

$$= \frac{n}{2}\left\{(n+1)(n^2+n+1) - (n-1)(n^2-n+1\right\} = \frac{n}{2}(4n^2+2) = n(2n^2+1).$$

EXAMPLE 12 *If* $S_k = \dfrac{1+2+\ldots+k}{k}$, *find the value of* $S_1{}^2 + S_2{}^2 + \ldots + S_n{}^2$.

SOLUTION We have,

$$S_k = \frac{1+2+\ldots+k}{k} = \frac{k(k+1)}{2k} = \frac{k+1}{2}$$

$$\therefore \quad S_1{}^2 + S_2{}^2 + \ldots + S_n{}^2 = \sum_{k=1}^{n} S_k{}^2 = \sum_{k=1}^{n}\left(\frac{k+1}{2}\right)^2 = \frac{1}{4}\sum_{k=1}^{n}(k+1)^2$$

$$= \frac{1}{4}\left\{2^2 + 3^2 + \ldots + (n+1)^2\right\} = \frac{1}{4}\left\{1^2 + 2^2 + \ldots + (n+1)^2 - 1^2\right\}$$

$$= \frac{1}{4}\left\{\frac{(n+1)(n+1+1)\{2(n+1)+1\}}{6} - 1\right\}$$

$$= \frac{1}{4}\left\{\frac{(n+1)(n+2)(2n+3)}{6} - 1\right\} = \frac{n}{24}(2n^2+9n+13)$$

EXAMPLE 13 *Sum to n terms of the series:* $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots$

SOLUTION Clearly, nth term of the given series is negative or positive according as n is even or odd respectively. So, the following cases arise:

Case I *When n is even :* In this case the given series is

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots + (n-1)^2 - n^2$$

$$= (1^2 - 2^2) + (3^2 - 4^2) + (5^2 - 6^2) + \ldots + ((n-1)^2 - n^2)$$

$$= (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + \ldots + \left\{(n-1) - (n)\right\}(n-1+n)$$

$$= -(1+2+3+4+\ldots+(n-1)+n) = -\frac{n(n+1)}{2}.$$

Case II *When n is odd :* In this case the given series is

$$(1^2 - 2^2) + (3^2 - 4^2) + \ldots + \{(n-2)^2 - (n-1)^2\} + n^2$$

$$= (1-2)(1+2) + (3-4)(3+4) + \ldots + \left\{(n-2) - (n-1)\right\}\left\{(n-2) + (n-1)\right\} + n^2$$

$$= -\{1+2+3+4+\ldots+(n-2)+(n-1)\}+n^2 = -\frac{(n-1)(n-1+1)}{2}+n^2 = \frac{n(n+1)}{2}.$$

EXAMPLE 14 *Find the sum of all possible products of the first n natural numbers taken two by two.*

SOLUTION We know that

$(x_1+x_2+\ldots+x_n)^2 = (x_1{}^2+x_2{}^2+\ldots+x_n{}^2)+2$ (Sum of all possible products taken two at a time)

or, $$\left(\sum_{i=1}^{n} x_i\right)^2 = \left(\sum_{i=1}^{n} x_i{}^2\right)+2\left(\sum_{i=1,\, i<j}^{n}\ \sum_{j=1}^{n} x_i\, x_j\right)$$

$$\Rightarrow \quad \sum_{i=1,\, i<j}^{n}\ \sum_{j=1}^{n} x_i\, x_j = \frac{1}{2}\left\{\left(\sum_{i=1}^{n} x_i\right)^2 - \left(\sum_{j=1}^{n} x_i^2\right)\right\}$$

Let S denote the required sum. Then,

$$S = \frac{1}{2}\left\{\left(\sum_{k=1}^{n} k\right)^2 - \left(\sum_{k=1}^{n} k^2\right)\right\} = \frac{1}{2}\left[\left\{\frac{n(n+1)}{2}\right\}^2 - \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{1}{2}\left[\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2}-\frac{2n+1}{3}\right\}\right] = \frac{n(n+1)}{4}\left\{\frac{3n^2+3n-4n-2}{6}\right\}$$

$$= \frac{n(n+1)(3n^2-n-2)}{24} = \frac{n(n+1)(n-1)(3n+2)}{24}$$

EXAMPLE 15 *Find the sum of the series:* $1.n+2.(n-1)+3.(n-2)+\ldots+(n-1).2+n.1$.

SOLUTION Let T_r be the rth term of the given series and S be its sum. Then,

$$T_r = r\{n-(r-1)\} = r(n-r+1) = r\{(n+1)-r\} = (n+1)r-r^2$$

And, $$S = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left\{(n+1)r-r^2\right\} = (n+1)\left(\sum_{r=1}^{n} r\right)-\left(\sum_{r=1}^{n} r^2\right)$$

$$= (n+1)\,\frac{n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(n+2)}{6}$$

EXAMPLE 16 *Find the sum of the series* $(3^3-2^3)+(5^3-4^3)+(7^3-6^3)+\ldots\ldots$ *to n terms*

[NCERT EXEMPLAR]

SOLUTION Let S be the sum of the given series. Then,

$$S = (3^3-2^3)+(5^3-4^3)+(7^3-6^3)+\ldots\ldots \text{ to } n \text{ terms.}$$

$$\Rightarrow \quad S = \sum_{r=1}^{n}\left\{(2r+1)^3-(2r)^3\right\} = \sum_{r=1}^{n}\left\{(2r+1)-(2r)\right\}\left\{(2r+1)^2+(2r+1)(2r)+(2r)^2\right\}$$

$$\Rightarrow \quad S = \sum_{r=1}^{n}(12r^2+6r+1) = 12\sum_{r=1}^{n} r^2+6\sum_{r=1}^{n} r+\sum_{r=1}^{n} 1$$

$\Rightarrow \quad S = 12 \times \dfrac{n(n+1)(2n+1)}{6} + 6 \times \dfrac{n(n+1)}{2} + n = 2n(n+1)(2n+1) + 3n(n+1) + n$

$\Rightarrow \quad S = n(4n^2 + 6n + 2 + 3n + 3 + 1) = n(4n^2 + 9n + 6)$

EXERCISE 20.1

BASIC

Find the sum of the following series to n terms: (1-7)

1. $1^3 + 3^3 + 5^3 + 7^3 + \ldots$
2. $2^3 + 4^3 + 6^3 + 8^3 + \ldots$
3. $1.2.5 + 2.3.6 + 3.4.7 + \ldots$
4. $1.2.4 + 2.3.7 + 3.4.10 + \ldots$

BASED ON LOTS

5. $1 + (1+2) + (1+2+3) + (1+2+3+4) + \ldots$
6. $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \ldots$ [NCERT]
7. $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots$ [NCERT]
8. Find the sum of the series whose nth term is:
   (i) $2n^3 + 3n^2 - 1$ (ii) $n^3 - 3^n$ (iii) $n(n+1)(n+4)$ [NCERT] (iv) $(2n-1)^2$ [NCERT]
9. Find the 20th term and the sum of 20 terms of the series:
   $2 \times 4 + 4 \times 6 + 6 \times 8 + \ldots$ [NCERT]

ANSWERS

1. $n^2(2n^2 - 1)$
2. $2\{n(n+1)\}^2$
3. $\dfrac{n}{12}(n+1)(3n^2 + 23n + 34)$
4. $\dfrac{n}{12}(n+1)(9n^2 + 25n + 14)$
5. $\dfrac{n(n+1)(n+2)}{6}$
6. $\dfrac{n}{3}(n+1)(n+2)$
7. $\dfrac{n}{6}(n+1)(3n^2 + 5n + 1)$
8. (i) $\dfrac{n}{2}(n^3 + 4n^2 + 4n - 1)$ (ii) $\left\{\dfrac{n(n+1)}{2}\right\}^2 - \dfrac{3}{2}(3^n - 1)$
   (iii) $\dfrac{n(n+1)}{12}(3n^2 + 23n + 34)$ (iv) $\dfrac{n}{3}(2n+1)(2n-1)$
9. 1680, 12320

HINTS TO SELECTED PROBLEMS

1. Clearly, $T_n = (2n-1)^3$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (2k-1)^3 = 8\sum_{k=1}^{n} k^3 - 12\sum_{k=1}^{n} k^2 + 6\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

2. Clearly, $T_n = (2n)^3$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (2k)^3 = 8\sum_{k=1}^{n} k^3 = 8\left\{\frac{n(n+1)}{2}\right\}^2 = 2n^2(n+1)^2$$

6. Let T_r be the r^{th} term of the given series and S_n denote the sum of its n terms. Then,

$T_r = r(r+1), \; r = 1, 2, 3, \ldots$

$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} (r^2 + r) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$\Rightarrow \quad S_n = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(2n+4)}{6} = \frac{n(n+1)(n+2)}{3}$

7. Let T_r be the r^{th} term of the series and S_n be the sum of its n terms. Then,

$T_r = (2r+1)r^2, r = 1, 2, 3, \ldots n$

$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (2r+1)r^2 = \sum_{r=1}^{n} (2r^3 + r^2) = 2\sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2$

$\Rightarrow \quad S_n = 2\left\{\frac{n(n+1)}{2}\right\}^2 + \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(3n^2+5n+1)}{6}$

8. (iii) We have, $T_n = n(n+1)(n+4) = n^3 + 5n^2 + 4n$

Let S_n be the sum of n terms. Then,

$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (r^3 + 5r^2 + 4r) = \sum_{r=1}^{n} r^3 + 5\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r$

$\Rightarrow \quad S_n = \left\{\frac{n(n+1)}{2}\right\}^2 + \frac{5n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} = \frac{n(n+1)(3n^2+23n+34)}{12}$

(iv) We have, $T_n = (2n-1)^2$. Let S_n be the sum of n terms of the given series. Then,

$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (2r-1)^2 = \sum_{r=1}^{n} (4r^2 - 4r + 1) = 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$\Rightarrow \quad S_n = 4\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n = \frac{n(2n-1)(2n+1)}{3}$

9. Let T_r be the r^{th} term of the given series. Then, $T_r = 2r(2r+2) = 4r^2 + 4r, r = 1, 2, 3, \ldots$

$\therefore \quad T_{20} = 4 \times 20^2 + 4 \times 20 = 1600 + 80 = 1680$

Let S_{20} be the sum of 20 terms. Then,

$S_{20} = \sum_{r=1}^{20} T_r = \sum_{r=1}^{20} (4r^2 + 4r) = 4\left(\sum_{r=1}^{20} r^2\right) + \left(\sum_{r=1}^{20} r\right)$

$\Rightarrow \quad S_{20} = 4 \times \frac{20(20+1)(40+1)}{6} + 4 \times \frac{20(20+1)}{2} = 12320$

20.2 METHOD OF DIFFERENCE

Sometimes the *nth* term of a series can not be determined by the methods discussed so far. If a series is such that the difference between successive terms are either in A.P. or in G.P., then we determine its nth term by the method of difference and then find the sum of the series by using the formulas for Σn, Σn^2 and Σn^3. The method of difference is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Find the sum to n terms of the series:* $3 + 15 + 35 + 63 + \ldots$

SOLUTION The difference between the successive terms are $15 - 3 = 12$, $35 - 15 = 20$, $63 - 35 = 28$, ... Clearly, these differences are in A.P.

Let T_n be the *nth* term and S_n denote the sum to n terms of the given series. Then,

$$S_n = 3 + 15 + 35 + 63 + \ldots + T_{n-1} + T_n \quad \ldots(i)$$

Also, $$S_n = \quad 3 + 15 + 35 + \ldots \quad + T_{n-1} + T_n \quad \ldots(ii)$$

Subtracting (ii) from (i), we get

$$0 = 3 + \left\{12 + 20 + 28 + \ldots + (T_n - T_{n-1})\right\} - T_n$$

$$\Rightarrow \quad T_n = 3 + \frac{(n-1)}{2}\left\{2 \times 12 + (n-1-1) \times 8\right\} = 3 + (n-1)(12 + 4n - 8)$$

$$\Rightarrow \quad T_n = 3 + (n-1)(4n+4) = 4n^2 - 1$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (4k^2 - 1)$$

$$\Rightarrow \quad S_n = 4\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} 1 = 4\left\{\frac{n(n+1)(2n+1)}{6}\right\} - n = \frac{n}{3}(4n^2 + 6n - 1)$$

REMARK *Instead of determining the nth term of a series by the method of difference as discussed in the above example, we can use the following steps to obtain the same.*

Step I *Obtain the terms of the series and compute the differences between* $T_2 - T_1, T_3 - T_2, T_4 - T_3, \ldots$ *etc. If these are in A.P., then take the nth term as* $T_n = an^2 + bn + c$, *where* a, b, c *are constants. Determine constants* a, b, c *by putting* $n = 1, 2, 3$ *and equating them with the values of corresponding terms of the given series.*

Step II *If the differences* $T_2 - T_1, T_3 - T_2, T_4 - T_3, \ldots$ *are in G.P. with common ratio r, then take* $T_n = ar^{n-1} + bn + c$ *and determine constants by putting* $n = 1, 2, 3$ *in* T_n.

Step III *If the differences of the differences computed in step I are in A.P., then take* $T_n = an^3 + bn^2 + cn + d$ *and find the values of* a, b, c, d *by putting* $n = 1, 2, 3, 4$.

Step IV *If the differences of the differences computed in step I are in G.P. with common ratio r, then take* $T_n = ar^{n-1} + bn^2 + cn + d$ *and find the values of* a, b, c, d *by putting* $n = 1, 2, 3, 4$.

EXAMPLE 2 *Find the sum to n terms of the series:* $1 + 5 + 12 + 22 + 35 + \ldots$

SOLUTION The sequence of differences between successive terms is 4, 7, 10, 13, ..., which is clearly an A.P. Let T_n be the nth term of the sequence and S_n be the sum of its n terms. Then,

$$S_n = 1 + 5 + 12 + 22 + 35 + \ldots + T_{n-1} + T_n \quad \ldots(i)$$

Also, $$S_n = \quad 1 + 5 + 12 + 22 + \ldots \quad + T_{n-1} + T_n \quad \ldots(ii)$$

Subtracting (ii) from (i), we get

$$0 = 1 + \left\{4 + 7 + 10 + 13 + \ldots + (T_n - T_{n-1})\right\} - T_n$$

$$\Rightarrow \quad T_n = 1 + \frac{(n-1)}{2}\left\{2 \times 4 + (n-1-1) \times 3\right\} = 1 + \left(\frac{n-1}{2}\right)(3n+2) = \frac{1}{2}(3n^2 - n)$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{2}(3k^2 - k) = \frac{3}{2}\sum_{k=1}^{n} k^2 - \frac{1}{2}\sum_{k=1}^{n} k$$

$$= \frac{3}{2}\left\{\frac{n(n+1)(2n+1)}{6}\right\} - \frac{1}{2}\left\{\frac{n(n+1)}{2}\right\} = \frac{n^2(n+1)}{2}$$

ALITER The given series is : $1 + 5 + 12 + 22 + 35 + \ldots$

The sequence of differences between successive terms is : $4, 7, 10, 13, \ldots$ Clearly, it is an A.P. So, let the nth term T_n of the given series be

$$T_n = an^2 + bn + c \qquad \text{...(i)}$$

Putting $n = 1, 2, 3$ successively we obtain

$$T_1 = a + b + c \Rightarrow a + b + c = 1 \qquad [\because T_1 = 1]$$
$$T_2 = 4a + 2b + c \Rightarrow 4a + 2b + c = 5 \qquad [\because T_2 = 5]$$
$$T_3 = 9a + 3b + c \Rightarrow 9a + 3b + c = 12 \qquad [\because T_3 = 12]$$

Solving these equations, we get: $a = \frac{3}{2}, b = -\frac{1}{2}$ and $c = 0$.

Substituting the values of a, b, c in (i), we get: $T_n = \frac{3}{2}n^2 - \frac{1}{2}n = \frac{1}{2}(3n^2 - n)$

$$\therefore \text{ Sum of the given series } = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} \frac{1}{2}(3r^2 - r) = \frac{3}{2}\sum_{r=1}^{n} r^2 - \frac{1}{2}\sum_{r=1}^{n} r$$

$$= \frac{3}{2}\left\{\frac{n(n+1)(2n+1)}{6}\right\} - \frac{1}{2}\left\{\frac{n(n+1)}{2}\right\} = \frac{n^2(n+1)}{2}$$

EXAMPLE 3 *Find the sum of first n terms of the following series:*

(i) $3 + 7 + 13 + 21 + 31 + \ldots$ [NCERT] (ii) $5 + 11 + 19 + 29 + 41 + \ldots$ [NCERT]

SOLUTION (i) The given series is: $3 + 7 + 13 + 21 + 31 + \ldots$

The sequence of the differences between the successive terms of this series is $4, 6, 8, 10, \ldots$ Clearly, it is an A.P. with common difference 2. So, let the nth term of the given series be

$$T_n = an^2 + bn + c \qquad \text{...(i)}$$

Putting $n = 1, 2, 3$, we get

$$T_1 = a + b + c \Rightarrow a + b + c = 3 \qquad [\because T_1 = 3]$$
$$T_2 = 4a + 2b + c \Rightarrow 4a + 2b + c = 7 \qquad [\because T_2 = 7]$$
$$T_3 = 9a + 3b + c \Rightarrow 9a + 3b + c = 13 \qquad [\because T_3 = 13]$$

Solving these equations, we get: $a = b = c = 1$

$$\therefore \quad T_n = n^2 + n + 1$$

The sum S_n of terms of the given series is given by

$$S = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (r^2 + r + 1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n = \frac{n}{3}(n^2 + 3n + 5)$$

(ii) The given series is: $5 + 11 + 19 + 29 + 41 + \ldots$. The sequence of the differences between the successive terms is: $6, 8, 10, 12, \ldots$. Clearly, it is an A.P. So, nth term of the given series is given by

$$T_n = an^2 + bn + c \quad \text{...(i)}$$

Putting $n = 1, 2, 3$, we get

$$T_1 = a + b + c \Rightarrow a + b + c = 5 \quad [\because T_1 = 5]$$
$$T_2 = 4a + 2b + c \Rightarrow 4a + 2b + c = 11 \quad [\because T_2 = 11]$$
$$T_3 = 9a + 3b + c \Rightarrow 9a + 3b + c = 19 \quad [\because T_3 = 19]$$

Solving these three equations, we get : $a = 1, b = 3$ and $c = 1$.

$$\therefore \quad T_n = n^2 + 3n + 1$$

Let S be the sum of n terms of the given series. Then,

$$S = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (r^2 + 3r + 1) = \sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + n = \frac{n(n+2)(n+4)}{3}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Sum the following series to n terms:* $5 + 7 + 13 + 31 + 85 + \ldots$

SOLUTION The sequence of differences between successive terms is $2, 6, 18, 54, \ldots$ Clearly, it is a G.P. Let T_n be the nth term of the given series and S_n be the sum of its n terms. Then,

$$S_n = 5 + 7 + 13 + 31 + 85 + \ldots + T_{n-1} + T_n \quad \text{...(i)}$$

Also, $S_n = 5 + 7 + 13 + 31 + \ldots + T_{n-1} + T_n$ ...(ii)

Subtracting (ii) from (i), we get

$$0 = 5 + \left\{2 + 6 + 18 + 54 + \ldots + (T_n - T_{n-1})\right\} - T_n$$

$$\Rightarrow \quad 0 = 5 + \frac{2(3^{n-1} - 1)}{(3-1)} - T_n \Rightarrow T_n = 5 + (3^{n-1} - 1) = 4 + 3^{n-1}$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (4 + 3^{k-1}) = \sum_{k=1}^{n} 4 + \sum_{k=1}^{n} 3^{k-1}$$

$$\Rightarrow \quad S_n = 4n + (1 + 3 + 3^2 + \ldots + 3^{n-1})$$

$$\Rightarrow \quad S_n = 4n + 1 \times \left(\frac{3^n - 1}{3 - 1}\right) = 4n + \left(\frac{3^n - 1}{2}\right) = \frac{1}{2}\left(3^n + 8n - 1\right)$$

ALITER The given series is: $5 + 7 + 13 + 31 + 85 + \ldots$ The sequence of the differences between successive terms is $2, 6, 18, 54, \ldots$ Clearly, it is a G.P. with common ratio 3. So, let the nth term of the given series be

$$T_n = a \cdot 3^{n-1} + bn + c \quad \text{...(i)}$$

Putting $n = 1, 2, 3$, we get

$$T_1 = a + b + c \Rightarrow a + b + c = 5 \quad [\because T_1 = 5]$$
$$T_2 = 3a + 2b + c \Rightarrow 3a + 2b + c = 7 \quad [\because T_2 = 7]$$
$$T_3 = 9a + 3b + c \Rightarrow 9a + 3b + c = 13 \quad [\because T_3 = 13]$$

Solving these equations, we get: $a = 1, b = 0$ and $c = 4$. Substituting the values of a, b, c in (i), we get

$$T_n = 3^{n-1} + 4$$

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (3^{r-1} + 4) = \sum_{r=1}^{n} 3^{r-1} + \sum_{r=1}^{n} 4$$

$$= (1 + 3 + 3^2 + \ldots + 3^{n-1}) + 4n = \frac{3^n - 1}{3 - 1} + 4n = \frac{3^n - 1}{2} + 4n = \frac{1}{2}(3^n - 1 + 8n)$$

20.3 SUM OF SOME SPECIAL SERIES

In this section, we shall discuss some problems for finding the sum of some series of the form

$$\frac{1}{a(a+d)} + \frac{1}{(a+d)(a+2d)} + \frac{1}{(a+2d)(a+3d)} + \ldots + \ldots + \frac{1}{(a+(n-2)d)(a+(n-1)d)}$$

In such kind of series the successive terms are reciprocals of two consecutive terms of an A.P. in succession. In order to find the sum of a finite number of terms of such series, we write its each term as the difference of reciprocals of two successive terms as given below:

$$\frac{1}{a(a+d)} = \frac{1}{d}\left(\frac{1}{a} - \frac{1}{a+d}\right), \quad \frac{1}{(a+d)(a+2d)} = \frac{1}{d}\left(\frac{1}{a+d} - \frac{1}{a+2d}\right),$$

$$\frac{1}{(a+2d)(a+3d)} = \frac{1}{d}\left(\frac{1}{a+2d} - \frac{1}{a+3d}\right) \quad \text{and so on.}$$

$$\therefore \quad \frac{1}{a(a+d)} + \frac{1}{(a+d)(a+2d)} + \frac{1}{(a+2d)(a+3d)} + \ldots + \frac{1}{\{a+(n-2)d\}\{a+(n-1)d\}}$$

$$= \frac{1}{d}\left\{\left(\frac{1}{a} - \frac{1}{a+d}\right) + \left(\frac{1}{a+d} - \frac{1}{a+2d}\right) + \left(\frac{1}{a+2d} - \frac{1}{a+3d}\right) + \ldots + \left(\frac{1}{a+(n-2)d} - \frac{1}{a+(n-1)d}\right)\right\}$$

$$= \frac{1}{d}\left\{\frac{1}{a} - \frac{1}{a+(n-1)d}\right\} = \frac{n-1}{a\{a+(n-1)d\}}$$

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Find the sum to n terms of the series:* $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n.(n+1)}$ [NCERT]

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = \frac{1}{r(r+1)} \Rightarrow T_r = \frac{1}{r} - \frac{1}{r+1}, \quad r = 1, 2, \ldots, n$$

The sum S of the given series is given by

$$\therefore \quad S = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

EXAMPLE 2 *Find the sum to n terms of the series:* $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots$

SOLUTION Let T_r be the rth term of the given series and S be the sum of its n terms. Then,

$$T_r = \frac{1}{(2r-1)(2r+1)} = \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right), r = 1, 2, 3, \ldots n$$

$$\therefore \quad S = \sum_{r=1}^{n} T_r = \frac{1}{2}\left\{\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right\}$$

$$= \frac{1}{2}\left\{1 - \frac{1}{2n+1}\right\} = \frac{n}{2n+1}$$

EXAMPLE 3 *Find the sum :* $\displaystyle\sum_{r=1}^{n} \frac{1}{(ar+b)(ar+a+b)}$

SOLUTION We have,

$$\sum_{r=1}^{n} \frac{1}{(ar+b)(ar+a+b)} = \sum_{r=1}^{n} \frac{1}{a}\left(\frac{1}{ar+b} - \frac{1}{ar+a+b}\right) = \frac{1}{a}\sum_{r=1}^{n}\left(\frac{1}{ar+b} - \frac{1}{ar+a+b}\right)$$

$$= \frac{1}{b}\left\{\left(\frac{1}{a+b} - \frac{1}{2a+b}\right) + \left(\frac{1}{2a+b} - \frac{1}{3a+b}\right) + \ldots + \left(\frac{1}{na+b} - \frac{1}{(n+1)a+b}\right)\right\}$$

$$= \frac{1}{a}\left\{\frac{1}{a+b} - \frac{1}{(n+1)a+b}\right\} = \frac{n}{(a+b)\{(n+1)a+b\}}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Find the sum to n terms of the series:* $\dfrac{3}{1^2.2^2} + \dfrac{5}{2^2.3^2} + \dfrac{7}{3^2.4^2} + \ldots$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = \frac{(2r+1)}{r^2(r+1)^2} = \frac{(r+1)^2 - r^2}{r^2(r+1)^2} = \left\{\frac{1}{r^2} - \frac{1}{(r+1)^2}\right\}, \ r = 1, 2, 3, \ldots$$

Let S_n be the sum to n terms of the given series. Then,

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left\{\frac{1}{r^2} - \frac{1}{(r+1)^2}\right\} = 1 - \frac{1}{(n+1)^2} = \frac{2n+n^2}{(n+1)^2}$$

EXAMPLE 5 *Find the sum to n terms of the series:* $\dfrac{1}{1+1^2+1^4} + \dfrac{2}{1+2^2+2^4} + \dfrac{3}{1+3^2+3^4} + \ldots$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = \frac{r}{1+r^2+r^4}, r = 1, 2, 3, \ldots, n$$

$$\Rightarrow \quad T_r = \frac{r}{(r^2+r+1)(r^2-r+1)} = \frac{1}{2}\left\{\frac{2r}{(r^2+r+1)(r^2-r+1)}\right\} = \frac{1}{2}\left\{\frac{(r^2+r+1)-(r^2-r+1)}{(r^2+r+1)(r^2-r+1)}\right\}$$

$$\Rightarrow \quad T_r = \frac{1}{2}\left\{\frac{1}{r^2-r+1} - \frac{1}{r^2+r+1}\right\}, r = 1, 2, \ldots, n$$

Let S_n be the sum to n terms of the given series. Then,

$$\therefore \qquad S_n = \sum_{r=1}^{n} T_r = \frac{1}{2}\left\{\sum_{r=1}^{n}\left(\frac{1}{r^2-r+1} - \frac{1}{r^2+r+1}\right)\right\}$$

$$= \frac{1}{2}\left\{\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{13}\right)+\ldots+\left(\frac{1}{n^2-n+1}-\frac{1}{n^2+n+1}\right)\right\}$$

$$= \frac{1}{2}\left\{1-\frac{1}{n^2+n+1}\right\} = \frac{n^2+n}{2(n^2+n+1)}$$

EXERCISE 20.2

BASIC

Sum the following series to n terms:

1. $3+5+9+15+23+\ldots$
2. $2+5+10+17+26+\ldots$
3. $1+3+7+13+21+\ldots$
4. $3+7+14+24+37+\ldots$
5. $1+3+6+10+15+\ldots$
6. $1+4+13+40+121+\ldots$
7. $4+6+9+13+18+\ldots$
8. $2+4+7+11+16+\ldots$
9. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots$
10. $\frac{1}{1.6}+\frac{1}{6.11}+\frac{1}{11.16}+\frac{1}{16.21}+\ldots+\frac{1}{(5n-4)(5n+1)}$

ANSWERS

1. $\frac{n}{3}(n^2+8)$
2. $\frac{n}{6}(2n^2+3n+7)$
3. $\frac{n(n^2+2)}{3}$
4. $\frac{n}{2}(n^2+n+4)$
5. $\frac{n}{6}(n+1)(n+2)$
6. $\frac{1}{4}(3^{n+1}-2n-3)$
7. $\frac{n}{6}(n^2+3n+20)$
8. $\frac{n}{6}(n^2+3n+8)$
9. $\frac{n}{3n+1}$
10. $\frac{n}{5n+1}$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If S_1 and S_2 denote respectively the sum of first 100 natural numbers and the sum of their cubes, then the relation between S_1 and S_2 is ....
2. Let S_n and S'_4 denote respectively the sum and the sum of the squares of first n natural numbers. If $a_n = \frac{S_n'}{S_n}, n \in N$. Then $a_1, a_2, a_3, \ldots\ldots, a_n, \ldots\ldots$ forms an .... with ....
3. The sum of first 25 odd natural numbers is ....
4. The value of $\sum_{r=1}^{n}\left\{(2r-1)+\frac{1}{2^r}\right\}$ is ....
5. The sum of n terms of the series $2^2+4^2+6^2+\ldots\ldots$, is ....
6. $1^4+2^4+3^4+\ldots\ldots\ldots+n^4 =$ ....
7. If S_2 and S_4 denote respectively the sum of the squares and the sum of the fourth powers of first n natural numbers, then $\frac{S_4}{S_2} =$ ....

8. The value of $\dfrac{1^3+2^3+3^3+\ldots\ldots+10^3}{1+2+3+\ldots\ldots+10}$ ....

9. If the sum of the squares of first n natural numbers exceeds their sum by 330, then $n =$ ....

10. The sum of n terms of the series $\dfrac{3}{1^2}+\dfrac{5}{1^2+2^2}+\dfrac{7}{1^2+2^2+3^2}+\ldots\ldots$ is ....

ANSWERS

1. $S_2 = S_1^2$
2. A.P. with common difference $\dfrac{2}{3}$
3. 625
4. $n^2+1-\dfrac{1}{2^n}$
5. $\dfrac{2n(n+1)(2n+1)}{3}$
6. $\dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
7. $\dfrac{3n^2+3n-1}{5}$
8. 55
9. 10
10. $\dfrac{6n}{n+1}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the sum of the series: $2+4+6+8+\ldots+2n$.
2. Write the sum of the series: $1^2-2^2+3^2-4^2+5^2-6^2+\ldots+(2n-1)^2-(2n)^2$.
3. Write the sum to n terms of a series whose r^{th} term is: $r+2^r$.
4. If $\sum\limits_{r=1}^{n} r = 55$, find $\sum\limits_{r=1}^{n} r^3$.
5. If the sum of first n even natural numbers is equal to k times the sum of first n odd natural numbers, then write the value of k.
6. Write the sum of 20 terms of the series: $1+\frac{1}{2}(1+2)+\frac{1}{3}(1+2+3)+\ldots$.
7. Write the 50th term of the series $2+3+6+11+18+\ldots$.
8. Let S_n denote the sum of the cubes of first n natural numbers and s_n denote the sum of first n natural numbers. Then, write the value of $\sum\limits_{r=1}^{n}\dfrac{S_r}{s_r}$.

ANSWERS

1. $n(n+1)$
2. $-n(2n+1)$
3. $\dfrac{n(n+1)}{2}+2^{n+1}-2$
4. 3025
5. $\dfrac{n+1}{n}$
6. 115
7. 49^2+2
8. $\dfrac{n(n+1)(n+2)}{6}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The sum to n terms of the series $\dfrac{1}{\sqrt{1}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{5}}+\dfrac{1}{\sqrt{5}+\sqrt{7}}+\ldots\ldots$ is

   (a) $\sqrt{2n+1}$ (b) $\frac{1}{2}\sqrt{2n+1}$ (c) $\sqrt{2n+1}-1$ (d) $\frac{1}{2}\left\{\sqrt{2n+1}-1\right\}$

2. The sum of the series : $\frac{1}{\log_2 4}+\frac{1}{\log_4 4}+\frac{1}{\log_8 4}+\ldots+\frac{1}{\log_{2^n} 4}$ is

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n+1)(2n+1)}{12}$ (c) $\frac{n(n+1)}{4}$ (d) none of these

3. The value of $\sum_{r=1}^{n}\left\{(2r-1)a+\frac{1}{b^r}\right\}$ is equal to

(a) $an^2+\frac{b^{n-1}-1}{b^{n-1}(b-1)}$ (b) $an^2+\frac{b^n-1}{b^n(b-1)}$

(c) $an^3+\frac{b^{n-1}-1}{b^n(b-1)}$ (d) none of these

4. If $\Sigma n=210$, then $\Sigma n^2=$

(a) 2870 (b) 2160 (c) 2970 (d) none of these

5. If $S_n=\sum_{r=1}^{n}\frac{1+2+2^2+\ldots \text{Sum to } r \text{ terms}}{2^r}$, then S_n is equal to

(a) 2^n-n-1 (b) $1-\frac{1}{2^n}$ (c) $n-1+\frac{1}{2^n}$ (d) 2^n-1

6. If $1+\frac{1+2}{2}+\frac{1+2+3}{3}+\ldots$ to n terms is S. Then, S is equal to

(a) $\frac{n(n+3)}{4}$ (b) $\frac{n(n+2)}{4}$ (c) $\frac{n(n+1)(n+2)}{6}$ (d) n^2

7. Sum of n terms of the series $\sqrt{2}+\sqrt{8}+\sqrt{18}+\sqrt{32}+\ldots$ is

(a) $\frac{n(n+1)}{2}$ (b) $2n(n+1)$ (c) $\frac{n(n+1)}{\sqrt{2}}$ (d) 1

8. The sum of 10 terms of the series $\sqrt{2}+\sqrt{6}+\sqrt{18}+\ldots$ is

(a) $121(\sqrt{6}+\sqrt{2})$ (b) $243(\sqrt{3}+1)$ (c) $\frac{121}{\sqrt{3}-1}$ (d) $242(\sqrt{3}-1)$

9. The sum of the series $1^2+3^2+5^2+\ldots$ to n terms is

(a) $\frac{n(n+1)(2n+1)}{2}$ (b) $\frac{n(2n-1)(2n+1)}{3}$ (c) $\frac{(n-1)^2(2n+1)}{6}$ (d) $\frac{(2n+1)^3}{3}$

10. The sum of the series $\frac{2}{3}+\frac{8}{9}+\frac{26}{27}+\frac{80}{81}+\ldots$ to n terms is

(a) $n-\frac{1}{2}(3^{-n}-1)$ (b) $n-\frac{1}{2}(1-3^{-n})$ (c) $n+\frac{1}{2}(3^n-1)$ (d) $n-\frac{1}{2}(3^n-1)$

11. Let S_n denote the sum of the cubes of the first n natural numbers and s_n denote the sum of the first natural numbers. Then $\sum_{r=1}^{n}\frac{S_r}{s_r}$ equals [NCERT EXEMPLAR]

(a) $\frac{n(n+1)(n+2)}{6}$ (b) $\frac{n(n+1)}{2}$ (c) $\frac{n^2+3n+2}{2}$ (d) None of these

ANSWERS

1. (d) 2. (c) 3. (b) 4. (a) 5. (c) 6. (a) 7. (c) 8. (a)
9. (b) 10. (b) 11. (a)

ACTIVITIES

ACTIVITY-1

OBJECTIVE To show that the sum of first n odd natural numbers is n^2 i.e. $\sum_{r=1}^{n}(2r-1)=n^2$.

MATERIALS REQUIRED Thermocol sheet, thermocol balls, pins, pencil, scale, adhesive, chart paper etc.

STEPS OF CONSTRUCTION

Step I Take a square sheet of thermocol and some thermocol balls.

Step II Fix a chart paper on the thermocol sheet.

Step III Draw horizontal vertical lines with pencil on the chart paper to make squares as shown in Fig 20.1.

Step IV Take a pin, fix a thermocol ball in it and fix it in the corner of first square in the top right most corner of the thermocol sheet as shown in Fig. 20.1.

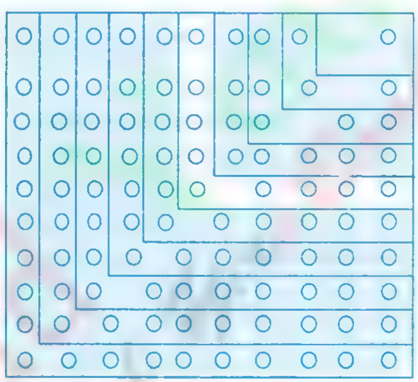

Fig. 20.1

STEPS OF DEMONSTRATION

Step I For $n=1$:

$$\sum_{r=1}^{n}(2r-1)=\sum_{r=1}^{1}(2r-1)=2\times 1-1=2-1=1$$

And, the number of balls in first square $=1=1^2$.

$$\therefore \sum_{r=1}^{1}(2r-1)=1^2$$

Step II For $n=2$:

$$\sum_{r=1}^{n}(2r-1)=\sum_{r=1}^{2}(2r-1)=(2\times 1-1)+(2\times 2-1)=1+3=4$$

And, the number of balls in the second square $=4=2^2$.

$$\therefore \quad \sum_{r=1}^{n} (2r-1) = 2^2.$$

Step III For $n = 3$:

$$\sum_{r=1}^{n} (2r-1) = \sum_{r=1}^{3} (2r-1) = (2\times1-1)+(2\times2-1)+(2\times3-1) = 1+3+5 = 9$$

And, the number of balls in the third square $= 9 = 3^2$

$$\therefore \quad \sum_{r=1}^{3} (2r-1) = 3^2$$

Continuing in this manner, we obtain

$$\therefore \quad \sum_{r=1}^{n} (2r-1) = \text{Number of balls in } n^{\text{th}} \text{ square} = n^2.$$

ACTIVITY-2

OBJECTIVE To show that the sum of the squares of first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$.

i.e. $\quad 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

MATERIALS REQUIRED Wooden cubes of size $1 \times 1 \times 1$ cubic unit, adhesive, nails etc.

STEPS OF CONSTRUCTION

Step I Take one wooden cube of size $(1 \times 1 \times 1)$ cubic unit as shown in Fig. 20.2

The volume of this cube is $= 1 = 1^3$ cubic unit.

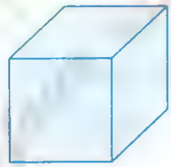

Fig. 20.2

Step II Take four wooden cubes of size $(1 \times 1 \times 1)$ cubic unit and fix them together as shown in Fig. 20.3. The volume of this structure is $= 2 \times 2 \times 1 = 2^2$ cubic units.

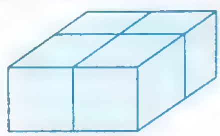

Fig. 20.3

Step III Take nine wooden cubes of size $(1 \times 1 \times 1)$ cubic units and fix them together as shown in Fig 20.4.

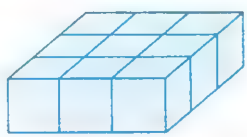

Fig. 20.4

The volume of this structure is $= 3 \times 3 \times 1 = 3^2$ cubic units.

Step IV Take 16 wooden cubes of size $(1 \times 1 \times 1)$ cubic units and fix them together as shown in Fig. 20.5.

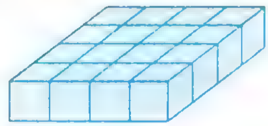

Fig. 20.5

The volume of this structure $= 4 \times 4 \times 1 = 4^2$ cubic units.

Step V Arrange the blocks formed in the above steps to form an echelon type of structure as shown in Fig. 20.6.

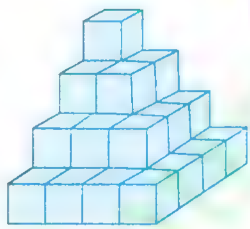

Fig. 20.6

The volume of this structure is $= (1^2 + 2^2 + 3^2 + 4^2)$ cubic units.

Step VI Make six such echelon type of pieces.

The total volume of six echelon type of pieces $= 6\,(1^2 + 2^2 + 3^2 + 4^2)$ cubic units.

Step VII Arrange six echelon type of pieces to form a bigger cuboidal structure of dimension $4 \times 5 \times 9$ units as shown in Fig. 20.7.

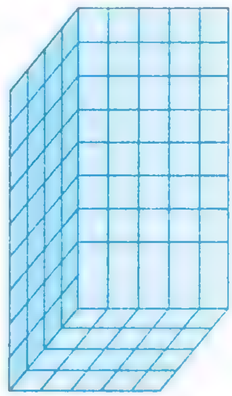

Fig. 20.7

The volume of the cuboidal block so formed $= 4 \times 5 \times 9$ cubic units.

STEPS OF DEMONSTRATION

Step I The total volume of six echelon type of pieces is $6(1^2+2^2+3^2+4^2)$ cubic units.

Step II The volume of the bigger cuboidal block $= 4 \times 5 \times 9$ cubic units.

Since 6 echelon type of pieces and the bigger cuboidal block are made of same number of unit cubes.

$$\therefore \quad 6(1^2+2^2+3^2+4^2) = 4 \times 5 \times 9$$

$$\Rightarrow \quad 6(1^2+2^2+3^2+4^2) = 4(4+1)(4 \times 2+1)$$

$$\Rightarrow \quad 1^2+2^2+3^2+4^2 = \frac{1}{6}4(4+1)(2 \times 4+1)$$

Continuing in this manner it can be shown that

$$1^2+2^2+3^2+4^2+5^2 = \frac{1}{6}5(5+1)(2 \times 5+1)$$

$$1^2+2^2+3^2+4^2+5^2+6^2 = \frac{1}{6}6(6+1)(2 \times 6+1)$$

and so on.

In general

$$1^2+2^2+3^2+4^2+\ldots\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1)$$

ACTIVITY-3

OBJECTIVE To show that $1^3+2^3+3^3+\ldots\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

MATERIALS REQUIRED Chart papers of different colours, themocol sheet, thermocol balls, thumpins, cutter, scissors, adhesive etc.

STEPS OF CONSTRUCTION

Step I Take a square sheet of thermocol of dimension 15×15 cm and paste a chart paper on it.

Step II Draw horizontal and vertical lines to form $15 \times 15 = 225$ squares of dimension 1×1 cm as shown in Fig. 20.8.

Step III Put a thermocol ball at the square on the upper most left corner as shown in Fig. 20.8.

Step IV Fix $2^3 = 8$ thermocol balls in the shell as shown in Fig. 20.8.

Step V Fix $3^3 = 27$ thermocol balls in the shell as shown in Fig. 20.8.

Step VI Fix $4^3 = 64$ thermocol balls in the shell 4 as shown in Fig. 20.8

Step VII Fix $5^3 = 125$ thermocol balls in the shell 5 as shown in Fig. 20.8

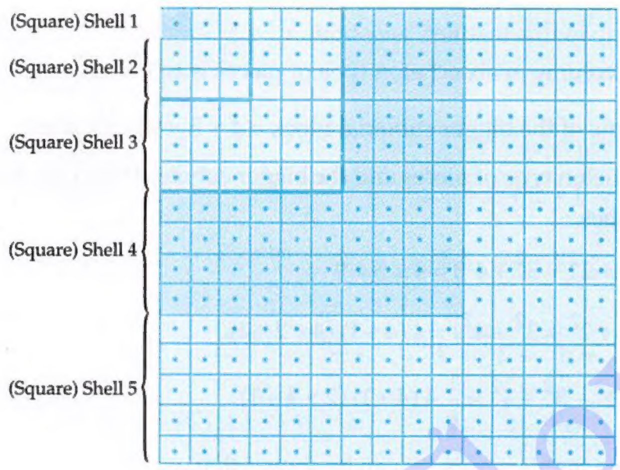

Fig. 20.8

STEPS OF DEMONSTRATION

Step I Number of thermocol balls in shell $I = 1 = 1^3 = \left\{\frac{1\,(1+1)^2}{2}\right\}$

Step III Number of thermocol balls in shell $II = 9 = 1^3 + 2^3$.

Also, $\left\{\frac{2\,(2+1)}{2}\right\}^2 = 9$

$\therefore \quad 1^3 + 2^3 = \left\{\frac{2\,(2+1)}{2}\right\}^2$

STEP III Number of thermocol balls in Shell III $= 36 = 1^3 + 2^3 + 3^3$

Also, $\left\{\frac{3\,(3+1)}{2}\right\}^2 = 36$

$\therefore \quad 1^3 + 2^3 + 3^3 = \left\{\frac{3\,(3+1)}{2}\right\}^2$

Step IV Number of thermocol balls in shell IV $= 100 = 1^3 + 2^3 + 3^3 + 4^3$

Also, $\left\{\frac{4\,(4+1)}{2}\right\}^2 = 100$

$\therefore \quad 1^3 + 2^3 + 3^3 + 4^3 = \left\{\frac{4\,(4+1)}{2}\right\}^2$

Step V Number of thermocol balls in shell V $= 225 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3$

Also, $\left\{\frac{5\,(5+1)}{2}\right\}^2 = 225$

$\therefore \quad 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = \left\{\frac{5\,(5+1)}{2}\right\}^2$

Continuing in this manner, we obtain $1^3+2^3+3^3+4^3+\ldots\ldots+n^3=\left\{\frac{n(n+1)}{2}\right\}^2$

SUMMARY

1. For any $n \in N$, we have

(i) $\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$

(ii) $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

(iv) $\sum_{k=1}^{n} k^4 = 1^4+2^4+3^4+\ldots+n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

2. In a series $a_1+a_2+a_3+a_4+\ldots+a_n+\ldots,$

(i) if the differences $a_2-a_1, a_3-a_2, a_4-a_3, \ldots$ are in A.P., then the nth term is given by
$$a_n = an^2+bn+c,$$ where a, b, c are constants.

(ii) if the differences $a_2-a_1, a_3-a_2, a_4-a_3, \ldots$ are in G.P. with common ratio r, then
$$a_n = ar^{n-1}+bn+c,$$ where a, b, c are constants.

To determine constants a, b, c we put $n=1, 2, 3$ and equate them with the values of corresponding terms of the given series.